BJ 342-M

A NOTE ON A FAMILY OF RESOLVABLE BALANCED INCOMPLETE BLOCK DESINGS[1]

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1. Introduction and Summary. In this paper we show that every resolvable balanced incomplete block design with parameters $b = v(v - 1)/6$, $v \equiv 3 (\text{mod } 6)$, $r = (v - 1)/2$, $k = 3$ and $\lambda = 1$ implies the existence of a set consisting of at least one pair of mutually orthogonal Latin squares of order v. The combinatorial structure of this set is different from those of known sets of orthogonal Latin squares in the literature and this might prove to be useful for the construction of other designs and combinatorial systems derivable from sets of mutually orthogonal Latin squares. The case $v = 15$ leads to a new result, namely the existence of a set consisting of three mutually orthogonal Latin squares of order 15.

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[1]Research supported by N. R. C. Grant #A.07204

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2. Basic Definitions. In the sequel we will need the following concepts:

(1) Let Σ be a set of v distinct treatments. Then a balanced incomplete block (BIB) design with parameters b, v, r, k, λ on Σ is a collection of b subsets of Σ, called blocks, such that the following conditions are satisfied: (i) each block contains k distinct treatments (ii) each treatment occurs in r distinct blocks (iii) each pair of distinct treatments occurs together in exactly λ different blocks.

A BIB design with parameters b, v, r, k, λ is said to be a resolvable BIB design if the following additional stipulation holds. The set of b blocks can be partitioned into r disjoint classes such that the totality of treatments in each class exhausts the set of treatments.

(2) Let Ω be an m-set. Then L is a Latin square of order m on Ω if L is an mxm matrix with the property that each row and column of L is an m-permutation of elements of Ω. A collection of m cells in L is said to form a transversal (directrix) for L if the entries of these cells exhaust the set Ω and every row and column of L is represented in this collection. Two transversals are said to be parallel if they have no cell in common. Let L_1 and L_2 be two Latin squares of order m on the m-set $\Omega_1 = \{a_1, a_2, \ldots, a_m\}$ and $\Omega_2 = \{b_1, b_2, \ldots, b_m\}$ respectively. Then we say L_2 is an orthogonal mate for L_1 if upon superposition of L_2 and L_1, a_i in L_1 appears with b_j in L_2 for all i, j = 1, 2, ..., m.

For more details on these concepts, see Hall [1] and Parker [5].

3. The Result. Ray-Chaudhuri and Wilson [7] have recently proved the sufficiency of $v \equiv 3 \pmod 6$ for the existence of a resolvable BIB design with parameters $b = v(v - 1)/6$, $v \equiv 3 \pmod 6$ $r = (v - 1)/2$, $k = 3$ and $\lambda = 1$. We now prove the following.

Theorem 3.1. *Let* Σ *be a set of* $v \equiv 3 \pmod 6$ *distinct treatments. Then every resolvable* BIB *design with parameters* $b = v(v - 1)/6$, v, $r = (v - 1)/2$, $k = 3$, $\lambda = 1$ on Σ *implies the existence of a set consisting of at least a pair of mutually orthogonal Latin squares of order* v.

To prove the theorem we need the following known Lemma:

LEMMA 3.1. *If L is a Latin square of order v, then L can have an orthogonal mate if and only if it has v-1 parallel transversals*.

Since L is a Latin square then $v - 1$ parallel transversals implies v parallel transversals. Now the proof that L can have an orthogonal mate follows directly from the definition of parallel transversals and orthogonality of Latin squares.

PROOF OF THEOREM: Let A be a $v \times v$ square. Associate with every row and column of A a unique element of Σ. Put in the cell corresponding to row x and column y the element z, where z is that element of Σ which together with x and y form a block. Put x in the cell with row and column indices x. Call the resulting square H. It is easy to see that H is a Latin square of order v on Σ. We now show that H has v parallel transversals. Partition the design into r disjoint classes C_i, i = 1, 2, ..., r as described earlier. Consider the α-th class and denote an arbitrary block in this class by $(x_{\alpha j}, y_{\alpha j}, z_{\alpha m})$, j = 1, 2, ..., v/3. Identify three cells in H by the 2-tuples $(x_{\alpha j}, y_{\alpha j})$, $(y_{\alpha j}, z_{\alpha j})$ and $(z_{\alpha j}, x_{\alpha j})$, the components of each 2-tuple being the row and column indices respectively. The entries in these cells are then, by the definition of H, $z_{\alpha j}$, $x_{\alpha j}$ and $y_{\alpha j}$ respectively. Now let j run through all the v/3 triples in C_α, then the corresponding 3.v/3 = v cells determined by the preceding rule form a transversal in H. Denote this transversal by $t_{\alpha 1}$. Another transversal $t_{\alpha 2}$ is obtained by concidering the three cells in H described by the 2-tuples $(y_{\alpha j}, x_{\alpha j})$, $(z_{\alpha j}, y_{\alpha j})$ and $(x_{\alpha j}, z_{\alpha j})$ and letting j run through the values 1, 2, ..., v/3. These exhibition rules guarantee that $t_{\alpha 1}$ is parallel to $t_{\alpha 2}$. Since there are (v - 1)/2 classes, we may in this way obtain from every class C_i, a pair of parallel transversals t_{i1} and t_{i2}. Moveover, t_{ik} (k = 1, 2) is parallel to $t_{i'k}$ (k - 1, 2) if $i \neq i'$, since every pair of distinct elements of Σ appears exactly once in the whole system. Hence, we have shown that H contains $2 \cdot (v - 1)/2 = v - 1$ parallel transversals so that by the lemma it has an orthogonal mate. Finally from the definition of H the reader should note that the v-th transversal is determined by the v cells with row and column indices (x,x).

4. Discussion. We emphasize that the combinatorial structure of orthogonal Latin squares constructed by the preceding theorem is completely different from those of known orthogonal Latin squares in the literature. This fact was confirmed with Parker [6]. For instance, every block determines a sub-Latin square of order 3 in H, and in this way H contains v(v - 1)/6 sub-Latin squares of order 3. Or in general whenever the given resolvable design contains a sub-BIB design of order t then the corresponding Latin square H contains a sub-Latin square of order t. Therefore, it is worthwhile to study these squares for the purpose of constructing new designs or even new finite projective planes. Hedayat [3] has shown that for the case v = 15, the corresponding Latin square H admits a special orthogonal mate which can be embedded into a set consisting of three mutually orthogonal Latin squares of order 15.

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