# POSSIBLE ABSOLUTE DETERMINANT VALUES FOR SQUARE (0,1)-MATRICES USEFUL IN FRACTIONAL REPLICATION†

by

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## Abstract

Let $D\big((n + 1) \times n\big)$ denote an $(n + 1) \times n$ (0,1)-matrix with distinct rows, and let $X^* = [\underline{1} \vdots D]$ where $\underline{1}$ denotes a vector with every element unity. There are $\binom{2^n}{n+1}$ possible matrices D, and hence $X^*$; a problem with important mathematical and statistical implications is to determine all possible values of $\|X^*\|$. A general solution to the problem appears intractable at the present time. In this paper we consider ten general methods of constructing D and obtain the possible values of $\|X^*\|$ from each. For $n \le 7$ the problem has been solved by enumeration, and for these cases the methods of construction easily produce all possible values. When $n = 7$, the possible values are all integers $\le 18$, 20, 24, and 32 with the unlisted values between 18 and 32 being unattainable. For $n = 8$, the values obtained include all integers $\le 33$, 36, 40, 44, 48, 56, and for $n = 9$ all integers $\le 64$, 66, 68, 69, 72, 76, 80, 81, 84, 88, 92, 96, 100, 104, 120, 144. For $n = 10$, 151 values were obtained with the largest being 264, and for $n = 11$, 302 different values were obtained, the largest being 1458. For $n > 7$, it is likely that these constructions will not produce all possible determinant values. It is shown that for $2^{p-1} \le n \le 2^p - 1$ that the possible ranks for D are $p, p + 1, \cdots, n$.

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## 1. Introduction and Summary.

Mathematical research on (0,1)-matrices and on the related (-1,1)-matrices has remained active over the years with many problems still unresolved. One area of research has been directed toward a study of those square (0,1)-matrices which are singular (e.g., see Metropolis and Stein [1967]) and which attain a maximum absolute value of the determinant (e.g., see Williamson [1946], Mood [1948], Ryser [1963]). Here it should be noted that Hadamard [1873] gave an upper bound for $k \times k$ (-1,1)-matrices as $\leq k^{k/2}$ with equality being achieved for Hadamard matrices. This result may be directly translated to (0,1)-matrices (see Raktoe and Federer [1970])where the upper bound on the maximal value of the determinant of a $k \times k$ (0,1)-matrix is $\leq k^{k/2}/2^{k-1}$ with equality being achieved for those (0,1)-matrices which are obtained from (-1,1)-Hadamard matrices by replacing the minus ones with zeros. Limited attention has been paid to the set of possible values for the determinant of square (0,1)-matrices (e.g., see Paik and Federer [1970b] and Wells [1971]). Federer et al. [1973] have discussed a number of unsolved problems related to these matrices.

In statistics, (0,1)-matrix theory is directly useful in the setting where n factors or variables are present and each factor takes on the values zero or one.

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If all combinations of the two levels of each factor are present, a $2^n$-factorial with $2^n$ distinct combinations result. In some settings only k of the $2^n$ combinations are used resulting in a fractional replicate plan. When $k = n + 1$ and when it is desired to estimate the mean and the main effect of each of the n factors, a saturated main effect plan results. Since there are $\binom{2^n}{n+1}$ possible such plans, a study of their properties is desirable in order that the investigator can select a plan which is best for his needs under some criterion. One criterion is to select plans having a specified value (usually the maximal) of the $(n + 1) \times (n + 1)$ determinant formed from the n + 1 combinations with a row of ones added in the first column. This results in a semi-normalized (0,1)-matrix.

An investigation of the possible absolute values that a square normalized (0,1)-matrix can take was given by Paik and Federer [1970b] for n = 2, 3, and 4 and by Wells [1971] for n = 3, 4, 5, 6, and 7. Both used complete enumerations. Wells [1971] considered $n \times n$ (0,1)-matrices with distinct rows and without the combination of all zeros whereas the former authors used $(n + 1) \times (n + 1)$ (0,1)-matrices where the last n elements of the first row were zeros. The same possible values of the determinants are obtained in both cases. Since it has not been possible to give all possible values of the determinants for any n, we shall present a number of methods to demonstrate values that are attainable for various (0,1)-semi-normalized matrices. Ten such methods have been found for obtaining values of the determinants. These are summarized in the main theorem of the paper as presented in the next section. Values obtained by these methods are presented for n = 8, 9, 10, and 11 in the third section. In the fourth section, we discuss possible ranks and maximum values of determinants, and in the final section some unsolved problems related to determinants of (0,1)-semi-normalized matrices, are presented.

## 2. Notations, the Main Theorem, and Constructions.

In this paper $X^*$ will denote a (0,1)-matrix with the first column having every element unity, denoted by $\underline{1}$. A column, other than the first, may be changed by interchanging 0's and 1's without changing the absolute value of $|X^*|$. Thus, the first row of $X^*$ can be brought to (1, 0, 0, ⋯, 0) and in this form $X^*$ will be said to be in standard form (see Paik and Federer [1970a], [1972]). Since the sign of the determinant can be changed by the interchange of two columns, we shall hereafter always refer to the absolute value of the determinant.

For compactness a product of two square bracket notations will be used to denote all possible products of quantities between the brackets. That is,

$$[a_1, a_2, \cdots, a_m][b_1, b_2, \cdots, b_n] = \{a_i b_j;\ i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n\}.$$

Further, the notation $[|U^*|][|V^*|]$ will be used to denote all possible products of all possible values of $|U^*|$ and $|V^*|$. Likewise, this notation may be extended for k products of determinants. The symbol $\bar{V}^*$ denotes the complement of the matrix $V^*$, that is, the zeros replace ones and ones replace zeros in $V^*$ to obtain $\bar{V}^*$.

Using this notation we now state the following theorem:

Theorem 2.1. *Let $X^*$ be an $(n + 1) \times (n + 1)$ matrix of zeros and ones with first column all ones and first row (1, 0, 0, ⋯, 0). Then, the following are possible values for $|X^*|$:*

(1) 0, 1, 2, ⋯, n - 1.

(2) *Sum composition: If $n = \sum_{i=1}^{k} n_i$, then $|X^*|$ may take values* $\prod_{i=1}^{k} [1, 2, \cdots, n_i - 2]$.

(3) *Kronecker Product: If $n + 1 = a \cdot b$, then $|X^*|$ may take values $2^{(a-1)(b-1)} \times [|U^*|]^b[|V^*|]^a$ where $|U^*|$ and $|V^*|$ may take any possible value for the $2^{a-1}$ and $2^{b-1}$ experiments, respectively.*

(4) Modified Kronecker Product: If $n + 1 = 2b$, then $|X^*|$ may take values $2^{b-1}[|U^*|][|V^*|]$ where $|U^*|$ and $|V^*|$ may take any of the possible values for the $2^{b-1}$ experiment.

(5) Modified Sum Composition: If $n + 1 = a + b$ with $a \geq b$, then possible values of $|X^*|$ are $2^{b-1}[|U^*|][|V^*|]$ where $|U^*|$ and $|V^*|$ may take any of the possible values for the $2^{a-1}$ and $2^{b-1}$ experiments, respectively.

(6) Hadamard Matrices: If $n = 4t - 1$, then $|X^*|$ may take values $(2t - k)t^{2t-1}$, $k = 0, 1, \cdots, 2t$, if a Hadamard matrix of order $4t$ exists.

(7) Cutting from Hadamard Matrix: If $n = 4t - 1 - k$, then possible values for $|X^*|$ are $t^{2t-k}$, $k = 1, 2, 3$, and $2t^{2t-4}$ if $k = 4$.

(8) Adjoining to Hadamard Matrices: If $n = 4t$, then $|X^*|$ may take values $(4t - k)t^{2t-1}$, $k = 1, 2, \cdots, 4t$, and if $n = 4t + 1$, $|X^*|$ may take values $(8t - 2 - k)t^{2t-1}$, $k = 1, 2, \cdots, 8t - 2$.

(9) Balanced Incomplete Block Designs: Suppose there exists a symmetric balanced incomplete block design $(v, k, \lambda)$. Then, if $n = v - 1$, a possible value for $|X^*|$ is $(k - \lambda)^{(v-1)/2}(v - 2k)$, and if $n = v$, possible values of $|X^*|$ are $|k - i|(k - \lambda)^{(v-1)/2}$, $i = 0, 1, \cdots, v$.

(10) Single Step Sum Composition: Suppose $U^*$ is an $n \times n$ (0,1)-matrix in standard form and let $U_i^*$ be obtained from $U^*$ by placing a one in the $(1, i+1)$ position. Then possible values for the $(n + 1) \times (n + 1)$ (0,1)-matrix $X^*$ include $|U^*| + |U_i^*|$, $i = 1, 2, \cdots, n - 1$.

The method of proof of the theorem is by construction. Hence, the method of proving the theorem provides a method of constructing a fractional replicate from the $2^n$-factorial resulting in a specified value of the determinant. The proof then is valuable in the construction of fractional replicates.

The essence of the theorem is not that it provides a large number of intricate constructions. Rather, it gives a few simple constructions that generate many different values for $|X^*|$.

The construction leading to the possible values stated in theorem 2.1 will be given here. The details of calculating the actual determinant is omitted to conserve space.

(1) The matrix $X^* = \begin{bmatrix} 1 & \underline{0}' \\ \underline{1} & I_n \end{bmatrix}$ has determinant one. If the zeros and ones of k of the columns of $I_n$ are interchanged, the determinant becomes $(k - 1)$, $k = 1, 2, \cdots, n$.

(2) Let

$$X^* = \left[\begin{array}{c|cccc} 1 & & \underline{0}' & & \\ \hline & X^*_{n_1 \times n_1} & 0 & \cdots & 0 \\ \underline{1} & 0 & X^*_{n_2 \times n_2} & & 0 \\ & \vdots & & & \vdots \\ & 0 & 0 & & X^*_{n_k \times n_k} \end{array}\right]$$

and the result follows immediately from part (1).

(3) Suppose $n + 1 = a \cdot b$ and let $U^*$ and $V^*$ denote (0,1)-matrices of size $a \times a$ and $b \times b$, respectively. Form U and V from $U^*$ and $V^*$ by replacing 0's by (-1)'s. Then, let $X = U \otimes V$ and form $X^*$ from X by replacing (-1)'s by 0's and convert to standard form.

(4) Suppose $n + 1 = 2b$ and let $U^*$ and $V^*$ be two (0,1)-matrices of size $b \times b$ in standard form. Then, take

$$X^* = \left[\begin{array}{c|c} U^* & U^* \\ \hline V^* & \bar{V}^* \end{array}\right].$$

(5) Suppose $n + 1 = a + b$ with $a \geq b$ and let $U^*$ be an $a \times a$ (0,1)-matrix partitioned as

$$U^* = \left[\underline{1} \;\vdots\; \overset{a-b}{U_1^*} \;\vdots\; \overset{b}{U_2^*}\right].$$

Then, for any $b \times b$ (0,1)-matrix $V^*$, $X^*$ is formed as:

$$X^* = \left[\begin{array}{c|c|c|c} 1 & U_1^* & U_2^* & U_2^* \\ \hline \underline{1} & 0 & V^* & \bar{V}^* \end{array}\right].$$

We note that any b columns of $U^*$ may be used to form $U_2^*$, each yielding the same values given in (5). However, if new $X^*$ matrices are formed by adding ones in the first row, different choices of columns may give rise to different values of the determinant.

(6) Suppose a Hadamard matrix of order 4t, $H_{4t}$, exists and is put into a form with first column all ones and first row (1, -1, -1, ···, -1). The (0,1)-matrix $X^*$ is obtained from $H_{4t}$ by replacing (-1)'s by 0's. If k of the zeros in the first row are changed to ones, we have $|X^*| = (2t - k)t^{2t-1}$.

(7) Suppose $H_{4t}$ exists. Cutting a single row and column from $H_{4t}$ and forming $X^*$ by replacing (-1)'s by 0's as before, we obtain $|X^*| = t^{2t-1}$. If we cut either two (or three) rows and columns and if the $2 \times 2$ (or $3 \times 3$) corner matrix cut out is nonsingular, we obtain $|X^*| = t^{2t-2}$ (or $t^{2t-3}$ cutting 3 rows and columns). If the $2 \times 2$ (or $3 \times 3$) submatrix deleted is singular, we have

$|X^*| = 0$. If we cut four rows and four columns so that the $4 \times 4$ matrix deleted is $H_4$, then the determinant of $X^*$ is $2t^{2t-4}$.

(8) Suppose $H_{4t}$ exists; suppose one forms $U^*$ from $H_{4t}$ by changing (-1)'s to 0's and puts $U^*$ in standard form. If $U^* = [\underline{1}, \underline{u}_1, \underline{u}_2, \cdots, \underline{u}_{4t-1}]$, then form $X^*$ as

$$X^* = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & & 0 & 0 \\ \underline{1} & \underline{u}_2 & \underline{u}_3 & \cdots & \underline{u}_i & \cdots & \underline{u}_{4t-1} & \bar{\underline{u}}_i \\ 1 & 0 & 0 & & 1 & \cdots & 0 & 1 \end{bmatrix}.$$

Then, $|X^*| = (4t-1)t^{2t-1}$. Additional ones in the last row produce the remaining values in (8) for $n = 4t$. If $n = 4t + 1$, form $X^*$ as

$$X^* = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \underline{1} & \underline{u}_2 & \underline{u}_3 & \cdots & \underline{u}_i & \cdots & \underline{u}_j & \cdots & \underline{u}_{4t-1} & \bar{\underline{u}}_i & \bar{\underline{u}}_j \\ 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & & 0 & & 1 & & 0 & 0 & 1 \end{bmatrix},$$

we obtain $|X^*| = (8t - 3)t^{2t-1}$ and the remaining values are obtained by interchanging ones and zeros in the last row.

The results of Dykstra [1971], Mitchell [1972], and Wynn [1970] are related to this method of construction.

(9) Suppose there exists a $(v, k, \lambda)$ design. If $n = v - 1$, the incidence matrix of the design itself may be put into standard form to yield an $X^*$ with $|X^*| = (k - \lambda)^{(v-1)/2}(v - 2k)$. If $n = v$, adjoin a column of all ones and a row of zeros to obtain an $X^*$ with $|X^*| = k(k - \lambda)^{(v-1)/2}$. The value $(k - i)(k - \lambda)^{(v-1)/2}$ is obtained by placing i ones in the first row.

(10) Consider the $n \times n$ (0,1)-matrix $U^*$ and let

$$X^* = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \underline{1} & \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_i & \cdots & \underline{u}_{n-1} & \bar{\underline{u}}_i \\ 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 1 \end{bmatrix}$$

to obtain the desired values of the determinant of $X^*$.

## 3. Specific Results for $n \leq 11$.

For $n \leq 7$ the possible values for $|X^*|$ are known, Wells [1971]. In this range, the constructions of theorem 2.1 easily produce all the possible values. When n = 2, 3, 4, 5, 6, the possible values include all integers from 0 through, respectively, 1, 2 = 1 + 1, 3 = 2 + 1, 5 = 3 + 2, and 9 = 5 + 4. For these values the modified sum composition one step at a time is sufficient to generate all possible values. The following array is presented to illustrate this construction and to provide a matrix yielding the maximum value for each $n \leq 6$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| n = 2 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| n = 3 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| n = 4 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| n = 5 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| n = 6 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |

(3.1)

The possible values for n = 7 are all integers 0 through 18, 20, 24, and 32. As one would expect, from array (3.1) we can obtain 17 = 9 + 8 via method 10 and adjoining the column (0, 1, 0, 1, 0, 1, 1, 1)' and row (1, 0, 0, 0, 1, 0, 0, 1). Suppose, however, that we write the row as $(1, a_1, a_2, \cdots, a_7)$ and calculate $|X^*|$ by expansion of the last row to obtain:

$$|X^*| = -5a_1 - 3a_2 - 2a_3 + 8a_4 - 2a_5 - 4a_6 + 9a_7.$$

The value 17 is obtained with $a_4 = a_7 = 1$ and the remaining $a_i$ zero. The value -16 is obtained by taking $a_4 = a_7 = 0$ and the remaining $a_i$ equal to one. It can be checked that all values, in absolute value, from 0 through 17 are obtainable by various selections of the $a_i$.

From method (7) and cutting four rows and columns from $H_{12}$, we obtain an $X^*$ with $|X^*| = 2 \cdot 3^2 = 18$. Change of 0 to 1 in the first row results in step-downs of magnitude 3, thus we obtain 18, 15, 12, 9, 6, 3, 0 from this construction. The value of 20 was obtained via sum composition with 8 = 5 + 3, and changes in the first row yielding step-downs of size 4, that is, 20, 16, 12, 8, 4, 0. The Hadamard matrix $H_8$ gives $|X^*| = 32$ and in step-downs of size 8, thus 32, 24, 16, 8, 0.

It is fairly easy to see what will occur in the next step for n = 8 using method (10). From the $X^*$ with $|X^*| = 17$ we will obtain a $9 \times 9$ matrix with determinant 32 = 17 + 15, and by various selections of $a_1, a_2, \cdots, a_8$ as before all integer values from 0 through 32. From the $X^*$ with $|X^*| = 18$ we obtain a $9 \times 9$ matrix with determinant 33 = 18 + 15 and step-downs of size 3. Adjoining to $H_8$, method (8), we obtain values $7 \cdot 2^3 = 56$, 48, 40, 32, 24, 16, 8, 0. A value of 36 = 20 + 16 is obtained from $|X^*| = 20$. Finally, the modified sum composition with one change of 0 to 1 in one row yields a value of 44. Thus, for n = 8 we obtain

all integers $\leq$ 33, 36, 40, 44, 48, 56.

It is possible, even likely, that other values are obtainable, perhaps even from the stated methods. It should be noted that for n = 8, Mitchell [1972] also obtained 56 as the maximal value.

The following table presents the values of $|X^*|$ obtained for n = 9, 10, and 11. Again, it is likely that some other values are attainable and perhaps even from the construction of theorem 2.1. For n = 9, Yang [1968] has shown that the maximum value is 144 obtained by our method (3).

Table 3.1. Possible Values of $|X^*|$, n = 9, 10, 11.

| | |
|---|---|
| n = 9 | all integers ≤ 64, 66, 68, 69, 72, 76, 80, 81, 84, 88, 92, 96, 100, 104, 120, 144. |
| n = 10 | all integers ≤ 121, 123, 124, 126, 128, 129, 132, 135, 136, 140, 144, 148, 152, 156, 160, 162, 164, 168, 172, 176, 180, 184, 192, 196, 200, 208, 216, 240, 243, 264. |
| n = 11 | all integers ≤ 238, 240, 243, 244, 246, 248, 249, 252, 255, 256, 258, 260, 261, 264, 267, 268, 270, 272, 276, 280, 284, 288, 292, 296, 297, 300, 304, 308, 312, 316, 320, 324, 328, 332, 336, 340, 344, 348, 352, 356, 360, 364, 368, 372, 376, 380, 384, 392, 400, 408, 432, 456, 480, 486, 504, 512, 560, 640, 720, 729, 800, 972, 1215, 1458. |

## 4. Results on the Possible Ranks of and Bounds on (0,1)-Matrices.

Let X* be written in standard form as

$$X^* = \begin{bmatrix} 1 & \underset{\sim}{0}'_{1\times n} \\ \underset{\sim}{1}_{n\times 1} & D^*_{n\times n} \end{bmatrix}.$$

The rank of X*, and indeed the possible ranks, is a matter of interest. The following theorem is in this direction.

Theorem 4.1. The possible ranks of the matrix $X^*$ are $p + 1, p + 2, \cdots, n + 1$ for $2^{p-1} \le n \le 2^p - 1$.

Proof: The proof for $n = 2^p - 1$ is by construction of $D^*$. Let

$$X^* = \left[\begin{array}{c|c} 1 & \underset{\sim}{0}'_{1\times n} \\ \hline \underset{\sim}{1}_{n\times 1} & D^*_{n\times n} \end{array}\right].$$

Construct $D^*$ equal to

$$\begin{bmatrix} I_{p\times p} & 0 \\ \underset{\sim}{LI}_{p\times p} & 0 \end{bmatrix}$$

where $LI_{p\times p}$ is a $(2^p - 1 - p) \times p$ matrix whose rows are linear combinations of $I_{p\times p}$ and all distinct. Hence all rows in $D^*$ are distinct combinations and the rank of $D^*$ equals rank of $I_{p\times p}$ which is p. Now substitute one combination which is not a linear combination of the rows of $I_{p\times p}$ for one of the last $2^p - 1 - p$ rows of $D^*$; the rank will be $p + 1$. Continue this process until full rank, $2^p - 1$, is attained.

The proof for $2^{p-1} \le n < 2^p - 1$ is by contradiction. We first select $I_{(p-1)\times(p-1)}$. Note that there are too few rows to construct the remaining rows of $D^*$ and that it is necessary to use $I_{p\times p}$ in order to construct the remaining rows of $D^*$ as linear combinations of the first p rows.

Therefore, the possible ranks of $X^*_{(n+1)\times(n+1)}$ are $p + 1, p + 2, \cdots, n + 1$.

An upper bound on the absolute value of the determinant of $X^*$ may be obtained from Hadamard's theorem as shown by Raktoe and Federer [1970] as

$$|X^*| \le 2^{-n}(n + 1)^{(n+1)/2}.$$

An improvement in this bound is obtained by noting that $|X^*|$ is an integer and hence

$$|X^*| \le \text{integer part of } 2^{-n}(n + 1)^{(n+1)/2}.$$

It was noted in theorem 2.1 that the maximum value of $|X^*|$ for n = k is a possible value for $|X^*|$ with n = k + 1. Thus the maximum value for n = k + 1 is at least as large as the maximum value for n = k. Table 4.1 indicates the largest values obtained from the constructions of theorem 2.1 for n ≤ 42. These values, and the corresponding values for larger n, serve as lower bounds on the maximum value of $|X^*|$.

Table 4.1. Largest Determinant Value Obtained.

| t | n = 4t - 1 | n = 4t | n = 4t + 1 | n = 4t + 2 |
|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 9 |
| 2 | 32 | 56 | 144 | 264 |
| 3 | $6(3)^5$ | $15(3)^5$ | $27(3)^5$ | $4^7$ |
| 4 | $8(4)^7$ | $20(4)^7$ | $29(4)^7$ | $5^9$ |
| 5 | $10(5)^9$ | $19(5)^9$ | $37(5)^9$ | $6^{11}$ |
| 6 | $12(6)^{11}$ | $42(6)^{11}$ | $96(6)^{11}$ | $7^{13}$ |
| 7 | $14(7)^{13}$ | $27(7)^{13}$ | $53(7)^{13}$ | $8^{15}$ |
| 8 | $16(8)^{15}$ | $31(8)^{15}$ | $61(8)^{15}$ | $9^{17}$ |
| 9 | $18(9)^{17}$ | $63(9)^{17}$ | $125(9)^{17}$ | $10^{19}$ |
| 10 | $20(10)^{19}$ | $39(10)^{19}$ | $77(10)^{19}$ | $11^{21}$ |

The values in Table 4.1 are not all the maximum attainable. If $n = r\cdot 4^{k+1}$, $k \ge 0$, $r \le 9$ is odd, and g(n) and g(n + 1) denote the maximal determinants of $X^*$, respectively, then

$$g(n + 1) \ge \tfrac{1}{2}\{1 + \tfrac{1}{4}(r + 3)\sqrt{n/r}\,\}g(n),$$

Schmidt [1973]. This bound is larger than the table value, for example, when $k = 0$, $r = 5,7$.

## 5. Discussion.

Since it was not possible to obtain the full spectrum of absolute values of the determinant of (0,1)-matrices, ten methods of construction were presented in theorem 2.1 which would produce many values of the spectrum. Also, it should be noted that the method of construction given by Schmidt [1973] could have been added to theorem 2.1 as method (11). Now, the question arises, how does one prove that the construction methods presented yield, or do not yield, the full spectrum of values? If not, what other construction methods are possible to obtain different values for the determinant for some n?

We have presented construction methods to obtain specified values of the determinant of (0,1)-matrices. We know from Wells [1971] for $n = 7$ that no (0,1)-matrices exist which produce a determinant value of 19, 21, 22, 23, 25, 26, 27, 28, 29, or 31. The method of proof was complete enumeration. This method of proof is impossible for large n. Hence, how does one prove analytically, even for $n = 7$, that certain values of the determinant are impossible? If one could prove that all values for $n = 9$ not in Table 3.1, are impossible, then this would provide the full spectrum of possible values. It appears that the area of non-existence of certain values for the determinant would be an important and fruitful direction of research on (0,1)-matrices.

From Paik and Federer [1970a, 1970b, 1972] we may note that for each standard form of the (0,1)-matrix, a permutation of zeros and ones within each column provides $2^n$ distinct plans for $n + 1 \neq 2^k$. Also, from the similar plans or designs given by Joiner [1973], we may note that a permutation of columns of the matrix produces additional distinct plans. Under a level permutation or a factor (column) permutation the value of the determinant remains invariant. Now, for any specified value of the determinant, how many distinct plans are there? For singular (0,1)-matrices with distinct rows and with the (00 ··· 0) row omitted, Metropolis and Stein [1967] were able to obtain a lower bound on the number of plans. How does one improve this bound? How does one obtain the number of distinct plans for any specified value of the determinant?

In order to simplify the construction problem of fractional replication, is there an algorithm for producing all possible plans from a specified plan such that all plans have the same absolute value of the determinant of $X^*$? Answers to these and other related questions would greatly simplify the construction problem in fractional replication from the $2^n$ factorial and would contribute to the mathematical and combinatorial theory of (0,1)-matrices.

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