Extended Generalized Right-Angular Designs*

by

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ABSTRACT

Extended generalized right-angular designs were introduced to obtain generalizations of generalized right-angular designs and to enlarge the scope of confounding for asymmetrical factorial designs In this paper, it is shown how to construct two series of this class of experimental designs and how to use these two series of designs in constructing confounded asymmetrical factorial designs. The relative loss of information due to confounding on each of the interactions is also presented.

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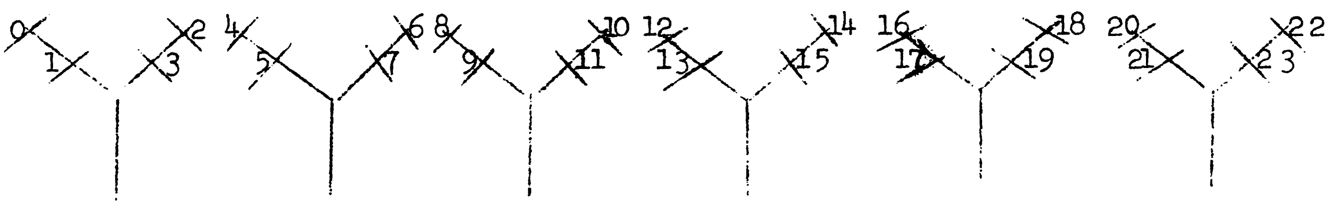
1. Definition
2. Eigen values and vectors of NN'
3. Constructions of two series
4. Applications as confounded asymmetrical factorial experiments

Definition

With a view to enlarge the scope of confounded asymmetrical factorial experiments, Raghavarao and Aggarwal (1971) introduced extended generalized right-angular (EGRA) designs, which are further generalizations of generalized right-angular (GRA) designs of Tharthare (1965).

To define EGRA association schemes, let there be $v = uv^*st$ symbols arranged as u symbols on each of the t branches of v^*s trees. Let the v^*s be arranged in an $s \times v^*$ rectangular array. Let the t branches of each tree be numbered 1,2,...,t from left to right. Two branches of different trees will be called parallel branches if they bear the same number, otherwise they will be called non-parallel branches. For $u = 2$, $v^* = 3$, $s = 2$, $t = 2$, the 24 symbols will be arranged as follows:



Now two symbols will be called

(1) first associates, if they occur on the same branch;

(2) second associates, if they occur on different branches of the same tree;

(3) third associates, if they occur on parallel branches of trees in the same row;

(4) fourth associates, if they occur on non-parallel branches of trees in the same row;

(5) fifth associates, if they occur on parallel branches of trees in the same column;

(6) sixth associates, if they occur on non-parallel branches of trees in the same column;

(7) seventh associates, if they occur on parallel branches of trees which are not in the same row and colunn; and

(8) eighth associates, otherwise.

For example, the 8 associate classes for the symbol 4 are (5), (6,7), (0,1,8,9), (2,3,10,11), (16,17), (18,19), (12,13,20,21), (14, 15,22,23).

Clearly the n_i parameters of EGRA schemes are

$$n_1 = u-1,\ n_2 = u(t-1),\ n_3 = u(v^*-1),\ n_4 = u(t-1)(v^*-1),$$

$$n_5 = u(s-1),\ n_6 = u(t-1)(s-1),\ n_7 = u(v^*-1)(s-1),\ n_8 = u(v^*-1)(s-1)(t-1).$$

The P^i_{jk} parameters of this scheme can be derived easily and are given in Aggarwal (1972).

If we replace the u symbols on every branch by a single symbol, the EGRA association scheme reduces to the extended group divisible association scheme with 7 associate classes discussed by Hinkelmann and Kempthorne (1963).

Eigen Values and vectors of NN'

Let N be the incidence matrix of EGRA designs. Then it can be verified that NN' have the eigen values θ_i with respective multiplicities α_i $(i = 0, 1, \ldots, 8)$.

$$\theta_0 = rk,\ \alpha_0 = 1;\ \theta_1 = r-\lambda_1,\ \alpha_1 = tv^* s(u-1);$$

$$\theta_2 = r-\lambda_1 + (n_1+1)(\lambda_1-\lambda_3) + n_2(\lambda_2-\lambda_4) + n_5(\lambda_5-\lambda_7) + n_6(\lambda_6-\lambda_8),\ \alpha_2 = v^*-1;$$

$$\theta_3 = r-\lambda_1 + (n_1+1)(\lambda_1-\lambda_2) + n_3(\lambda_3-\lambda_4) + n_5(\lambda_5-\lambda_6) + n_7(\lambda_7-\lambda_8),\ \alpha_3 = t-1;$$

$$\theta_4 = r-\lambda_1 + (n_1+1)(\lambda_1-\lambda_5) + n_2(\lambda_2-\lambda_6) + n_3(\lambda_3-\lambda_7) + n_4(\lambda_4-\lambda_6),\ \alpha_4 = s-1;$$

$$\theta_5 = r-\lambda_1 + (n_1+1)(\lambda_1-\lambda_2-\lambda_3+\lambda_4) + n_5(\lambda_5-\lambda_6-\lambda_7+\lambda_8),\ \alpha_5 = (v^*-1)(t-1);$$

$$\theta_6 = r-\lambda_1 + (n_1+1)(\lambda_1-\lambda_3-\lambda_5+\lambda_7) + n_2(\lambda_2-\lambda_4-\lambda_6+\lambda_8),\ \alpha_6 = (v^*-1)(s-1);$$

$$\theta_7 = r-\lambda_1 + (n_1+1)(\lambda_1-\lambda_2-\lambda_5+\lambda_6) + n_3(\lambda_3-\lambda_4-\lambda_7+\lambda_8),\ \alpha_7 = (t-1)(s-1); \text{ and}$$

$$\theta_8 = r-\lambda_1 + (n_1+1)(\lambda_1-\lambda_2-\lambda_3+\lambda_4-\lambda_5+\lambda_6+\lambda_7-\lambda_8),\ \alpha_8 = (t-1)(s-1)(v^*-1)\ .$$

The eigen vectors associated with the eigen values θ_2 through θ_8 can be seen to be the vector of coefficients of treatment combinations corresponding to main effects and interactions of an asymmetrical factorial experiment $v^* \times t \times s$.

Constructions of two series

Let there be $v = v^* s^m t^n$ symbols and let them be identified by the treatment combinations of a pseudo factorial experiment $v^* \times s^m \times t^n$. Let s,t be primes or prime powers. Let the factors of the pseudo factorial experiment be F, $G_1, \ldots, G_m$, $H_1, \ldots, H_n$; F being at v^* levels, each G being at s levels and each H being at t levels. Let Z_1 be some pencil which can be confounded among the factors G_i $(i = 1, 2, \ldots, m)$ and let the pencil Z_1 partition the s^m treatment combinations into s-flats $X_0, X_1, \ldots, X_{s-1}$ each containing s^{m-1} treatment combinations. In a similar way, let the pencil Z_2 partition the t^n treatment combinations into t-flats $Y_0, Y_1, \ldots, Y_{t-1}$ each containing t^{n-1} treatment combinations.

Let there exist a BIB design with the parameters v^*, b^*, r^*, k^* and λ^*. We identify the v^* levels of this BIB design with the v^* levels of the factor F. We associate the flats $X_0, X_1, \ldots, X_{s-1}$ with the v^* levels of factor F as described by Tharthare (1965) and get a GRA design with $b^* s(s-1)$ sets. If S_i is a set of the original BIB design, then the GRA design will contain a pair of the form

$$(1) \qquad \left(S_i \otimes X_j,\ \bar{S}_i \otimes X_{j'}\right)$$

$$\left(S_i \otimes X_{j'},\ \bar{S}_i \otimes X_j\right); \quad j, j' = 0, 1, \ldots, s-1;\ j \neq j';$$

where $S_i \otimes X_j$ is the symbolic Kronecker product of the symbols of S_i with the treatment combinations of X_j and $\bar{S}_i$ is the complementary set of S_i with respect to the v^* symbols of the BIB design.

From the pair of sets of the form (1) we generate $t(t-1)$ sets by extending the technique and forming the sets

$$(2) \qquad \left(S_i \otimes X_j \otimes Y_k,\ \bar{S}_i \otimes X_{j'} \otimes Y_k,\right.$$

$$\left. S_i \otimes X_{j'} \otimes Y_{k'},\ \bar{S}_i \otimes X_j \otimes Y_{k'}\right);$$

$$k, k' = 0, 1, \ldots, t-1;\ k \neq k' .$$

In the GRA design obtained in the beginning, there are $b^* s(s-1)/2$ pairs of sets of the type (1) and every pair of these sets contributes $t(t-1)$ sets of the EGRA design. Thus there are in all $b^* s(s-1)t(t-1)/2$ sets in the EGRA design. The other parameters of this EGRA design are given by (3) and can be verified easily.

$$v = v^{*}s^{m}t^{n},\ b = b^{*}s(s-1)t(t-1)/2,\ r = b^{*}(t-1)(s-1),$$
$$k = 2v^{*}s^{m-1}t^{n-1},\ n_1 = u-1,\ n_2 = u(t-1),\ n_3 = u(v^{*}-1),$$
$$n_4 = u(t-1)(v^{*}-1),\ n_5 = u(s-1),\ n_6 = u(s-1)(t-1),$$
$$n_7 = u(s-1)(v^{*}-1),\ n_8 = u(t-1)(v^{*}-1)(s-1),$$
$$\lambda_1 = r,\ \lambda_2 = 0,\ \lambda_3 = (s-1)(t-1)[b^{*}-2(r^{*}-\lambda^{*})], \tag{3}$$
$$\lambda_4 = 2(s-1)(r^{*}-\lambda^{*}),\ \lambda_5 = 0,\ \lambda_6 = b^{*},$$
$$\lambda_7 = 2(t-1)(r^{*}-\lambda^{*}) \text{ and } \lambda_8 = b^{*}-2(r^{*}-\lambda^{*}),$$

where $u = s^{m-1}t^{n-1}$.

Another series of EGRA design with parameters is

$$v = v^{*}s^{m}t^{n},\ b = st(s-1)(t-1),$$
$$r = (s-1)(t-1),\ k = v^{*}s^{m-1}t^{n-1},$$
$$n_1 = u-1,\ n_2 = u(t-1),\ n_3 = u(v^{*}-1), \tag{4}$$
$$n_4 = u(t-1)(v^{*}-1),\ n_5 = u(s-1),$$
$$n_6 = u(s-1)(t-1),\ n_7 = u(s-1)(v^{*}-1),$$
$$n_8 = u(t-1)(v^{*}-1)(s-1),\ \lambda_1 = r,$$
$$\lambda_2 = \lambda_3 = \cdots = \lambda_7 = 0,\ \lambda_8 = 1$$

where $u = s^{m-1}t^{n-1}$ exists when $v^{*} < s,t$ and s,t being primes or prime powers and the construction is as follows:

We identify the symbols $v^{*}s^{m}t^{n}$ with the treatment combinations of a pseudo factorial experiment $v^{*} \times s^{m} \times t^{n}$ and define X_j $(j = 0,1,\ldots,s-1)$, Y_k $(k = 0,1,\ldots,t-1)$ as in the first case.

Since $v^{*} < s$, a prime or a prime power, a partially balanced array A in v^{*} constraints, s symbols of strength two in $s(s-1)$ assemblies exist with $\lambda(x,y) = 0$, if $x = y$ and 1 if $x \neq y$. Let the symbolic inner product of two vectors $(\alpha_1,\alpha_2,\ldots,\alpha_s)$ and $(\beta_1, \beta_2,\ldots,\beta_s)$ be defined as $(\alpha_1\beta_1, \alpha_2\beta_2,\ldots,\alpha_s\beta_s)$. Then the $s(s-1)$ columns, of the symbolic inner product of $(0,1,\ldots,v^*-1)$ with each column of the partially balanced array A treated as sets, gives an EGD design with parameters

$$\begin{aligned} &v = v^*s,\ b = s(s-1),\ r = s-1,\ k = v^*, \\ (5)\qquad &n_{10} = v^*-1,\ n_{01} = s-1,\ n_{11} = (v^*-1)(s-1), \\ &\lambda_{10} = \lambda_{01} = 0,\ \lambda_{11} = 1\ . \end{aligned}$$

Again a partially balanced array B exists in v^* constraints, t symbols of strength two in $t(t-1)$ assemblies with $\lambda(2,y) = 0$ if $x = y$ and 1, if $x \neq y$. By taking the symbolic inner produce of $t(t-1)$ of B with the sets of the design (5) we get another EGD design with parameters

$$\begin{aligned} &v = v^*st,\ t = st(s-1)(t-1),\ r = (s-1)(t-1), \\ &k = v^*,\ n_{100} = v^*-1,\ n_{010} = s-1,\ n_{001}= t-1, \\ (6)\qquad &n_{110} = (v^*-1)(s-1),\ n_{101} = (v^*-1)(t-1), \\ &n_{001} = (s-1)(t-1),\ n_{111} = (v^*-1)(s-1)(t-1), \\ &\lambda_{100} = \lambda_{010} = \lambda_{001} = \lambda_{110} = \lambda_{101} = \lambda_{011} = 0,\ \lambda_{111} = 1\ . \end{aligned}$$

Let the v^*st symbols of the EGD design (6) be represented by i j k ($i = 0,1,\ldots,v^*-1$; $j = 0,1,\ldots,s-1$, $k = 0,1,\ldots,t-1$). Replacing symbol i j k by the set of symbols $i \otimes X_j \otimes Y_k$ in the design and association scheme of EGD design we get an EGRA design.

Applications

The first series of EGRA designs can be used as confounded asymmetrical $v^* \times s^m \times t^n$ experiments in blocks of size $2v^*s^{m-1}t^{n-1}$. The relative loss of information on each of (s-1) degrees of freedom of Z_1 is $b^*(s-2)(t-1)/2r$; on each of (t-1) degrees of freedom of Z_2 is $b^*(s-1)(t-2)/2r$; on each of (s-1)(t-1) degrees of freedom for the interaction Z_1Z_2 is $\left(v^*b^*[(s-1)(t-1) + 1] - 2st(v^*-1)(r^*-\lambda^*)\right)/2v^*r$ and on each of $(v^*-1)(s-1)(t-1)$ degrees of freedom for the interaction FZ_1Z_2 is $st(r^*-\lambda^*)/v^*r$.

For the second series of designs used as confounded asymmetrical factorial experiments, the relative loss of information on different interactions is:

interaction	df	Relative loss of information on each degree
Z_1	s-1	$(s-v^*)/v^*(s-1)$
Z_2	t-1	$(t-v^*)/v^*(s-1)$
Z_1Z_2	(s-1)(t-1)	$\frac{(s-1)(t-1) + (v^*-1)}{v^*(s-1)(t-1)}$
FZ_1	$(v^*-1)(s-1)$	$s/v^*(s-1)$
FZ_2	$(v^*-1)(t-1)$	$t/v^*(t-1)$
FZ_1Z_2	$(v^*-1)(s-1)(t-1)$	$\frac{(s-1)(t-1)-1}{v^*(s-1)(t-1)}$

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