ON DEVIATIONS FROM THE MAXIMUM
IN A STOCHASTIC PROCESS

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ABSTRACT

Let T be the first time a stochastic process $\{X(t)\}$ drops a units below its maximum to date. We determine the transform of the joint distribution of T and $X(T)$ for integer valued stochastic processes that are continuous in the sense that only jumps of ± 1 occur. We concentrate on spatially homogeneous processes, that is, when $\{X(t)\}$ has stationary independent increments. In this case, the first two moments of T and $X(T)$ are derived, and $X(T) + a$ is shown to have a geometric distribution, the parameter of which is given. Applications in queuing theory and to two armed bandit problems are sketched.

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1. Introduction and Summary. For a fixed value a let T be the first time a stochastic process $X(t)$ drops a units below its maximum to date. That is, setting $M(t) = \max\{X(u); 0 \leq u \leq t\}$, let

$$T = \inf\{t \geq 0; M(t) - X(t) \geq a\}.$$

In [3] the transform of the joint distribution of T and $X(T)$ was determined for $\{X(t); t \geq 0\}$ a Brownian motion having arbitrary drift and diffusion, and applications of this formula in economics and quality control were discussed. Later Lehoczky [1] extended the approach and obtained the transform for more general diffusion processes.

Here we derive the analogous formula when (i) $\{X(t); t = 0,1,2,...\}$ is a random walk for which

$$\Pr\{X(t+1) = i + 1 | X(t) = i\} = p_i,$$

$$\Pr\{X(t+1) = i | X(t) = i\} = r_i,$$

$$\Pr\{X(t+1) = i - 1 | X(t) = i\} = q_i,$$

where $p_i + r_i + q_i = 1$, and when (ii) $X(t)$ is the minimal process associated with the infinitesimal parameters

$$\Pr\{X(t+\Delta t) = i + 1 | X(t) = i\} = \mu_i \Delta t + o(\Delta t),$$

$$\Pr\{X(t+\Delta t) = i - 1 | X(t) = i\} = \nu_i \Delta t + o(\Delta t),$$

and

$$\Pr\{|X(t+\Delta t) - i| > 1 | X(t) = i\} = o(\Delta t).$$

In both cases the state space is the set of all integers and the process starts from $X(0) = 0$.

The key to the derivation is a simple observation: In order that $M(T)$ equal k, the process must first reach state 1 before hitting state $-a$, then, starting afresh from state 1, it must next reach state 2 before hitting state $-a+1$, and so on until finally, again starting afresh from state k, the process must reach state $k-a$ before hitting state $k+1$. We want to derive the joint transform $E[e^{\alpha X(T)-\beta T}]$, and while the factor $e^{-\beta T}$ complicates the analysis somewhat, nevertheless, the task reduces to solving a series of gambler's ruin type problems as just indicated.

Explicit results are obtained when the processes are spatially homogeneous, that is, when $p_i = p$ for all i, $\mu_i = \mu$ for all i, etc. Then

$$E[e^{\alpha X(T)-\beta T}] = \frac{e^{-\alpha a}(\lambda_+ - \lambda_-)}{\lambda_-^{-a}(\lambda_+ - e^{\alpha}) - \lambda_+^{-a}(\lambda_- - e^{\alpha})}$$

which holds for $\beta > 0$ and

$$\alpha < -\ln\{(\lambda_-^{-a} - \lambda_+^{-a})/(\lambda_+ \lambda_-^{-a} - \lambda_- \lambda_+^{-a})\},$$

and where

$$\lambda_\pm = \frac{(1-re^{-\beta}) \pm \{(1-re^{-\beta})^2 - 4pqe^{-2\beta}\}^{1/2}}{2pe^{-\beta}} \tag{1.2}$$

in Case (i), and

$$\lambda_\pm = \frac{(\beta+\nu+\mu) \pm \{(\beta+\nu+\mu)^2 - 4\nu\mu\}^{1/2}}{2\mu} \tag{1.3}$$

in Case (ii).

In both cases, when the processes are spatially homogeneous, $M(T)$ has a geometric distribution, $\Pr\{M(T) \geqq k\} = \theta^k$, and the parameter θ is identified.

The formula can be used to determine the moments of T and $X(T)$ and some asymptotic distributions as was done in [3].

Applications of some of these results in queuing theory and to the two-armed bandit problem are sketched in a later section.

2. *The Derivation*. To begin that part of the derivation which is common to both cases, let $X(t)$ be an integer-valued Markov process continuous in the sense that only jumps of ± 1 are permitted. Fix a positive integer a. Because the process is continuous, $X(T) = M(T) - a$ and the quantity we seek differs from $E[e^{\alpha M(T)-\beta T}]$, which we will study first, by the constant factor $e^{\alpha a}$.

Confine k to integer values and let $\tau(k) = \inf\{t \geqq 0;\ X(t) = k\}$ be the hitting time to k. Introduce the notation $E[\cdot;\ A] = \int_A \cdot \Pr\{d\omega\}$ for expectation restricted to an event A. The suggested decomposition into gambler's ruin type problems motivates us to write

$$E[e^{\alpha M(T)-\beta T}] = \sum_{k=0}^{\infty} e^{\alpha k}\, E[e^{-\beta T};\ M(T) = k]$$

$$= \sum_{k=0}^{\infty} e^{\alpha k} \left\{ \prod_{i=0}^{k-1} E[e^{-\beta\tau(i+1)};\ \tau(i+1) < \tau(i-a)\ |X(0) = i] \right\}$$

$$\times\ E[e^{-\beta\tau(k-a)};\ \tau(k-a) < \tau(k+1)\ |X(0) = k].$$

Set

$$\delta_i = \delta_i(\beta) = E[e^{-\beta\tau(i+1)};\ \tau(i+1) < \tau(i-a)\ |X(0) = i]$$

and

$$\gamma_i = \gamma_i(\beta) = E[e^{-\beta\tau(i-a)};\ \tau(i-a) < \tau(i+1)\ |X(0) = i]$$

so that

$$E[e^{\alpha M(T)-\beta T}] = \sum_{k=0}^{\infty} e^{\alpha k} \{\prod_{i=0}^{k-1} \delta_i\}\gamma_k\ .$$

The connection with gambler's ruin problems becomes clearer if we bring in a random lifetime ζ, independent of the $X(\cdot)$ process and having the exponential distribution

$$\Pr\{\zeta > t\} = e^{-\beta t},\ t \geqq 0,$$

and write $\underset{\sim}{1}_A = \underset{\sim}{1}(A)$ for the indicator random variable associated with an event A. Then, in view of the assumed independence

$$\begin{aligned}\delta_i &= E[e^{-\beta\tau(i+1)}\ \underset{\sim}{1}\{\tau(i+1) < \tau(i-a)\}|X(0) = i]\\ &= E[\underset{\sim}{1}\{\tau(i+1) < \tau(i-a)\}E[e^{-\beta\tau(i+1)}|X(\cdot)]|X(0) = i]\\ &= E[\underset{\sim}{1}\{\tau(i+1) < \tau(i-a)\}E[\underset{\sim}{1}\{\tau(i+1) < \zeta\}|X(\cdot)]|X(0) = i]\\ &= \Pr\{\tau(i+1) < \tau(i-a) \wedge \zeta\ |X(0) = i\}\end{aligned}$$

where $a \wedge b = \min\{a, b\}$, and similarly

$$\gamma_i = \Pr\{\tau(i-a) < \tau\ (i+1) \wedge \zeta|X(0) = i\}.$$

Thus δ_i (and γ_i) may be derived by solving a system of linear equations derived from a "first step analysis." Fix a state i and set $u(j) = u_i(j) = \Pr\{\tau(i+1) < \tau(i-a) \wedge \zeta | X(0) = j\}$. Then $\delta_i = u_i(i)$. In Case (i), $u_i(j)$ for $i - a \leq j \leq i + 1$ is determined by

$$u(i-a) = 0;\ u(i+1) = 1$$

while

$$u(j) = e^{-\beta}(p_j u(j+1) + r_j u(j) + q_j u(j-1)), \quad \text{for } i - a < j < i + 1.$$

Similar equations determine γ_i in Case (i). In Case (ii), again

$$u(i-a) = 0;\ u(i+1) = 1$$

while conditioning on the first jump yields

$$u(j) = \frac{1}{\mu_j+\nu_j+\beta} \{\mu_j u(j+1) + \nu_j u(j-1)\}, \text{ for } i - a < j < i + 1,$$

and again, similar equations determine γ_i.

3. Distribution of $M(T)$. It is easily seen from the foregoing analysis that

$$\Pr\{M(T) = k\} = (\prod_{j=0}^{k-1} \delta_j)\gamma_k$$

where $\delta_j = \delta_j(\beta=0) = 1 - \gamma_j$.

Explicitly,

$$\gamma_j = \Pr\{\tau(j-a) < \tau\ (j+1) | X(0) = j\}$$

$$= 1/(\sum_{i=0}^{a} \rho_{ji})$$

where $\rho_{j0} = 1$

and

$$\rho_{ji} = \frac{p_j \, p_{j-1} \times \ldots \times p_{j-i+1}}{q_j \, q_{j-1} \times \ldots \times q_{j-i+1}}, \quad \text{for} \quad 0 < i \leqq a,$$

in Case (i) while in Case (ii),

$$\rho_{ji} = \frac{\mu_j \, \mu_{j-1} \times \ldots \times \mu_{j-i+1}}{\nu_j \, \nu_{j-1} \times \ldots \times \nu_{j-i+1}} .$$

4. Spatially homogeneous processes. By far the most interesting results appear when the parameters $p_i = p$, $\mu_i = \mu$, etc. do not depend on the state i. From now on, we consider only this case.

Under this assumption, $\delta_i = \delta$ and $\gamma_i = \gamma$ are also constant and so

$$E[e^{\alpha M(T)-\beta T}] = \sum_{k=0}^{\infty} e^{\alpha k} \delta^k \gamma$$

$$= \gamma/(1-\delta e^{\alpha}) \tag{4.1}$$

for $\alpha < \ln(1/\delta)$.

In both of our cases we will determine distinct values $\lambda = \lambda_{\pm}$ for which

$$\begin{aligned} Z(t) &= \lambda^{X(t)}, \quad t < \zeta \\ &= 0 \quad\;\;\;\;, \quad t \geqq \zeta \end{aligned}$$

is a martingale, and use this to evaluate γ and δ. Invoking the martingale optional stopping theorem at the Markov time $T \wedge \tau(1) = \min\{T, \tau(1)\}$ we have

$$1 = E[Z(\tau(1) \wedge T)]$$

$$= E[Z(\tau(1)); \tau(1) \leqq T] + E[Z(T); T < \tau(1)]$$

$$= \lambda\delta + \lambda^{-a} \gamma.$$

Using the distinct values $\lambda = \lambda_+$ and $\lambda = \lambda_-$ we may solve for δ and γ to obtain

$$\gamma = \frac{\lambda_+ - \lambda_-}{\lambda_+\lambda_-^{-a} - \lambda_-\lambda_+^{-a}}$$

and

$$\delta = \frac{\lambda_-^{-a} - \lambda_+^{-a}}{\lambda_+\lambda_-^{-a} - \lambda_-\lambda_+^{-a}} .$$

Finally

$$E[e^{\alpha X(T)-\beta T}] = e^{-\alpha a}\left\{\frac{\gamma}{1 - e^{\alpha}\delta}\right\}$$

$$= \frac{e^{-\alpha a}(\lambda_+ - \lambda_-)}{\lambda_-^{-a}(\lambda_+ - e^{\alpha}) - \lambda_+^{-a}(\lambda_- - e^{\alpha}),} \tag{4.2}$$

which holds for $\beta > 0$ and $\alpha < \ell n(1/\delta)$.

It remains only to prescribe $\lambda = \lambda_\pm$.

Case (i): $X(t) = \xi_1 + \ldots + \xi_t$, $t = 1,2,\ldots$ where $\xi_1, \xi_2 \ldots$ are independent and share the common distribution

$$\begin{aligned} \Pr\{\xi_n = k\} &= p \quad \text{for} \quad k = 1, \\ &= r \quad \text{for} \quad k = 0, \\ &= q \quad \text{for} \quad k = -1, \end{aligned}$$

with $p + q + r = 1$. In this case

$$\begin{aligned} Z(t) &= \lambda^{X(t)}, \quad t < \zeta, \\ &= 0 \quad\;\;\;\, , \quad t \geqq \zeta, \end{aligned}$$

is a martingale provided

$$\lambda = \lambda_\pm = \frac{1 - re^{-\beta} \pm \{(1 - re^{-\beta})^2 - 4pqe^{-2\beta}\}^{1/2}}{2pe^{-\beta}} \tag{4.3}$$

because these λ solve $\lambda^k = e^{-\beta}[p\lambda^{k+1} + r\lambda^k + q\lambda^{k-1}]$ which is the computation that checks the martingale condition

$$E[Z(t+1)|Z(t) = \lambda^k] = \lambda^k.$$

Case (ii): $X(t) = U(t) - V(t)$ where $U(t)$ and $V(t)$ are homogeneous Poisson processes having rates μ and ν respectively. In this case

$$Z(t) = \lambda^{X(t)}, \quad t < \zeta$$
$$= 0 \quad , \quad t \geqq \zeta$$

is a martingale when

$$\lambda = \lambda_{\pm} = \frac{\beta+\nu+\mu \pm \{(\beta+\nu+\mu)^2 - 4\nu\mu\}^{1/2}}{2\mu} \tag{4.4}$$

5. *Some Marginal Distributions and Moments.* By setting $\alpha = 0$ we obtain the Laplace transform

$$E[e^{-\beta T}] = \frac{\lambda_+ - \lambda_-}{\lambda_-^{-a}(\lambda_+ -1) + \lambda_+^{-a}(1 -\lambda_-)}. \tag{5.1}$$

We will do nothing further with this now. Later, in connection with an application, we will develop the asymptotic distribution of T as $a \to \infty$ in Case (ii).

The marginal distribution of $X(T)$ has a nice explicit description. We have $X(T) = M(T) - a$ and $M(T)$ has a geometric distribution in which $\Pr\{M(T) \geqq k\} = \theta^k$, $k = 0,1,\ldots$ and $\theta = \delta|_{\beta=0} = \Pr\{\tau(1) \leqq T\}$ is a gambler's ruin probability. To see this, correspond $\zeta = \infty$ with $\beta = 0$ and write

$$(\delta_{\beta=0})^k = \Pr\{\tau(k) \leqq T\}$$
$$= \Pr\{M(T) \geqq k\}$$

To obtain θ, note that $\tau(1) \leq T$ if and only if the process reaches state 1 before state $-a$. The well know gambler's ruin probabilities are:

Case (i):

$$\begin{aligned} \theta &= a/(1+a) && \text{if } p = q \\ &= \frac{1-(q/p)^{-a}}{(q/p)-(q/p)^{-a}} && \text{if } p \neq q, \end{aligned} \tag{5.2}$$

and

Case (ii):

$$\begin{aligned} \theta &= a/(1+a) && \text{if } \nu = \mu \\ &= \frac{(\nu/\mu)^{a}-1}{(\nu/\mu)^{a+1}-1} && \text{if } \nu \neq \mu \end{aligned} \tag{5.3}$$

The mean of the geometrically distributed $M(T)$ is $E[M(T)] = \theta/(1-\theta)$ whence

$$E[X(T)] = \frac{\theta}{1-\theta} - a \tag{5.4}$$

with θ given in (5.2) and (5.3). We also obtain the variances

$$\mathrm{Var}[X(T)] = \mathrm{Var}[M(T)] = \theta/(1-\theta)^2. \tag{5.5}$$

Finally we may use Wald's equation $E[X(T)] = E[T]E[X(1)]$, to get

$$\begin{aligned} E[T] &= \frac{1}{p-q}\left(\frac{\theta}{1-\theta} - a\right) && \text{in Case (i) with } p \neq q, \\ &= \frac{1}{\mu-\nu}\left(\frac{\theta}{1-\theta} - a\right) && \text{in Case (ii) with } \mu \neq \nu. \end{aligned} \tag{5.6}$$

When $p = q$ or $\mu = \nu$ we use $E[X(T)^2] = E[T]E[X(1)^2]$ to get

$$E[T] = \frac{\theta}{(1-r)(1-\theta)^2}, \quad \text{Case (i)}, \quad p = q,$$

$$= \frac{\theta}{2\nu(1-\theta)^2}, \quad \text{Case (ii)}, \quad \nu = \mu.$$

6. Applications. Here are brief sketches of two applications.

Queuing theory. In Case (ii), $Y(t) = M(t) - X(t)$ evolves like the queue length in an M/M/1 queue system having arrival rate ν and service rate μ. Then T is the first time the queue holds a customers.

Referring to (5.3) and (5.6), we see the mean is

$$E[T] = \frac{1}{\mu-\nu}\left\{\left(\frac{\mu}{\nu}\right)^a \frac{1-(\nu/\mu)^a}{1-(\nu/\mu)} - a\right\}$$

When the service rate μ exceeds the arrival rate ν, for large a the dominant term is $[\mu/(\mu-\nu)^2](\mu/\nu)^a$.

In general, we have the Laplace transform

$$E[e^{-\beta T}] = \gamma/(1-\delta)$$

$$= \frac{\lambda_+ - \lambda_-}{\lambda_-^{-a}(\lambda_+ - 1) - \lambda_+^{-a}(\lambda_- - 1)}$$

with $\lambda_\pm$ given by (4.4).

We will derive the asymptotic distribution of T as $a \to \infty$ when the service rate μ exceeds the arrival rate ν. Indeed, we will show that asymptotically $U_a = (\nu/\mu)^a T$ is exponentially distributed with parameter $(\mu-\nu)^2/\mu$. To do this we need only show the Laplace transforms corresponding to $U = U_a$ have the limit

$$\lim_{a\to\infty} E[e^{-\beta(\nu/\mu)^a T}] = \frac{(\mu-\nu)^2/\mu}{(\mu-\nu)^2/\mu + \beta.}$$

To begin the proof, stipulating that $\mu > \nu$ and using elementary calculus, derive the Taylor series expansions

$$\lambda_+(\beta) = 1 + \frac{1}{\mu-\nu}\beta + o(\beta)$$

and

$$\lambda_-(\beta) = \frac{\nu}{\mu}\left(1 - \frac{1}{\mu-\nu}\beta\right) + o(\beta)$$

for $\lambda_\pm = \{\beta+\mu+\nu \pm [(\beta+\mu+\nu)^2 - 4\mu\nu]^{1/2}\}/2\mu$ when $\mu > \nu$. Then

$$\lim_{a\to\infty} E[e^{-\beta U}] = \lim_{a\to\infty} E[e^{-\beta(\nu/\mu)^a T}]$$

$$= \lim_{a\to\infty} \frac{\lambda_+((\frac{\nu}{\mu})^a\beta) - \lambda_-((\frac{\nu}{\mu})^a\beta)}{\dfrac{\lambda_+((\frac{\nu}{\mu})^a\beta)-1}{[\lambda_-((\frac{\nu}{\mu})^a\beta)]^a} + \dfrac{1-\lambda_-((\frac{\nu}{\mu})^a\beta)}{[\lambda_+((\frac{\nu}{\mu})^a\beta)]^a}}$$

$$= \frac{1 - (\nu/\mu)}{\lim\limits_{a\to\infty}\left\{\dfrac{\frac{1}{\mu-\nu}\beta(\frac{\nu}{\mu})^a}{(\frac{\nu}{\mu})^a\left[1-\frac{1}{\mu-\nu}\beta(\frac{\nu}{\mu})^a\right]^a}\right\} + 1-(\nu/\mu)}$$

$$= \frac{1 - (\nu/\mu)}{\beta/(\mu-\nu) + 1 - (\nu/\mu)} = \frac{(\mu-\nu)^2/\mu}{\beta + (\mu-\nu)^2/\mu}$$

as claimed.

In summary, roughly speaking, the time it takes to completely fill a large waiting room holding a customers in an M/M/1 queueing system is approximately exponentially distributed with mean $[\mu/(\mu-\nu)^2]\cdot(\mu/\nu)^a$.

<u>Two armed bandit problems.</u> A play on a one armed bandit wins a dollar with probability p and loses a dollar with probability $q = 1 - p$. Suppose we face two such machines having unknown win probabilities p_1 and p_2 respectively. We begin with machine 1 and continue to play it as long as wins are secured. At the first loss, we switch to machine 2, and now play it as long as we win. When we first lose on machine 2 we revert to machine 1 and repeat the cycle. This strategy is called the <u>Play-the-winner rule</u>. In the notation of this article, it corresponds to playing a machine until time $T = T_a$ for $a = 1$. That is, we continue on a machine as long as our fortune increases and new maxima are reached on each play. At the first loss, when our fortune first drops one unit below its maximum to date, we stop play and switch to the other machine. It is an obvious question to look at the behavior of the strategies $T = T_a$ for $a > 1$ in this context.

Let E_1 and E_2 denote the expectations under p_1 and p_2 respectively. An easy argument shows that the long run average gain per play is

$$G(a) = \frac{E_1[X(T_a)] + E_2[X(T_a)]}{E_1[T_a] + E_2[T_a]}$$

$$= \frac{(p_1-q_1)E_1[T_a] + (p_2-q_2)E_2[T_a]}{E_1[T_a] + E_2[T_a]} .$$

Let us suppose that at least one of the games is favorable, that is, at least one win probability exceeds one-half. To be definite, suppose $p_1 = \max\{p_1,p_2\} > 1/2$. As a increases, the dominant term in

$$E_1[T_a] = \frac{1}{p_1-q_1}\left\{\left(\frac{p_1}{q_1}\right)^a\left[\frac{1-(q_1/p_1)^a}{1-q_1/p_1}\right] - a\right\}$$

is

$$E_1[T_a] \sim \frac{p_1}{(p_1-q_1)^2}\left(\frac{p_1}{q_1}\right)^a$$

That is, $E_1[T_a]$ grows geometrically with a at rate p_1/q_1. The same argument shows that $E_2[T_a]$ can grow at most geometrically at rate $p_2/q_2 < p_1/q_1$. In fact $E_2[T_a]$ asymptotically grows linearly in a if $p_2 < 1/2$. In either case we have

$$\lim_{a\to\infty} \{E_2[T_a]/E_1[T_a]\} = 0$$

and

$$\lim_{a\to\infty} G(a) = \lim_{a\to\infty} \frac{(p_1-q_1) + (p_2-q_2)E_2[T_a]/E_1[T_a]}{1 + E_2[T_a]/E_1[T_a]}$$

$$= p_1-q_1,$$

the gain rate if machine 1 were played continuously! That is, as $a \to \infty$ the strategy T_a is asymptotically optimal, and in particular, does significantly better than the Play-the-winner rule. Would the same improvement over Play-the-winner appear if criteria other than average gain were used? In particular, what about criteria more appropriate to the medical trials problem [2]? These are provocative open questions.

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