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A GENERALIZED PROCEDURE FOR CONSTRUCTING FRACTIONAL REPLICATES

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ABSTRACT

A generalized method of constructing fractional replicates from a complete factorial is developed in the present paper. Special reference is made to the construction of saturated fractional replicates for a specified set of parameters. The method of construction involves a special ordering of the treatment observations and of the single degree of freedom parameter contrasts. Prior to presentation of the method, a generalized inverse method is used on the estimates of parameters and the corresponding variances. Also, a Kronecker product representation is given for the design matrix of any n-factor factorial composed of linear contrasts and some relationships between the design matrices and corresponding orthogonal arrays are investigated. Various saturated main effect plans for 2^4 and 3^3 factorial are presented.

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SUMMARY

A generalized method of constructing fractional replicates from a complete factorial is developed in the present paper. Special reference is made to the construction of saturated fractional replicates for a specified set of parameters. The method of construction involves a special ordering of the treatment observations and of the single degree of freedom parameter contrasts. Prior to presentation of the method, a generalized inverse method is used on the estimates of parameters and the corresponding variances. Also, a Kronecker product representation is given for the design matrix of any n-factor factorial composed of linear contrasts and some relationships between the design matrices and corresponding orthogonal arrays are investigated. Various saturated main effect plans for 2^4 and 3^3 factorials are presented.

1. INTRODUCTION

Raktoe and Federer [1966] have shown how to obtain unsaturated and saturated non-orthogonal main effect and resolution V plans using a single replicate of a lattice design for 2^n treatments in incomplete blocks of size two. A special ordering of the 2^{n-1} incomplete blocks was used. Then, from this ordering they

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obtained a set of fractional replicates. It is the purpose of this paper to present a method of construction of saturated and unsaturated fractional replicates for any specified set of p rameters from any complete factorial.

First we shall need to develop and define a notation. Then, some of the results of Banerjee and Federer [1963, 1964, 1966] on the estimates of parameters and their variances will be obtained using a generalized inverse procedure. This alternative development may be useful in other connections. In the next section the single degree of freedom contrast design matrix will be presented as a Kronecker product of the linear contrasts of the levels of each of the n factors. Special orderings of the observations and of the parameter contrasts are used in this Kronecker representation, and some relationships between the design matrices and corresponding orthogonal arrays are investigated. With the Kronecker representation, the method of construction of fractional replicates is then developed and illustrated with several examples. Various saturated non-orthogonal main effect plans for a 2^4 and a 3^3 factorial are given.

2. NOTATION

Let Y represent a column vector of N random observation variables $y_1, y_2, \cdots, y_N$, let $\underline{B}$ represent a column vector of N unknown parameters $b_1, b_2, \cdots, b_N$, and let the known linear orthogonal comparison matrix X (treatment design matrix) in the complete factorial be composed of N rows and N columns. Then, the observational equation may be represented as:

$$Y = X\underline{B} + e \quad , \tag{2.1}$$

where e is an N x 1 column vector of random error components, $e_1, e_2, \cdots, e_N$, $E(Y) = X\underline{B}$, $E(ee') = I\sigma^2$, and I is the N x N identity matrix.

Consider the following expression

$$Y = [X_1 \; X_2] \begin{bmatrix} \underline{B}_p \\ \underline{B}_{N-p} \end{bmatrix} + e \quad , \tag{2.2}$$

where $\underline{B}'_p = [b_1, b_2, \cdots, b_p]$ is a given parameter vector, $p \le N$, X_1 is an N x p matrix, and X_2 is an N x (N-p) matrix. Since $r(X) = N$ and $r(X_1) = p$, then there exists at least one non-singular p x p matrix X_{11} in X_1.

After rearranging row orders in Y, $[X_1 \; X_2]$ and e respectively, we obtain the following matrix equation

$$\begin{bmatrix} Y_p \\ Y_{N-p} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} \underline{B}_p \\ \underline{B}_{N-p} \end{bmatrix} + \begin{bmatrix} e_p \\ e_{N-p} \end{bmatrix} \quad , \tag{2.3}$$

where X_{11} is a non-singular p x p matrix. Then,

$$Y_p = [X_{11} \; X_{12}] \begin{bmatrix} \underline{B}_p \\ \underline{B}_{N-p} \end{bmatrix} + e_p \tag{2.4}$$

and the observations in Y_p yield a saturated fractional replicate for the given parameters in $\underline{B}_p$.

3. USE OF GENERALIZED INVERSE

Banerjee and Federer [1963, 1964, 1966] have shown how to obtain estimates of parameters and corresponding variances from a non-orthogonal fractional replicate. We shall obtain their results using a generalized inverse method.

Theorem 1. For a given parameter vector $\underline{B}_p$, there always exists a fractional replicate as given by equation (2.4) from a complete factorial replicate equation (2.1), and $X_{11}^{-1} Y_p$ is the best linear unbiased estimator of the $\underline{B}_p + X_{11}^{-1} X_{12} \underline{B}_{N-p}$.

Proof: Existence of a fractional replicate given the parameters is obvious from the section 2. To show estimability, using the least squares method, the matrix expression of the normal equations for the fractional replicate given by equation (2.4) is:

$$[X_{11}\ X_{12}]'[X_{11}\ X_{12}] \begin{bmatrix} \hat{\underline{B}}_p \\ \hat{\underline{B}}_{N-p} \end{bmatrix} = [X_{11}\ X_{12}]' Y_p$$

$$\begin{bmatrix} X_{11}' X_{11} & X_{11}' X_{12} \\ X_{12}' X_{11} & X_{12}' X_{12} \end{bmatrix} \begin{bmatrix} \hat{\underline{B}}_p \\ \hat{\underline{B}}_{N-p} \end{bmatrix} = \begin{bmatrix} X_{11}' \\ X_{12}' \end{bmatrix} \qquad (3.1)$$

One of the generalized inverses G of $\begin{bmatrix} X_{11}' X_{11} & X_{11}' X_{12} \\ X_{12}' X_{11} & X_{12}' X_{12} \end{bmatrix}$ is

$$G = \begin{bmatrix} (X_{11}' X_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.2}$$

The proof of (3.2) follows easily, i.e.,

$$\begin{bmatrix} X_{11}' X_{11} & X_{11}' X_{12} \\ X_{12}' X_{11} & X_{12}' X_{12} \end{bmatrix} G \begin{bmatrix} X_{11}' X_{11} & X_{11}' X_{12} \\ X_{12}' X_{11} & X_{12}' X_{12} \end{bmatrix}$$

$$= \begin{bmatrix} X_{11}' X_{11} & X_{11}' X_{12} \\ X_{12}' X_{11} & X_{12}' X_{11}(X_{11}' X_{11})^{-1} X_{11}' X_{12} \end{bmatrix}$$

Since X_{11} is non-singular

$$X_{11}(X_{11}' X_{11})^{-1} = X_{11} X_{11}^{-1} X_{11}'^{-1} = X_{11}'^{-1}$$

then

$$X_{12}' X_{11}(X_{11}' X_{11})^{-1} X_{11}' X_{12} = X_{12}' X_{12} .$$

Hence, (3.2) is proven.

We define

$$H = G[X_{11} \; X_{12}]'[X_{11} \; X_{12}] = \begin{bmatrix} I & (X_{11}' X_{11})^{-1} X_{11}' X_{12} \\ 0 & 0 \end{bmatrix} , \tag{3.3}$$

then

$$\begin{bmatrix} \hat{\underline{B}}_p \\ \hat{\underline{B}}_{N-p} \end{bmatrix} = G \begin{bmatrix} X'_{11} \\ X'_{12} \end{bmatrix} Y_p + (H - I_{NxN})Z$$

$$= \begin{bmatrix} (X'_{11} X_{11})^{-1} X'_{11} \\ 0 \end{bmatrix} Y_p + \begin{bmatrix} (X'_{11} X_{11})^{-1} X'_{11} X_{12} \\ - I_{(N-p)x(N-p)} \end{bmatrix} Z^* \qquad (3.4)$$

From equation (3.4)

$$Z^* = -\hat{\underline{B}}_{N-p} \qquad (3.5)$$

then

$$\hat{\underline{B}}_p + (X'_{11} X_{11})^{-1} X'_{11} X_{12} \hat{\underline{B}}_{N-p} = (X'_{11} X_{11})^{-1} X'_{11} Y_p \qquad (3.6)$$

or

$$\hat{\underline{B}}_p + X_{11}^{-1} X_{12} \hat{\underline{B}}_{N-p} = X_{11}^{-1} Y_p \quad . \qquad (3.7)$$

Then, $X_{11}^{-1} Y_p$ is the best linear unbiased estimator of the $\underline{B}_p + X_{11}^{-1} X_{12} \underline{B}_{N-p}$, and the theorem is proven.

Since X'X is a diagonal matrix, if X_{22}^{-1} exists, then X_{11}^{-1} exists and we may write (Banerjee and Federer [1964]):

$$X = \begin{bmatrix} X_{11} & X_{12} \\ \lambda' X_{11} & X_{22} \end{bmatrix} , \quad \text{where} \quad \lambda = -X_{12} X_{22}^{-1} \quad .$$

Since $(X_1' X_1)^{-1} X_1' X_1 = I_{pxp}$

$$(X_1' X_1)^{-1} [X_{11}' \; X_{11}' \lambda] \begin{bmatrix} X_{11} \\ \lambda' X_{11} \end{bmatrix} = I_{pxp}$$

and

$$(X_1' X_1)^{-1} X_{11}'(I + \lambda\lambda') = X_{11}^{-1} \quad . \tag{3.9}$$

Hence, we rewrite (3.7) as follows:

$$\begin{aligned} &\underline{\hat{B}}_p + (X_1' X_1)^{-1} X_{11}'(I + \lambda\lambda') X_{12} \underline{\hat{B}}_{N-p} \\ &\qquad = (X_1' X_1)^{-1} X_{11}'(I + \lambda\lambda') Y_p \end{aligned} \tag{3.10}$$

From Searle [1966], e.g., we note that

$$\text{var} \begin{bmatrix} \underline{\hat{B}}_p \\ \underline{\hat{B}}_{N-p} \end{bmatrix} = G\sigma^2 = \begin{bmatrix} (X_{11}' X_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \sigma^2 \quad ; \tag{3.11}$$

then

$$\text{var}(\underline{\hat{B}}_p) = (X_{11}' X_{11})^{-1} \sigma^2 \quad . \tag{3.12}$$

These results are equivalent to those of Banerjee and Federer [1963, 1964].

4. KRONECKER PRODUCT CONSTRUCTION OF THE DESIGN MARTIX X

Consider a 3 x 2 factorial arrangement of treatments, and suppose factor A is represented at the three levels 0, 1, and 2, and factor B at the two levels 0 and 1; then, in Table 4.1, we obtain the coefficients for the 6 orthogonal contrasts among 6 treatments by using the Kronecker product of the two matrices L_{3_A} and L_{2_B} (e.g., see Yates [1937] and Robson [1959]) where

$$L_{3_A} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad L_{2_B} = \begin{bmatrix} 1 & -1 \\ & \\ 1 & 1 \end{bmatrix}$$

Table 4.1. The coefficients for single degree of freedom comparisons in a 3 x 2 factorial.

Treatment combination	M	B	A_L	A_LB	A_Q	A_QB*
00	1	-1	-1	1	1	-1
01	1	1	-1	-1	1	1
10	1	-1	0	0	-2	2
11	1	1	0	0	-2	-2
20	1	-1	1	-1	1	-1
21	1	1	1	1	1	1

* Later on, we shall use the notation A^0B^0, A^0B^1, A^1B^0, A^1B^1, A^2B^0 and A^2B^1 to replace M, B, A_L, A_LB, A_Q, and A_QB respectively.

If we represent the matrix of coefficients given in Table 4.1 by L_{3x2}, then

$$L_{3x2} = L_{3_A} \otimes L_{2_B} \quad ,$$

where $\otimes$ refers to the Kronecker product. L_{3x2} is the design matrix X of a complete 3 x 2 factorial for the parameter vector $\underline{B}$.

In general, if we denote the contrast matrix as L_{q_h}, where q_h refers to the number of levels associated with the h^{th} factor F_h, the representation of the design matrix is:

$$X = \prod_{h=1}^{n} \otimes L_{q_h} = L_{\prod_{h=1}^{n} q_h} \tag{4.1}$$

and define the product order as follows:

$$\prod_{h=1}^{n} \otimes L_{q_h} = L_{q_1} \otimes \left(\prod_{h=2}^{n} \otimes L_{q_h} \right) = L_{q_1} \otimes \left(L_{q_{h_2}} \otimes \left(\prod_{h=3}^{n} \otimes L_{q_h} \right) \right) \tag{4.2}$$

where

$$L_{q_h} = \begin{bmatrix} \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0,q_h-1} \\ \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1,q_h-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{q_h-1,0} & \gamma_{q_h-1,1} & \cdots & \gamma_{q_h-1,q_h-1} \end{bmatrix} \tag{4.3}$$

where $\gamma_{i,0} = 1$ for $i=0,1,\cdots,q_h-1$, and

$$\sum_{i=0}^{q_h-1} \gamma_{ij}\gamma_{ik} = 0 \text{ for } j \neq k \text{ and } j,k = 0,1,\cdots,q_h-1 \;.$$

Particularly, if $q_h = s$ for $h=1,2,\cdots,n$, then

$$X = L_{s^n} = \begin{bmatrix} L_{s^{n-1}} & \gamma_{01}L_{s^{n-1}} & \cdots & \gamma_{0,s-1}L_{s^{n-1}} \\ L_{s^{n-1}} & \gamma_{11}L_{s^{n-1}} & \cdots & \gamma_{1,s-1}L_{s^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ L_{s^{n-1}} & \gamma_{s-1,2}L_{s^{n-1}} & \cdots & \gamma_{s-1,s-1}L_{s^{n-1}} \end{bmatrix} \tag{4.4}$$

The column vector corresponding to the n factor interaction component $F_1^{c_1} F_2^{c_2} \cdots F_n^{c_n}$ in X, say $\underline{g}$, may be written as follows:

$$\underline{g} = \prod_{h=1}^{n} \otimes \begin{bmatrix} \gamma_{0c_h} \\ \gamma_{1c_h} \\ \vdots \\ \gamma_{q_h-1,c_h} \end{bmatrix} \tag{4.5}$$

If $c_i=j$, $j \neq 0$, for $i=h$ and $c_i=0$ for $i \neq h$,

$$\underline{g}_j(h) = \mathbf{1}_t \otimes \begin{bmatrix} \gamma_{0j} \\ \gamma_{1j} \\ \vdots \\ \gamma_{q_h-1,j} \end{bmatrix} \otimes \mathbf{1}_u = \mathbf{1}_t \otimes \begin{bmatrix} \gamma_{0j}\,\mathbf{1}_u \\ \gamma_{1j}\,\mathbf{1}_u \\ \vdots \\ \gamma_{q_h-1,j}\,\mathbf{1}_u \end{bmatrix} \tag{4.6}$$

where $\mathbf{1}_t$ is a t x 1 column vector with all elements equal to one, $t = \prod_{i=1}^{h-1} q_i$ and $u = \prod_{i=h+1}^{n} q_i$. If $c_i = 0$ for $i=1,2,\cdots,n$

$$\underline{g}_0 = \prod_{h=1}^{n} \otimes \underline{1}_{q_h} = \underline{1}_N$$

The ordering of the treatments (it may be called a combination or an assembly) in the treatment combination array [Y] is as follows: Set the first n-1 factors at the first level and run through all levels of the n^{th} factor consecutively; then set all levels of the first n-2 factors at the first level and set the level of the $n-1^{st}$ factor at the second level and run through all levels of the n^{th} factor in consecutive order; continue this process until all levels of the $n-1^{st}$ factor have been exhausted in consecutive order; then run through levels of the $n-2^{nd}$ factor in the manner for the $n-1^{st}$ factor; continue this process for the $n-3^{rd}$ up to and including the first factor which exhausts all the combinations in the n-factor factorial. The parameter order is such that the mean and n^{th} factor contrast appear first, then the first contrast of the $n-1^{st}$ factor and interaction with the n^{th} factor contrasts appear next, etc.

If the h^{th} factor F_h has q_h levels, then the h^{th} column vector of the N x n matrix of subscripts of the observations in [Y], say $\underline{f}_h$, may be expressed as follows:

$$\underline{f}_h = \underline{1}_t \otimes \begin{bmatrix} 0 \\ 1 \\ \vdots \\ q_h-1 \end{bmatrix} \otimes \underline{1}_u = \underline{1}_t \otimes \begin{bmatrix} (0)\ \underline{1}_u \\ (1)\ \underline{1}_u \\ \vdots \\ (q_k-1)\ \underline{1}_u \end{bmatrix} . \qquad (4.8)$$

The $k+1^{st}$ treatment yield subscript in [Y] and $k+1^{st}$ parameter may be expressed as:

$$(\alpha_1,\alpha_2,\cdots,\alpha_n) \qquad (4.9)$$

$$F_1^{\alpha_1},F_2^{\alpha_2},\cdots,F_n^{\alpha_n} \quad , \text{ respectively} \qquad (4.10)$$

where

$$\alpha_i = [k_{j-1}/\prod_{h=j+1}^{n} q_h] \quad \text{for } j=1,2,\cdots,n-1$$

$$\alpha_n = k_{n-1}$$

where $[k_{j-1}/\prod_{k=j+1}^{n} q_h]$ denotes the largest integer less than or equal to $k_j/\prod_{h=j+1}^{n} q_h$ and $k_0 = k$ and $k_{j-2} = k_{j-1} \pmod{\prod_{h=j}^{n} q_h}$.

5. REARRANGING THE TREATMENT ORDER

If we recall the solution (3.7) or (3.10), we note the inverse of X_{11} or X_{22} is needed to obtain the solution. Also, we see later that if the size of the fraction is less than s^{n-1} in an s^n factorial, then we can use the s^{n-1} x s^{n-1} orthogonal matrix X^*_{11} (in the sense that $(X^*_{11})'X^*_{11}$ is diagonal) instead of the s^n x s^n matrix to obtain a solution such as (3.7) or (3.10). Also, we shall see in this case that the method of constructing a saturated fractional replicate resolves itself into the problem of selecting the smallest number of treatments from those corresponding to the orthogonal matrix X^*_{11}. Here we also recall that, in (4.4), $L_{s^{n-1}}$ is already an orthogonal matrix; then, we can construct a saturated replicate from the first s^{n-1} treatment observations in the vector Y. However, in this case, the mean effect will be confounded with the main effect F_1. This is the reason for rearranging the treatment order in the vector Y with some higher order defining contrast before constructing a fractional plan; i.e., the mean effect is required to be unconfounded with the main effects.

Now consider rearranging the treatment order in vector Y with some defining contrast in the s^n factorial (s is a prime number). If we use the expression (4.9) for the treatment combinations, then the numbers α_h take on values from 0 to s-1. The s^n-1 degrees of freedom among the s^n treatment combinations may be partitioned into $(s^n-1)/(s-1)$ sets of s-1 degrees of freedom. Each set of s-1 degrees of freedom is given by the contrast among the s sets of s^{n-1} treatment combinations specified by the following equations:

$$
\begin{aligned}
c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n &= 0 \\
c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n &= 1 \\
&\;\;\vdots \\
c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n &= s-1
\end{aligned}
\qquad (5.1)
$$

where the right-hand sides of these equations are elements of the Galois Field GF(s). The c_i's are positive integers between 0 and s-1, not all equal to zero, and all addition and multiplication is done within the Galois Field GF(s), then the interaction $F_1^{c_1} F_2^{c_2} \cdots F_n^{c_n}$ corresponds to the equation whose left-hand side subscript is $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$.

For a defining contrast

$$M \doteq F_1^1 F_2^{c_2} \cdots F_n^{c_n} ,$$

where $\doteq$ means confounded with (c_1 is always 1 for convenience) the identity relationships are written as:

$$\begin{aligned} M_0 &\doteq (F_1 F_2^{c_2} \cdots F_n^{c_n})_0 \\ M_1 &\doteq (F_1 F_2^{c_2} \cdots F_n^{c_n})_1 \\ &\vdots \\ M_{s-1} &\doteq (F_1 F_2^{c_2} \cdots F_n^{c_n})_{s-1} \end{aligned} \tag{5.2}$$

Let the set of treatments for fixed $\alpha_1=\beta$, $\beta=0,1,\cdots,s-1$, be $\{\beta,\alpha_2,\cdots,\alpha_n\}$, then, from (5.1) and (5.2) we find the following relationships: If the k^{th} treatment corresponds to M_i in the set of $\{0,\alpha_2,\cdots,\alpha_n\}$, then the $(k + \beta s^{m-1})^{th}$ treatment corresponds to $M_{i+\beta=j}$ in the set of $\{\beta,\alpha_2,\cdots,\alpha_n\}$, where j is an element of the Galois Field GF(s).

It is understood that an orthogonal array of strength d, of size N*, with n factors each at s levels, consists of a set of N* treatment combinations from an s^n factorial arrangement with the property that all s^d treatment combinations corresponding to any d factors, chosen from n, occur an equal number of times, say λ times, in the subset. The orthogonal arrays are denoted by:

$$(N^*, n, s, d, \lambda) .$$

Then it follows that:

$$N^* = \lambda s^d \quad .$$

Let $\{y\}_i$ whose elements are in [Y], be an $s^{n-1} \times n$ matrix corresponding to $M_i \doteq (F_1 F_2^{c_2} \cdots F_n^{c_n})_i$, then $\{y\}_i$ is an orthogonal array such that

$$(s^{n-1},\ n,\ s,\ d = \text{at least } 2,\ \lambda) \tag{5.3}$$

$$\text{for} \quad i=0,1,\cdots,s-1 \quad .$$

Theorem 2. In an s^n factorial (s is a prime number or power of prime number), if the treatment order in Y is rearranged to correspond to the defining contrast $M_i \doteq (F_1 F_2^{c_2} \cdots F_n^{c_n})_i$, as follows:

$$\begin{matrix} \{y\}_0 \\ \{y\}_1 \\ \vdots \\ \{y\}_{s-1} \end{matrix}$$

then the following form of the corresponding linear orthogonal comparisons matrix X* can be obtained by rearranging the row vector order in X, i.e.,

$$X^* = \begin{bmatrix} X^*_{11} & X^*_{12} & \cdots & X^*_{1s} \\ X^*_{11} & X^*_{22} & \cdots & X^*_{2s} \\ \vdots & & & \\ X^*_{11} & X^*_{s2} & \cdots & X^*_{ss} \end{bmatrix} \tag{5.4}$$

<u>where</u> $X^*_{11} = L_{s^{n-1}}$ <u>and</u> X^*_{ij}, $i,j=1,2,\cdots,s$, <u>are all</u> $s^{n-1} \times s^{n-1}$ <u>matrices</u>.

<u>Proof</u>: Let $L^{(\beta)}_{s^{n-1}}$ be a matrix corresponding to $\{\beta,\alpha_2,\cdots,\alpha_n\}$ in L_{s^n} and let $\{k^{(\beta)}\}_i$ be the sequence of the row order numbers in $L^{(\beta)}_{s^{n-1}}$ corresponding to M_i.

Suppose one of the elements of the $\{k^{(\beta)}\}_i$ is equal to one of the elements of the $\{k^{(\delta)}\}_i$ for β,δ such that $\beta < \delta$ where $\beta,\delta = 0,1,\cdots,s-1$. Then

$$i + (\delta-\beta) = i \mod s \quad .$$

This implies

$$\delta - \beta = rs \quad , \quad r=0,1,\cdots$$

while β and δ are positive integers such that $\beta < s$ and $\delta < s$. Then $r = 0$ and this implies $\beta = \delta$. This contradicts the assumption. Hence, any element of the $\{k^{(\beta)}\}_i$ is not equal to one of the elements of the $\{k^{(\delta)}\}_i$ if $\beta \neq \delta$.

From the fact that $\{y\}_i$ is an orthogonal array such as (5.3), each number of elements of the $\{k^{(\beta)}\}_i$ is the same for β, $i=0,1,\cdots,s-1$. Then the set of sequences

$$\left\{\{k^{(0)}\}_i,\{k^{(1)}\}_i,\cdots,\{k^{(s-1)}\}_i\right\} \quad , \quad \text{given } i, \tag{5.5}$$

consists of s^{n-1} positive integers less than or equal to s^{n-1}, and none of the integers is equal to another one. Then

$$\left\{\{k^{(0)}\}_i,\{k^{(1)}\}_i,\cdots,\{k^{(s-1)}\}_i\right\} = \left\{\{k^{(0)}\}_0,\{k^{(0)}\}_1,\cdots,\{k^{(0)}\}_{s-1}\right\} \tag{5.6}$$

Let $\{\underline{k}^{(\beta)}\}_i$ be the set of the row vectors corresponding to M_i in $L^{(\beta)}_{s^{n-1}}$, then

$$\begin{bmatrix} \{\underline{k}^{(0)}\}_i \\ \{\underline{k}^{(1)}\}_i \\ \vdots \\ \{\underline{k}^{(s-1)}\}_i \end{bmatrix} \sim \begin{bmatrix} \{\underline{k}^{(0)}\}_0 \\ \{\underline{k}^{(0)}\}_1 \\ \vdots \\ \{\underline{k}^{(0)}\}_{s-1} \end{bmatrix} = L^{(0)}_{s^{n-1}}$$

where the notation $\sim$ means that if we rearrange the row vector order properly in the left-hand side matrix of the $\sim$ notation, then this matrix will be the same as $L^{(0)}_{s^{n-1}}$. This proves the theorem.

Theorem 3. In an s^n factorial, let $X^*_{1.} = [X^*_{11}\ X^*_{12} \cdots X^*_{1s}]$ be the $s^{n-1} \times s^n$ matrix corresponding to $\{y\}_0$ with defining contrast $M_0 \doteq (F_1\ F_2^{c_2} \cdots F_n^{c_n})_0$, where at least two of $c_2, \cdots, c_n$ are not zero, then mean and main effect columns in $X^*_{1.}$ are orthogonal to each other.

Proof: From (4.8) and (4.6), we find the following correspondence between the column vector $\underline{f}_h$ in [Y] and the column vector $\underline{g}_j(h)$ in X:

$$\underline{f}_h \qquad\qquad \underline{g}_j(h)$$

$$1_t \otimes \begin{bmatrix} (0)\ 1_u \\ (1)\ 1_u \\ \vdots \\ (s-1)\ 1_u \end{bmatrix} \Leftrightarrow 1_t \otimes \begin{bmatrix} \gamma_{0j}\ 1_u \\ \gamma_{1j}\ 1_u \\ \vdots \\ \gamma_{s-1,j}\ 1_u \end{bmatrix} \tag{5.7}$$

Let U_{11} be a matrix which is constructed using the mean and main effect columns in $X^*_{1\cdot}$ and $\underline{u}_j(h)$ be the column vector corresponding to $F_h^{c_h}$ in U_{11}, and define $\underline{u}_0 = 1_s$.

Since $\{y\}_0$ is an orthogonal array such as (5.3), (i) in each column of $\{y\}_0$, each level number occurs an equal number of times, say μ times; (ii) all s^2 treatment combinations correspond to any two factors, chosen from n, occur an equal number of times, say ν times, in the $\{y\}_0$.

Then, from (5.7), in U_{11}, the following holds:

$$\underline{u}_0 \cdot \underline{u}_j(h) = \mu \sum_{i=0}^{s-1} \gamma_{ij}(h) = 0 \quad \text{for } j=0,1,\cdots,s-1 \ ; \quad h=1,2,\cdots,n$$

$$\underline{u}_j(h) \cdot \underline{u}_g(h) = \mu \sum_{i=0}^{s-1} \gamma_{ij}(h)\gamma_{ig}(h) \quad \text{for } j \neq g \ ; \ j,g=0,1,\cdots,s-1 \ ; \text{ and } h=1,2,\cdots,n$$

$$\underline{u}_j(h) \cdot \underline{u}_g(k) = \nu \sum_{i=1}^{s-1} \sum_{m}^{s-1} \gamma_{ij}(h)\gamma_{mg}(k) \quad \text{for } h \neq k \ ; \ j,g=0,1,\cdots,s-1 \ ; \text{ and } h,k=1,2,\cdots,n \ .$$

The theorem is proven.

Example 5.1. 3^3 factorial.

Let

$$L_3 = \begin{bmatrix} 1 & \alpha_0 & \beta_0 \\ 1 & \alpha_1 & \beta_1 \\ 1 & \alpha_2 & \beta_2 \end{bmatrix}$$

where $\sum_{i=0}^{2} \alpha_i = \sum_{i=0}^{2} \beta_i = 0$ and $\sum_{i=0}^{2} \alpha_i \beta_i = 0$, then $\{y\}_0$ and U_{11} with defining contrast $M \doteq ABC^2$ are as follows:

$$
\begin{array}{ccc}
 & \{y\}_0 & \\
A & B & C \\
 & & \\
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & U_{11} & & & \\
A^0B^0C^0 & A^1B^0C^0 & A^2B^0C^0 & A^0B^1C^0 & A^0B^2C^0 & A^0B^0C^1 & A^0B^0C^2 \\
\underline{u}_0 & \underline{u}_1(A) & \underline{u}_2(A) & \underline{u}_1(B) & \underline{u}_2(B) & \underline{u}_1(C) & \underline{u}_2(C) \\
\end{array}
$$

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 2 & 2 \\
1 & 0 & 1 \\
1 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 2 \\
2 & 1 & 0 \\
2 & 2 & 1
\end{bmatrix}
\Longleftrightarrow
\begin{bmatrix}
1 & \alpha_0 & \beta_0 & \alpha_0 & \beta_0 & \alpha_0 & \beta_0 \\
1 & \alpha_0 & \beta_0 & \alpha_1 & \beta_1 & \alpha_1 & \beta_1 \\
1 & \alpha_0 & \beta_0 & \alpha_2 & \beta_2 & \alpha_2 & \beta_2 \\
1 & \alpha_1 & \beta_1 & \alpha_0 & \beta_0 & \alpha_1 & \beta_1 \\
1 & \alpha_1 & \beta_1 & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\
1 & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \alpha_0 & \beta_0 \\
1 & \alpha_2 & \beta_2 & \alpha_0 & \beta_0 & \alpha_2 & \beta_2 \\
1 & \alpha_2 & \beta_2 & \alpha_1 & \beta_1 & \alpha_0 & \beta_0 \\
1 & \alpha_2 & \beta_2 & \alpha_2 & \beta_2 & \alpha_1 & \beta_1
\end{bmatrix}
$$

then

$$\underline{u}_0 \cdot \underline{u}_j(h) = 0 \text{ and } \underline{u}_j(h) \cdot \underline{u}_g(k) = 0 \quad \text{for} \quad j,g=1,2 \text{ and } h,k=A,B,C.$$

Theorem 4. *Let $X^*_{1.} = [X^*_{11}\ X^*_{12}]$ be a $2^{n-1} \times 2^n$ matrix corresponding to $\{y\}_0$ with defining contrast $M_0 \doteq (F_1^{c_1} F_2^{c_2} \cdots F_n^{c_n})_0$, $c_1=1$, $c_h=0$ or 1 for $h \neq 1$, in a 2^n factorial, then the $X^*_{1.}$ can be rearranged as follows:*

$$[X^*_{11} \pm X^*_{11}] \qquad (5.8)$$

where the parameter order in (5.8) is $M, F_n, \cdots, F_2F_3 \cdots F_n$; $W, F_nW, \cdots, F_2F_3 \cdots, F_nW$, where $W = F_1^{c_1} F_2^{c_2} \cdots F_n^{c_n}$.

Proof: In a 2^n factorial, (4.5) becomes as follows:

$$\underline{g} = \prod_{h=1}^{n} \otimes \begin{bmatrix} \gamma_{0c_h} \\ \\ \gamma_{1c_h} \end{bmatrix} \tag{5.9}$$

where $\gamma_{0c_h} = 1$ if $c_h = 0$ and $\gamma_{0c_h} = -1$ if $c_h = 1$ and $\gamma_{1c_h} = 1$ for all h. Define a product of two matrices $A_{mxn} = (a_{ij})$ and $B_{mxn} = (b_{ij})$ such as:

$$A : B = \begin{bmatrix} a_{11}\, b_{11} & a_{12}\, b_{12} & \cdots & a_{1n}\, b_{1n} \\ a_{21}\, b_{21} & a_{22}\, b_{22} & \cdots & a_{2n}\, b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\, b_{m1} & a_{m2}\, b_{m2} & \cdots & a_{mn}\, b_{mn} \end{bmatrix}, \tag{5.10}$$

then (5.9) may be expressed as follows:

$$\begin{aligned} \underline{g} &= \begin{bmatrix} -1 \\ \\ 1 \end{bmatrix} \otimes 1_{2^{n-1}} : 1_2 \otimes \begin{bmatrix} \gamma_{0c_2} \\ \\ 1 \end{bmatrix} : \cdots : 1_{2^{n-1}} \otimes \begin{bmatrix} \gamma_{0c_n} \\ \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1_{2^{n-1}} \\ \\ 1_{2^{n-1}} \end{bmatrix} : 1_2 \otimes \begin{bmatrix} \gamma_{0c_2}\, 1_{2^{n-2}} \\ \\ 1_{2^{n-2}} \end{bmatrix} : \cdots : 1_{2^{n-1}} \otimes \begin{bmatrix} \gamma_{0c_n} \\ \\ 1 \end{bmatrix} \end{aligned} \tag{5.11}$$

From (5.7)

$$\underline{g} = \underline{g}_1(1) : \underline{g}_{c_2}(2) : \cdots : \underline{g}_{c_n}(n) \quad , \qquad (5.12)$$

then, if $c_h = 0$

$$\underline{g}_{c_h}(h) = 1_{2^n} \quad . \qquad (5.13)$$

On the other hand, from (4.8)

$$\underline{f}_h = 1_{2^{h-1}} \otimes \begin{bmatrix} 0_{2^{n-h}} \\ 1_{2^{n-h}} \end{bmatrix} \quad , \qquad (5.14)$$

then, if $c_h = 0$

$$c_h \underline{f}_h = 0_{2^n} \qquad (5.15)$$

where 0_{2^n} is a 2^n x 1 column vector with all elements equal to zero.

Let

$$\underline{f}^* = \underline{f}_1 + c_2\underline{f}_2 + \cdots + c_n\underline{f}_n \quad , \quad \text{mod } 2 \ ,$$

then

$$\underline{f}^* = \begin{bmatrix} 0_{2^{n-1}} \\ 1_{2^{n-1}} \end{bmatrix} + 1_2 \otimes \begin{bmatrix} 0_{2^{n-2}} \\ 1_{2^{n-2}} \end{bmatrix} c_2 + \cdots + 1_{2^{n-1}} \otimes \begin{bmatrix} 0 \\ c_n \end{bmatrix} c_n, \text{ mod } 2 .$$

(5.16)

Let G and F* be the 2^n x n matrices such that

$$G = [\underline{g}_1(1)\ \underline{g}_{c_2}(2) \cdots \underline{g}_{c_n}(n)]$$

$$F^* = [\underline{f}_1 \quad c_2\underline{f}_2 \quad \cdots \quad c_n\underline{f}_n \quad]$$

and suppose m of c_h's are zero, then m column vectors in G may be $\mathbf{1}_{2^n}$.

If the k^{th} element of $\underline{f}^*$ is 0, then the k^{th} row vector in F* has an even number, say r, of 1 elements, and the corresponding k^{th} row vector in G may have (n-m-r) of (-1) elements from (5.1), (5.10), (5.11), (5.13), and (5.15). From (5.12), the k^{th} element of $\underline{g}$ is $(-1)^{n-m-r} = (-1)^{n-m}$. Then, if n-m is an even number,

$$\underline{w}_M = \mathbf{1}_{2^{n-1}} \quad ,$$

where $\underline{w}_M$ is a 2^{n-1} column vector corresponding to $W = F_1 F_2^{c_2} \cdots F_n^{c_n}$ in $X^*_{1\cdot}$.
Hence

$$\underline{w}_n = \underline{f}^*_n : \mathbf{1}_{2^{n-1}} = \underline{f}^*_n$$

$$\underline{w}_{2,3,\cdots,n} = \underline{f}^*_{2,3,\cdots,n} : \mathbf{1}_{2^{n-1}} = \underline{f}^*_{2,3,\cdots,n}$$

where $\underline{w}_n, \cdots, \underline{w}_{2,3,\cdots,n}$ are the $2^{n-1} \times 1$ column vectors corresponding to the effect W, $F_n W_r$ $\cdots$, $F_2^{c_1} F_3^{c_3} \cdots F_n^{c_n} W$ in X^*_{12} respectively and $\underline{f}^*_n, \cdots, \underline{f}^*_{2,3,\cdots,n}$ are the $2^{n-1} \times 1$ column vectors corresponding to the effect $F_n, \cdots, F_2^{c_2} F_3^{c_3} \cdots F_n^{c_n}$ in X^*_{11} respectively.

If n-m is an odd number, then

$$\underline{w}_M = -\mathbf{1}_{2^{n-1}} \quad .$$

Hence

$$\underline{w}_n = -\underline{f}^*_n$$

$$\cdots$$

$$\underline{w}_{2,3,\cdots,n} = -\underline{f}^*_{2,3,\cdots,n} \quad .$$

This proves the theorem.

6. CONSTRUCTION OF FRACTIONAL REPLICATES

We shall consider mostly the method of constructing saturated main effect plans in an s^n factorial. Although we could always construct various saturated non-orthogonal plans for any given parameter set, the general steps of the construction method may not be too instructive. The following steps, however, will be common in constructing any fractional replicate for the specified parameters. Special cases will be illustrated in the following examples.

Step 1. Given the design matrix and parameter and observation vectors $X\underline{B} = E(Y)$ in any fashion and not necessarily that of the previous section, we now rearrange the parameter matrix such that the p parameters, $p < N$, are arranged to have the p parameters of interest first and N-p parameters not of interest last to obtain $\underline{B}$ rearranged $(\underline{B}^{*\prime}_p \, \underline{B}^{*\prime}_{N-p})$. This also rearranges the columns of X such that

$$X^* \, \underline{B}^* = E(Y) \tag{6.1}$$

$$\underset{N\times p}{(X^*_1} \quad \underset{N\times(N-p)}{X^*_2)} \begin{bmatrix} \underline{B}^*_p \\ \\ \underline{B}^*_{N-p} \end{bmatrix} = E(Y) \tag{6.2}$$

Step 2. Search through rows of X_1^* until there is an X_{11}, p x p, which is non-singular.

Step 3. Corresponding to the rows in X_{11} will be rows in X_1^* and observations in Y. Rearrange the observations in Y into

$$\begin{bmatrix} Y_p^* \\ Y_{N-p}^* \end{bmatrix}$$

corresponding to the rows in X_{11} from X_1^*. The observations in Y_p yield a saturated design for the parameters in $\underline{B}_p^*$. This obtained set is one of the possible sets. All possible sets are found by defining all X_{11} which have an inverse.

Example 6.1: Saturated main effect plans in a 3 x 2 factorial.

From Table (4.1), we obtain a matrix X_1^* for parameters M, A_L, A_Q, B as follows:

$$X_1^* = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & -2 & -1 \\ 1 & 0 & -2 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Let $\underline{t}_{ij}$ be the row vector corresponding to treatment combination (ij) in X_1^*, then by using the Schmidt method of orthogonalizing the rows, we obtain

$$\underline{t}^*_{00} = \underline{t}_{00}$$

$$\underline{t}^*_{01} = \underline{t}_{01} - \frac{\underline{t}_{00} \cdot \underline{t}_{01}}{\|\underline{t}_{00}\|^2} \underline{t}_{00}$$

$$= \frac{1}{2} (1 \quad -1 \quad 1 \quad 3)$$

$$\underline{t}^*_{10} = \underline{t}_{10} - \frac{\underline{t}_{00} \cdot \underline{t}_{10}}{\|t_{00}\|^2} \underline{t}_{00} - \frac{\underline{t}^*_{01} \cdot \underline{t}_{10}}{\|t^*_{01}\|^2} \underline{t}^*_{01}$$

$$= \frac{1}{3} (4 \quad -1 \quad -5 \quad 0)$$

$$\underline{t}^*_{11} = \underline{t}_{11} - \frac{\underline{t}_{00} \cdot \underline{t}_{11}}{\|\underline{t}_{00}\|^2} \underline{t}_{00} - \frac{\underline{t}^*_{01} \cdot \underline{t}_{11}}{\|\underline{t}^*_{01}\|^2} \underline{t}^*_{01} - \frac{\underline{t}^*_{10} \cdot \underline{t}_{11}}{\|\underline{t}^*_{10}\|^2} \underline{t}^*_{10}$$

$$= (0 \quad 0 \quad 0 \quad 0) .$$

Then $\underline{t}_{11}$ is not orthogonal to the set of vectors $\underline{t}_{00}$, $\underline{t}_{01}$, and $\underline{t}_{10}$.

Take vector $\underline{t}_{20}$.

$$\underline{t}^*_{20} = \underline{t}_{20} - \frac{\underline{t}_{00} \cdot \underline{t}_{20}}{\|\underline{t}_{00}\|^2} \underline{t}_{00} - \frac{\underline{t}^*_{01} \cdot \underline{t}_{20}}{\|\underline{t}^*_{01}\|^2} t^*_{01} - \frac{\underline{t}^*_{10} \cdot \underline{t}_{20}}{\|\underline{t}^*_{10}\|^2} \underline{t}^*_{10}$$

$$= \frac{3}{7} (2 \quad 3 \quad 1 \quad 0) .$$

Hence one of the saturated main effect plans in a 3 x 2 factorial is:

$$\begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{matrix} .$$

Example 6.2: Saturated main effect plans in a 2^4 factorial.

If we consider a 2^4 factorial design matrix L_{2^4} with the defining contrast $M \doteq ABCD$, then the alias scheme is as follows:

$$M \doteq ABCD, \quad A \doteq BCD, \quad B \doteq ACD, \quad C \doteq ABD, \quad D \doteq ABC$$

$$AB \doteq CD, \quad AC \doteq BD, \quad BC \doteq AD.$$

After rearranging the rows and columns under consideration of the above alias scheme and from Theorems 2 and 4, we obtain the following matrix X*:

$$X^* = \begin{bmatrix} X^*_{11} & X^*_{11} \\ & \\ X^*_{11} & -X^*_{11} \end{bmatrix} = \left[\begin{array}{rrrrrrrr rrrrrrrr}
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & & & & \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{array}\right] \quad (6.3)$$

where the treatment order is

$$\begin{array}{l} 0000,\ 0011,\ 0110,\ 0101,\ 1010,\ 1001,\ 1100,\ 1111; \\ 1000,\ 1011,\ 1110,\ 1101,\ 0010,\ 0001,\ 0100,\ \text{and}\ 0111, \end{array} \tag{6.4}$$

and the parameter order is

$$\begin{array}{l} M,\ D,\ C,\ CD,\ B,\ BC,\ BC,\ BCD; \\ ABCD,\ ABC,\ ABD,\ AB,\ ACD,\ AC,\ AD,\ \text{and}\ A. \end{array} \tag{6.5}$$

Consider the following fraction of a 2^4 factorial

$$Y_p = X^*_{1.}\underline{B} + e_p\ , \quad p < 8 \tag{6.6}$$

where Y_p is a p x 1 vector from the vector Y, $\underline{B}$ is a column vector of N = 16 unknown parameters reordered such as (6.5), $X^*_{1.}$ is a design matrix for given Y_p and $\underline{B}$, and e_p is a p x 1 column vector of random error components.

Suppose the following partition matrix of X is possible after rearranging the column vectors in X*,

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} X_{11} & X_{1211} & X_{1212} \\ X_{2111} & X_{2211} & X_{2212} \\ \hline X_{2121} & X_{2221} & X_{2222} \end{array}\right] \tag{6.7}$$

where parameter order corresponding to columns in X is as follows:

M, A, B, C, D, CD, BD, BC; ABCD, BCD, ACD, ABD, ABC, AB, AC, AD.

Let

$$X_{11}^{*} = \begin{bmatrix} X_{11} & X_{1211} \\ X_{2111} & X_{2211} \end{bmatrix} \qquad (6.8)$$

where X_{11} is a p x p (p < 8) non-singular matrix, X_{2111} and X'_{1211} are each p x (8-p) matrices, X_{2111} is an (8-p) x (8-p) matrix, X_{2121} and X'_{1212} are each 8 x p matrices, and X_{2221} and X'_{2212} are each 8 x (8-p) matrices.

We know from theorems 3 and 4 that

$$X_{1212} = [X_{11} \vdots X_{1211}] , \qquad (6.9)$$

and since $X_{11}^{*\prime}X_{11}^{*}$ is diagonal, if X_{11} is non-singular, then X_{2211} is also non-singular, and from (3.9)

$$X_{11}^{-1} = (U'_{11}U_{11})^{-1} X'_{11}(1 + \lambda_1\lambda'_1) \qquad (6.10)$$

where

$$U_{11} = \begin{bmatrix} X_{11} \\ X_{2111} \end{bmatrix} \qquad (6.11)$$

and

$$\lambda_1 = -X_{1211} X_{2211}^{-1} \tag{6.12}$$

then

$$X_{11}'(I+\lambda_1\lambda_1')X_{12} = X_{11}'(I+\lambda_1\lambda_1') [X_{1211} \vdots X_{11} \vdots X_{1211}]$$

$$= [X_{11}'(I+\lambda_1\lambda_1')X_{1211} \vdots 8I_{5x5} \vdots X_{11}'(I+\lambda_1\lambda_1')X_{1211}] \quad ,$$

then, from (3.7) we will obtain the following solution for (6.6):

$$\begin{aligned} \underline{\hat{B}}_p + (U_{11}'U_{11})^{-1} [X_{11}'(I+\lambda_1\lambda_1')X_{1211} \vdots 8I \vdots X_{11}'(I+\lambda_1\lambda_1')X_{1211}] \underline{\hat{B}}_{16-p} \\ = (U_{11}'U_{11})^{-1} X_{11}'(I+\lambda_1\lambda_1')Y_p \end{aligned} \tag{6.13}$$

This solution indicates that the solution depends **only** on λ_1. This further means that the solution depends only on X_{2211}.

Now consider the saturated main effect plans in a 2^4 factorial. Let the treatments be arranged such as (6.4) and the corresponding row vectors in X be numbered $1,2,\cdots,16$ respectively, and let

$$U_{12} = \begin{bmatrix} X_{1211} \\ X_{2211} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{6.14}$$

In the matrix U_{12}, we can find easily three independent rows, i.e., the following combinations of rows make non-singular 3 x 3 matrices.

(1,2,3), (1,2,4), (1,2,5), (1,2,6), (1,3,4), (1,3,5), (1,3,7), (1,4,6),

(1,4,7), (1,5,6), (1,5,7), (1,6,7), (2,3,4), (2,3,5), (2,3,8), (2,4,6),

(2,4,8), (2,5,6), (2,5,8), (2,6,8), (3,4,7), (3,4,8), (3,5,7), (3,5,8),

(3,7,8), (4,6,7), (4,6,8), (4,7,8), (5,6,7), (5,6,8), (5,7,8), (6,7,8),

where the numbers indicate the row numbers in matrix U_{12}, then the following 32 treatment combinations will be the saturated main effect plans in a 2^4 factorial

(1)	(2)	(3)	(4)	(5)	(6)
0101	0110	0110	0110	0011	0011
1010	1010	0101	0101	1010	0101
1001	1001	1001	1010	1001	1001
1100	1100	1100	1100	1100	1100
1111	1111	1111	1111	1111	1111

(7)	(8)	(9)	(10)	(11)	(12)
0011	0011	0011	0011	0011	0011
0101	0110	0110	0110	0110	0110
1010	1010	1010	0101	0101	0101
1001	1100	1001	1100	1001	1010
1111	1111	1111	1111	1111	1111

(13)	(14)	(15)	(16)	(17)	(18)
0000	0000	0000	0000	0000	0000
1010	0101	0101	0110	0110	0110
1001	1001	1010	1010	1010	0101
1100	1100	1001	1100	1001	1100
1111	1111	1100	1111	1100	1111

(6.15)

(19)	(20)	(21)	(22)	(23)	(24)
0000	0000	0000	0000	0000	0000
0110	0110	0011	0011	0011	0011
0101	0101	1010	1010	0101	0101
1001	1010	1001	1001	1001	1001
1100	1100	1111	1100	1111	1100

(25)	(26)	(27)	(28)	(29)	(30)
0000	0000	0000	0000	0000	0000
0011	0011	0011	0011	0011	0011
0101	0110	0110	0110	0110	0110
1010	1010	1010	1010	0101	0101
1100	1111	1100	1001	1111	1100

(31)	(32)
0000	0000
0011	0011
0110	0110
0101	0101
1001	1010

Let (n_1,n_2,n_3,n_4,n_5), where n_i is the treatment order number in (6.4), be one of the above 32 plans, then by recalling theorems 2 and 4 we know the following treatment combinations are also saturated main effect plans in a 2^4 factorial, i.e.,

$$(n_1+8,\ n_2+8,\ n_3+8,\ n_4+8,\ n_5+8) \tag{6.16}$$

From (6.16) and (6.7) we know, for example, that the 10^{th}, 11^{th}, 12^{th}, and the 16^{th} row vectors in X_{2121} form a set of independent row vectors; then, by adding another independent row vector to this set from U_{11}, we can construct the following eight satruated main effect plans:

0000	0011	0110	0101
1011	1011	1011	1011
1110	1110	1110	1110
1101	1101	1101	1101
0111	0111	0111	0111

1010	1001	1100	1111
1011	1011	1011	1011
1110	1110	1110	1110
1101	1101	1101	1101
0111	0111	0111	0111

Example 6.3: Saturated main effect plans in a 3^3 factorial.

In a 3^3 factorial, after rearranging the row order for the defining contrast $M \doteq ABC^2$, we obtain the following matrix:

$$X^* = \begin{bmatrix} X^*_{11} & X^*_{21} & X^*_{31} \\ X^*_{11} & X^*_{22} & X^*_{32} \\ X^*_{11} & X^*_{23} & X^*_{33} \end{bmatrix} \quad , \qquad (6.17)$$

where each X^*_{ij} is a 9 x 9 square matrix and the treatment order is 000, 011, 022, 101, 112, 120, 202, 210, 221; 100, 111, 122, 201, 212, 220, 002, 010, 021; 200, 211, 222, 001, 012, 020, 102, 110, and 121, and the parameter order is M, C_L, C_Q, B_L, B_LC_L, B_LC_Q, B_Q, B_QC_L, B_QC_Q; A_L, A_LC_L, A_LC_Q, A_LB_L, $A_LB_LC_L$, $A_LB_LC_Q$, A_LB_Q, $A_LB_QC_L$, $A_LB_QC_Q$; A_Q, A_QC_L, A_QC_Q, A_QB_L, $A_QB_LC_L$, $A_QB_LC_Q$, A_QB_Q, $A_QB_QC_L$, and $A_QB_QC_Q$, but we could not obtain a solution such as (6.13), because the effects B_LC_L, B_LC_Q, B_QC_L, and B_QC_Q are confounded with both main effects A_L and A_Q, respectively, i.e.,

$$\begin{aligned} B_LC_L &\doteq -\frac{1}{3} A_L \doteq \frac{1}{3} A_Q \\ B_LC_Q &\doteq -\frac{1}{3} A_L \doteq - A_Q \\ B_QC_L &\doteq \frac{1}{3} A_L \doteq A_Q \\ B_QC_Q &\doteq - A_L \doteq A_Q \quad . \end{aligned}$$

However, we will find that each X^*_{ij} is a non-singular matrix and if we rearrange the column order to obtain M, A_L, A_Q, B_L, B_Q, C_L, C_Q, B_LC_L, B_LC_Q, ···, and let the first 9 x 9 matrix of the rearranged matrix be A_{11}, then

$$A_{11} = \begin{array}{c} \begin{matrix} M & A_L & A_Q & B_L & B_Q & C_L & C_Q & B_LC_L & B_LC_L \end{matrix} \\ \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & 0 & 0 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 & 1 & 0 & -2 & 0 & 2 \\ 1 & 0 & -2 & 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & -2 & 0 & -2 \end{bmatrix} \end{array} \qquad (6.18)$$

If we use the symbols $\underline{M}$, $\underline{A_L}$, $\underline{A_Q}$, $\underline{B_L}$, $\underline{B_Q}$, $\underline{C_L}$, $\underline{C_Q}$, $\underline{B_LC_L}$ and $\underline{B_LC_Q}$ as the symbol of each corresponding column vectors respectively, then, from the theorem 3, the column vectors $\underline{M}$, $\underline{A_L}$, $\underline{A_Q}$, $\underline{B_L}$, $\underline{B_Q}$, $\underline{C_L}$, and $\underline{C_Q}$ are orthogonal to each other and also $\underline{M}$, $\underline{B_L}$, $\underline{B_Q}$, $\underline{C_L}$, $\underline{C_Q}$, $\underline{B_LC_L}$ and $\underline{B_LC_Q}$ are orthogonal to each other. Hence, we can say that matrix A_{11} is non-singular, and then we can make $\underline{B_LC_L}$ and $\underline{B_LC_Q}$ orthogonal vectors with the first 7 column vectors. Let such new vectors of $\underline{B_LC_L}$, $\underline{B_LC_Q}$ be $\underline{Z}_1$ and $\underline{Z}_2$ respectively, then by using the Schmidt method of orthogonalizing the columns we obtain:

$$[\underline{Z}_1 \quad \underline{Z}_2] = \begin{bmatrix} 1 & -1 \\ -2 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ -2 & 0 \\ -2 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{ignoring the common factor.} \qquad (6.19)$$

Now, if we find a non-singular 2 x 2 matrix from the 9 x 2 matrix, then we can construct a corresponding information matrix X_{11} for saturated main effect plans.

Consider the partitioned matrix X_{27x27}

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

where X_{11} is p x p ($p < 9$), X_{12} and X'_{21} are p x (27 - p) each, X_{22} is (27 - p) x (27 - p).

Now, consider the following fraction of a 3^3 factorial

$$Y_p = X_1\underline{B} + e_p \quad , \qquad p < 9$$

where Y'_p = (000, 011, 022, 101, 112, 120, 202), then from (3.7)

$$\hat{\underline{B}}_p + X_{11}^{-1} X_{12} \hat{\underline{B}}_{27-p} = X_{11}^{-1} Y_p \quad . \qquad (6.20)$$

Now, let

$$A^*_{11} = \begin{array}{c} \underline{M} \;\; \underline{A}_L \;\; \underline{A}_Q \;\; \underline{B}_L \;\; \underline{B}_Q \;\; \underline{C}_L \;\; \underline{Z}_1 \;\; \underline{Z}_2 \\ \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & 0 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 & 1 & 0 & 1 & 1 \\ 1 & 0 & -2 & 0 & -2 & 1 & 1 & -1 \\ 1 & 0 & -2 & 1 & 1 & -1 & -2 & 0 \\ 1 & 1 & 1 & -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & 0 & -2 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & -1 \end{bmatrix} \end{array} = \begin{bmatrix} X_{11} & Z_{12} \\ X_{1121} & Z_{22} \end{bmatrix} ,$$

then $A^{*\prime}_{11} A^*_{11}$ is diagonal and $Z_{22} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is non-singular. Hence,

$$X^{-1}_{11} = (U'_{11}U_{11})^{-1} X'_{11}(I+\mu\mu')$$

where

$$U_{11} = \begin{bmatrix} X_{11} \\ X_{1121} \end{bmatrix} ,$$

$$\mu = -Z_{12}Z^{-1}_{22} ,$$

then (6.20) becomes

$$\underline{\hat{B}}_p + (U'_{11}U_{11})^{-1} X'_{11}(I+\mu\mu')X_{12}\underline{\hat{B}}_{27-p}$$

$$= (U'_{11}U_{11})^{-1} X'_{11}(I+\mu\mu')Y_p .$$

The following 27 saturated main effect plans are constructed from the set $\{(ABC^2)_0\}$ in a 3^3 factorial:

(1)	(2)	(3)	(4)	(5)	(6)
022	011	011	011	011	011
101	101	022	022	022	022
112	112	112	101	101	101
120	120	120	112	112	112
202	202	202	202	120	120
210	210	210	210	210	202
221	221	221	221	221	221

(7)	(8)	(9)	(10)	(11)	(12)
000	000	000	000	000	000
101	022	022	022	022	011
112	112	101	101	101	101
120	120	120	112	112	120
202	202	202	120	120	202
210	210	210	202	202	210
221	221	221	221	210	221

(13)	(14)	(15)	(16)	(17)	(18)
000	000	000	000	000	000
011	011	011	011	011	011
101	101	101	022	022	022
112	112	112	120	112	112
202	120	120	202	202	120
210	210	202	210	210	210
221	221	210	221	221	221

(19)	(20)	(21)	(22)	(23)	(24)
000	000	000	000	000	000
011	011	011	011	011	011
022	022	022	022	022	022
112	101	101	101	101	101
120	202	120	120	112	112
202	210	210	202	202	202
210	221	221	221	221	210

(25)	(26)	(27)
000	000	000
011	011	011
022	022	022
101	101	101
112	112	112
120	120	120
221	210	202

As stated in the introduction to this section, the method presented is useful in constructing fractional replicates from any $q_1 \times q_2 \times \cdots \times q_n$ factorial. Two special cases were considered in that saturated main effect plans were constructed from 2^n and 3^n factorials. The method is applicable directly to s^n factorials. If saturated main effect and two-factor interaction plans were desired, the same general procedure would be applicable. For example, 11 treatments would be needed to obtain a saturated main effect and two-factor interaction plan from a 2^4 factorial.

Special attention has been given to saturated fractional replicates, but the procedure applies equally well to the construction of unsaturated fractional replicates. For example, suppose that it is desired to construct a $\frac{3}{4}$ replicate of a 2^4 factorial or $\frac{4}{9}$ replicates of 3^4 factorial for the parameter set involving mean, main effects, and two-factor interactions. This could be accomplished following the above procedure.

Criteria for goodness of fractional replicates would need to be developed to determine which of the several fractional replicates is "best". Consideration of efficiency (see Banerjee and Federer [1963, 1964, 1966], aliasing structure, equality of variance for effects of a given order, etc. would need to be considered. The use of any criterion above, or others would need to be justified.

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