# A Note on Positive Definite Matrices

by

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Abstract

The following theorem is proved and its significance discussed:

Theorem: The square matrix A is positive (non-negative) definite if and only if the symmetric matrix A+A' is positive (non-negative) definite.

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An n-square matrix C is said to be positive (non-negative) definite if for every n-vector $X \neq 0$, $X'CX > (\geq) 0$. To test if a symmetric matrix C is positive definite one simply calculates all the leading principal minors of C to see if they are positive. If so, then C is positive definite. That this test may fail for non-symmetric matrices is seen by observing that the matrix

$$C = \begin{pmatrix} 2.02 & -2 \\ -1 & 1.01 \end{pmatrix}$$

has positive leading principal minors but is not positive definite. That C is not positive definite follows from the fact that for $X = \left( 1, \sqrt{2} \right)'$ , $X'CX < 0$ . The following theorem provides a test for positive definiteness of any square matrix, symmetric or not.

Theorem: The square matrix A is positive (non-negative) definite if and only if the symmetric matrix $B = A+A'$ is positive (non-negative) definite.

<u>Proof</u>:

1. If A is positive (non-negative) definite then for any vector $X \neq 0$, $X'A'X$ is a scalar and thus equal to its transpose, $X'AX$ so that

$$X'BX = X'AX + X'A'X = 2X'AX > (\geq)\ 0 .$$

2. If $B = A+A'$ is positive (non-negative) definite, then for any vector $X \neq 0$ ,

$$0 < (\leq)\ X'BX = 2X'AX \quad \text{so that} \quad X'AX > (\geq)\ 0 .$$

The significance of the theorem is that to test any square matrix A for positive definiteness, we need only apply the leading principal minors test to the symmetric matrix $A+A'$ . For example

$$A = \begin{pmatrix} 6 & -4 \\ -2 & 3 \end{pmatrix}$$

is positive definite because all the leading principal minors of

$$B = A+A' = \begin{pmatrix} 12 & -6 \\ -6 & 6 \end{pmatrix}$$

are positive. Similarly, for the matrix C above,

$$C+C' = \begin{pmatrix} 4.04 & -3 \\ -3 & 2.02 \end{pmatrix}$$

has negative determinant so that C is not positive definite.