ON THE EQUIVALENCE OF KIRKMAN-STEINER TRIPLE SYSTEMS AND SETS OF MUTUALLY ORTHOGONAL LATIN SQUARES*

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ABSTRACT

It is shown that for every Kirkman-Steiner triple system of order $n \equiv 3 \pmod{6}$, there exists at least one pair of orthogonal Latin Squares of order n.

1. BASIC DEFINITIONS

In the following we need the following concepts:

(i) Let Σ be an n-set, $n \equiv 1, 3 \pmod{6}$. Then a Steiner triple system of order n on Σ is a collection of unordered triplets (x,y,z), x, y, z in Σ such that every pair of distinct elements of Σ belongs to exactly one triple.

For example:

$$S = \{(0,1,3),(1,2,4),(2,3,5),(3,4,6),(4,5,0),(5,6,1),(6,0,2)\}$$

is a Steiner triple system on $\Sigma = \{0,1,\ldots,6\}$.

(ii) A triple system of order $n \equiv 3 \pmod{6}$ is said to be a Kirkman-Steiner triple system of order n if it is a Steiner triple system of order n with the following additional stipulation. The set of triplets can be partitioned into r dis-

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joint classes such that the totality of elements in each class exhausts the set Σ on which the system is defined.

For example:

$$K = \left\{ \begin{matrix} (1,2,3) & (1,4,7) & (1,5,9) & (1,6,8) \\ (4,5,6) & (2,5,8) & (2,6,7) & (2,4,9) \\ (7,8,9) & (3,6,9) & (3,4,8) & (3,5,7) \\ \underbrace{}_{\text{class 1}} & \underbrace{}_{\text{class 2}} & \underbrace{}_{\text{class 3}} & \underbrace{}_{\text{class 4}} \end{matrix} \right\}$$

is a Kirkman-Steiner triple system of order 9 on $\Sigma = \{1,2,\ldots,9\}$.

(iii) Let Ω be an m-set. Then L is a Latin square of order m on Ω if L is an mxm matrix with the property that each row and column of L is an m-permutation of elements of Ω. A collection of m cells in L is said to form a transversal (directrix) for L if the entries of these cells exhaust the set Ω and every row and column of L is represented in this collection. Two transversals are said to be parallel if they have no cell in common. Let L_1 and L_2 be two Latin squares of order m on the m-set $\Omega_1 = \{a_1,a_2,\ldots,a_m\}$ and $\Omega_2 = \{b_1,b_2,\ldots,b_m\}$ respectively. Then we say L_2 is an orthogonal mate for L_1 if upon superposition of L_2 and L_1, a_i in L_1 appears with b_j in L_2 for all $i,j=1,2,\ldots,m$. In the following a set consisting of two orthogonal Latin squares of order m will be denoted by $O(m,2)$.

For example:

$$\left\{ \begin{matrix} 123 \\ 231 \\ 312 \end{matrix} \;,\; \begin{matrix} 123 \\ 312 \\ 231 \end{matrix} \right\}$$

is an $O(3,2)$.

2. THE RESULT

Let K be a Kirkman-Steiner triple system of order $n \equiv 3$ (mod. 6) on an n-set Σ. Then we prove the following theorem

THEOREM. $K \Longrightarrow O(n,2)$.

To prove the theorem we need the following Lemma:

LEMMA. If L is a Latin square of order n, then L can have an orthogonal mate if and only if it has n-1 parallel transversals.

Since L is a Latin square then n-1 parallel transversals implies n parallel transversals. Now the proof that L can have an orthogonal mate follows directly from the definition of parallel transversals and orthogonality of Latin Squares.

PROOF OF THEOREM: Let A be an nxn square. Associate with every row and column of A a unique element of Σ. Put in the cell corresponding to row x and column y the element z, where z is that element of Σ which together with x and y form a triple. Put x in the cell with row and column indices x. Call the resulting square H. It is easy to see that H is a Latin square of order n on Σ. We now show that H has n parallel transversals. Let $r = (n-1)/2$ and partition K into r disjoint classes $C_i, i=1,2,\ldots,r$ as described earlier. Consider the ν-th class and denote an arbitrary triple in this class by $(x_{\nu j}, y_{\nu j}, z_{\nu j})$, $j = 1,2,\ldots,n/3$. Identify three cells in H by the 2-tuples $(x_{\nu j}, y_{\nu j})$, $(y_{\nu j}, z_{\nu j})$ and $(z_{\nu j}, x_{\nu j})$, the components of each 2-tuple being the row and column indices respectively. The entries in these cells are then, by the definition of H, $z_{\nu j}$, $x_{\nu j}$ and $y_{\nu j}$ respectively. Now let j run through all the n/3 triples in C_ν, then the corresponding $3.n/3 = n$ cells determined by the preceding rule form a transversal in H. Denote this transversal by $t_{\nu 1}$. Another transversal $t_{\nu 2}$ is obtained by considering the three cells in H described by the 2-tuples $(y_{\nu j}, x_{\nu j})$, $(z_{\nu j}, y_{\nu j})$ and $(x_{\nu j}, z_{\nu j})$ and letting j run through the values $1,2,\ldots,n/3$. These exhibition rules guarantee that $t_{\nu 1}$ is parallel to $t_{\nu 2}$. Since there are $(n-1)/2$ classes, we may in this way obtain from every class C, a pair of parallel transversals t_{i1} and t_{i2}. Moveover, t_{ik} $(k=1,2)$ is parallel to t_{ik} $(k-1,2)$ if $i \neq i'$, since every pair of distinct elements of Σ appears exactly once in the whole triple system. Hence, we have shown that H contains $2\ (n-1)/2 = n-1$ parallel transversals so that by the lemma it has an orthogonal mate. Finally from the definition of H the reader should note that the n-th transversal is determined by the n cells with row and column indices (x,x).

DISCUSSION. Latin Square H and its orthogonal mate constructed by the preceding theorem have very peculiar combinatorial structures. For instance, every triple determines a sub-Latin square of order 3 in H, and H itself contains $n(n-1)/6$ sub-Latin Squares of order 3, because the Kirkman-Steiner triple system has this many triples. We hope to discuss this matter in more detail in a later paper. The sufficiency of $n \equiv 3$ (mod. 6) for the existence of a Kirkman-Steiner triple system of order n has been shown, though in a long paper, by Ray-Chaudhuri and Wilson [1]. Presently we are working towards an alternative and possibly shorter proof for the existence and construction of Kirkman-Steiner triple systems via orthogonal Latin Squares.

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