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Self orthogonal latin square designs and their importance

A. Hedayat

Cornell University

Abstract

A latin square is said to be self orthogonal if it is orthogonal to its transpose. The purpose of this paper is three fold. First we shall highlight the known results concerning the existence and nonexistence of these designs. Second we shall point out the usefulness of these designs for the construction of other statistical designs and also efficient cataloging pairs of orthogonal latin square designs. Finally we shall give a table of self orthogonal latin squares of orders 4, 5, 7, 8, 9, 10, 11, and 13.

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Summary

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1. Introduction

The usefulness of latin squares and orthogonal latin square designs in agriculture, education, industry, marketing, medecine, nutrition, psychology, tiling, and many other fields have been pointed out in a large body of published literature and partially being compiled in several statistical design texts such as Cochran and Cox (1966), Cox (1966), Davies (1954), Federer (1963), Finney (1963), Fisher (1935), and Kempthorne (1965). These designs are not only important in their own right but in addition many other useful statistical designs can be derived from them. From this point of view, the interested reader is referred to Hedayat and Shrikhande (1972) and Hedayat (1972). An important family of latin squares are those which are orthogonal to their own transposes. Thus if L is a latin square in this family then L is orthogonal to L^t. Hereafter we say L is self orthogonal if L is orthogonal to L^t. Self orthogonal latin squares form a very interesting and useful class of designs as will be demonstrated in section 3.

2. Known Results on Self Orthogonal Latin Squares

Whether or not self orthogonal latin squares exist for all possible orders, is an unsolved problem to-date. The following highlight the known results.

(1) It is impossible to construct self orthogonal latin squares for orders 2, 3, and 6.

(2) If L_i is a self orthogonal latin square of order n_i, $i=1, 2, \ldots, t$. then $L_1 \otimes L_2 \otimes \ldots \otimes L_t$ is a self orthogonal latin square of order $n_1 n_2 \ldots n_t$.

(3) If $n \not\equiv 0$ (mod 2 or 3), then the operation $a*b = a+\lambda b =$ (mod n) defines a quasigroup whose multiplication table is a self orthogonal latin square of order n whenever $(\lambda-1,n)=(\lambda,n)=(\lambda+1,n)=1$, where (x,y) represents the greatest common divisor of x and y [Mendelsohn (1971)].

(4) Let F be a finite field of order $n \neq 2,3$. Let $\lambda \in F$ such that $\lambda \neq 0, 1, \frac{1}{2}$. Then the operation $a \circ b = \lambda a+(1-\lambda)b$ defines a quasigroup whose multiplication table is a self orthogonal latin square of order n [Mendelsohn (1971)].

(5) From (2) and (4) one can conclude that if $n \not\equiv 2$ (mod 4), or $n \not\equiv 3, 6$ (mod 9) then there exists a self orthogonal latin square of order n.

(6) The existence of a self orthogonal latin square of order n_1 and a self orthogonal latin square of order n_2 with a sub latin square of order $\alpha=n_2-n_3 \neq 2,6$ implies the existence of a self orthogonal latin square of order $n=\alpha n_1+n_3$ [Horton (1970)].

(7) If n is odd and if an Abelian group of order n having certain properties exists, then there is a self orthogonal latin square of order n [Mullin and Nemeth (1970)].

(8) There exists a self orthogonal latin square of order 10 [Hedayat (1972)].

3. Some Useful Properties of Self Orthogonal Latin Squares

Self orthogonal latin squares form a very interesting and useful family of designs. We shall here point out three properties of this family. (i)Many experimental designs can be constructed via these squares which cannot be constructed from arbitrary pairs of orthogonal latin squares. For instance, all self orthogonal latin squares can be utilized for the construction of the experimental designs of types O:OT:TOO and O:OO:SSS considered by Clarke (1963), Freeman (1964), Hoblyn, Pearce, and Freeman (1954), Pearce (1960), and Pearce (1963). This is because if L is self orthogonal then the pair L and L^t has a common parallel transversal which runs from the upper left corner to the lower right corner in each square. Therefore the set $\{L, L^t\}$ can be transformed to these designs as has been shown by Hedayat, Parker, and Federer (1970). (ii)These squares can also serve as partially replicated latin square designs considered by Hedayat and Federer (1970), Scheffe (1959), and Youden and Hunter (1959). (iii)Self orthogonal latin square designs are very useful for efficient cataloging pairs of orthogonal latin square designs in the sense that one square is sufficient for each order.

4. A Table of Self Orthogonal Latin Squares for Orders 4, 5, 7, 8, 9, 10, 11, 13

In the range of $n \leq 15$, as we said before, there are no self orthogonal latin squares for orders 2, 3, and 6. Whether or not self orthogonal latin squares of orders 12, 14, and 15 exist, is still an open problem. Here we shall exhibit a self orthogonal latin square for each remaining order.

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

0	1	2	3	4	5	6
3	4	5	6	0	1	2
6	0	1	2	3	4	5
2	3	4	5	6	0	1
5	6	0	1	2	3	4
1	2	3	4	5	6	0
4	5	6	0	1	2	3

0	1	2	3	4	5	6	7
7	5	6	4	3	1	2	0
6	4	7	5	1	3	0	2
2	3	0	1	5	4	7	6
1	0	3	2	6	7	4	5
4	6	5	7	0	2	1	3
5	7	4	6	2	0	3	1
3	2	1	0	7	6	5	4

0	1	2	3	4	5	6	7	8
7	8	4	1	5	2	0	6	3
6	3	5	8	2	4	7	0	1
5	6	3	7	1	8	4	2	0
8	4	7	2	6	0	1	3	5
2	0	1	6	8	3	5	4	7
4	7	8	0	3	1	2	5	6
3	5	6	4	0	7	8	1	2
1	2	0	5	7	6	3	8	4

0	1	2	3	4	5	6	7	8	9
3	6	5	4	0	8	9	1	7	2
7	0	1	9	8	6	3	2	5	4
9	7	6	5	2	3	1	0	4	8
8	2	7	1	9	4	0	5	6	3
1	3	4	7	5	2	8	6	9	0
2	5	0	8	7	9	4	3	1	6
5	4	9	6	3	7	2	8	0	1
6	9	8	2	1	0	7	4	3	5
4	8	3	0	6	1	5	9	2	7

0	1	2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	10	0	1	2
6	7	8	9	10	0	1	2	3	4	5
9	10	0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	9	10	0
4	5	6	7	8	9	10	0	1	2	3
7	8	9	10	0	1	2	3	4	5	6
10	0	1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10	0	1
5	6	7	8	9	10	0	1	2	3	4
8	9	10	0	1	2	3	4	5	6	7

0	1	2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	0	1	2
6	7	8	9	10	11	12	0	1	2	3	4	5
9	10	11	12	0	1	2	3	4	5	6	7	8
12	0	1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12	0	1
5	6	7	8	9	10	11	12	0	1	2	3	4
8	9	10	11	12	0	1	2	3	4	5	6	7
11	12	0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11	12	0
4	5	6	7	8	9	10	11	12	0	1	2	3
7	8	9	10	11	12	0	1	2	3	4	5	6
10	11	12	0	1	2	3	4	5	6	7	8	9

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