QUADRATIC FORMS in SINGULAR NORMAL VARIABLES.[1]

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Abstract

A unified method is given for handling certain problems relating to quadratic forms of normally distributed random variables, especially in the case where the variance-covariance matrix is singular. Necessary and sufficient conditions for a quadratic form of jointly distributed normal variables to have a non-central chi-square distribution are obtained, as well as conditions for the independence both of two quadratic forms and of a linear and a quadratic form.

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Graybill (1961), in his fourth chapter, gives several theorems relating to the distribution of quadratic forms. His Theorem 4.9 states that if X is a vector of normally distributed random variables having vector of means μ and variance-covariance matrix V, then the quadratic form $X'AX$ is distributed as a non-central chi-square with q degrees of freedom and non-centrality parameter $\frac{1}{2}\mu'A\mu$ if and only if AV is idempotent, q being the rank of A. Implicit in the proof of the theorem is the non-singularity of the variance-covariance matrix C. We here extend this theorem to the case when V is singular and consider implications thereof.

1. Preliminary results.

If Y is a vector of r random variables having a multivariate normal distribution with mean vector μ and variance-covariance matrix Σ then since the integral of a probability distribution function is unity, it follows that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp -\tfrac{1}{2}(y - \mu)' \Sigma^{-1} (y - \mu) dy = (2\pi)^{\frac{1}{2}r} \sqrt{|\Sigma|} \ .$$

A consequence of this is, that for any linear function of the elements of y, $\ell'y$ say,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}\, \underline{y}' \sum{}^{-1} \underline{y} + \underline{\ell}'\underline{y}\right] = (2\pi)^{\frac{1}{2}r} \sqrt{\left|\sum\right|}\ \exp(\tfrac{1}{2}\underline{\ell}' \sum \underline{\ell}) \,. \tag{1}$$

(Putting $\underline{\ell} = \sum{}^{-1} \underline{\mu}$ in the first result achieves this.) The identity (1) plays a fundamental role in obtaining the results that follow. Another expression of importance is the moment generating function of a non-central χ^2 distribution having p degrees of freedom and non-centrality parameter λ (see for example Graybill 1961):

$$M(t) = e^{\frac{2\lambda t}{1-2t}} (1-2t)^{-\frac{1}{2}p} \,. \tag{2}$$

Expansion and collection of like terms shows that the k'th cumulant K_k of the non-central χ^2 is

$$K_k = 2^{k-1}(k-1)!(2k\lambda + p) \tag{3}$$

Throughout this paper $\underline{X}_{n \times 1}$ will represent a vector of n random variables having a multivariate normal distribution with mean vector μ and variance covariance matrix V. In particular we consider the case when V is singular, not of full rank, and we therefore refer to $\underline{X}$ as having a singular multivariate normal distribution, abbreviating this by saying that $\underline{X}$ is $SN(\underline{\mu},V)$. (Non-singular V will be considered as a special case.)

Whether V is singular or not it is, by nature, always positive semi-definite. Hence, when the rank of V is $r \le n$, a matrix L can always be found such that $V = LL'$, where L is $n \times r$ of rank r. Then, if $\underline{Y}_{r \times 1}$ is a vector of normal variables having zero means and variance-covariance matrix I, we can always write $\underline{X} = \mu + L\underline{y}$. This result is often taken to be the definition of the singular multivariate normal distribution (Anderson, 1958, for example.) By this means,

problems involving $\underline{X}$ can be put in terms of $\underline{Y}$.

2. Conditions for non-central χ^2.

Consider now the quadratic form $Q = \underline{X}'A\underline{X}$ where $\underline{X}$ is $SN(\mu,V)$. In terms of $\underline{Y}$ we can write

$$Q = \underline{X}'A\underline{X} = \underline{Y}'L'AL\underline{Y} + 2\mu'AL\underline{Y} + \mu'A\mu$$

so that the moment generating function of Q is

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}r} \exp[-\tfrac{1}{2}\underline{y}'\underline{y} + t\underline{y}'L'AL y + 2t\mu'AL\underline{y} + \mu'A\mu]d\underline{y}$$

which by a simple application of (1) is

$$| I - 2tL'AL |^{-\frac{1}{2}} \exp[t\mu'A\mu + 2t^2\mu'AL(I-2tL'AL)^{-1} L'A\mu] . \tag{4}$$

We seek the k'th cumulant of Q, to identify it with (3). From (4) we see that the cumulant generating function (c.g.f.) of Q is

$$-\tfrac{1}{2}\log|I - 2tL'AL| + t\mu'A\mu + 2t^2\mu'AL(I - 2tL'AL)^{-1} L'A\mu .$$

Now, it is easily established by diagonalizing $L'AL$ [see Lancaster (1954)] that, for t sufficiently small,

$$-\tfrac{1}{2}\log|I - 2tL'AL| = \sum_{j=1}^{\infty} \frac{t^j 2^{j-1} \mathrm{tr}(L'AL)^j}{j} \tag{5}$$

where in general, $\mathrm{tr}(M)$ is the trace of the square matrix M, the sum of its diagonal elements. Also, for sufficiently small t, direct expansion gives

$$(I - 2tL'AL)^{-1} = \sum_{j=0}^{\infty} (2tL'AL)^j , \tag{6}$$

using the standard result that the zero power of any matrix is the identity matrix. Hence the last term in the c.g.f. of Q is

$$2t^2 \underline{\mu}'AL(I - 2tL'AL)^{-1}L'A\underline{\mu} = \tfrac{1}{2}\underline{\mu}'\left\{ \sum_{j=0}^{\infty} (2t)^{j+2} AL(L'AL)^j L'A \right\} \underline{\mu} . \qquad (7)$$

Combining (5) and (7) gives the c.g.f. of Q' as

$$t[\underline{\mu}'A\underline{\mu} + tr(L'AL)] + \sum_{j=2}^{\infty} t^j 2^{j-1} [\underline{\mu}'AL(L'AL)^{j-2} L'A\underline{\mu} + tr(L'AL)^j / j]. \qquad (8)$$

Now, utilizing $V = LL'$ and the general result $tr(AB) = tr(BA)$, it is easy to establish (by induction) that in the second term of (8)

$$AL(L'AL)^{j-2} L'A = A(VA)^{j-1}$$

and

$$tr(L'AL)^j = tr(VA)^j .$$

Hence the c.g.f. of Q becomes

$$t[\underline{\mu}'A\underline{\mu} + tr(VA)] + \sum_{j=2}^{\infty} t^j 2^{j-1} [\underline{\mu}'A(VA)^{j-1}\underline{\mu} + tr(VA)^j / j]$$

or, more simply,

$$\sum_{j=1}^{\infty} t^j 2^{j-1} [\underline{\mu}'A(VA)^{j-1}\underline{\mu} + tr(VA)^j / j . \qquad (9)$$

It follows that the k'th cumulant of Q is

$$k!2^{k-1}[\underline{\mu}'A(VA)^{k-1}\underline{\mu} + tr(VA)^k/k]$$

$$= 2^{k-1}(k-1)![k\underline{\mu}'A(VA)^{k-1}\underline{\mu} + tr(VA)^k], \qquad (10)$$

for k = 1,2,... . We note that although the c.g.f. of Q was originally written in terms of L it is not, in fact, dependent on L at all, as seen in (9).

Since the moments, or equivalently the cumulants, uniquely determine the distribution when their respective generating functions exist we have the following theorem.

Theorem 1: If $\underline{X}$ is SN($\underline{\mu}$,V), a necessary and sufficient condition that the quadratic form $\underline{X}'A\underline{X}$ has a non-central χ^2- distribution with q degrees of freedom and non-centrality parameter $\frac{1}{2}\underline{\mu}'A\underline{\mu}$ is that

$$k\underline{\mu}'A(VA)^{k-1}\underline{\mu} + tr(VA)^k = k\underline{\mu}'A\underline{\mu} + q \text{ for } k = 1,2,... . \quad (11)$$

Proof: Equate the cumulant expressions (3) and (10).

Unfortunately, in applications (11) is not readily amenable to verification. Equivalent, but more appealling conditions will therefore be derived, based on the result (11). First, by putting k = 1 in (11) we have

$$tr(VA) = q.$$

Second, since A is non-negative definite it can be written in the form $A = A_1A_1'$, where A_1 is n × p. With A in this form it is easy to show that

$$A(VA)^{k-1} = A_1(A_1'VA_1)^{k-1}A_1'$$

and $$tr(VA)^k = tr(A_1'VA_1)^k$$

Hence (11) can be written as

$$k\underline{\mu}'A_1(A_1'VA_1)^{k-1}A_1'\underline{\mu} + tr(A_1'VA_1)^k = k\underline{\mu}'A_1A_1'\underline{\mu} + tr(A_1'VA_1).$$

Let us now take P as an orthogonal matrix such that $P'A_1'VA_1P = \lambda$ is diagonal with diagonal elements λ_i, $i = 1,2,\ldots,p$. Then, for $\gamma = P'A_1'\underline{\mu}$ having elements γ_i for $i = 2,\ldots p$ (11) becomes

$$k\sum_{j=1}^{p}\gamma_i^2(\lambda_i^{k-1}-1) + \sum_{i=1}^{p}(\lambda_i^{k-1}-1)\lambda_i = 0$$

or

$$\sum_{i=1}^{p}(\gamma_i^2 + \lambda_i/k)(\lambda_i^{k-1}-1) = 0.$$

Since this equation must hold for all integer values k, it is clear that the only permissible values of the λ_i are ones or zeros (and when λ_i is zero so is the corresponding γ_i). This, coupled with the result $tr(A_1'VA_1) = q$ indicates that the characteristic roots of $A_1'VA_1$ must consist of q 1's and p-q zeros. Since the non-zero characteristic roots of $A_1'VA$ are identical to those of VA, the latter must each be unity, there being q of them. Thus, returning to (11) we see that it holds if, and only if,

$$k\underline{\mu}'A(VA)^{k-1}\underline{\mu} = k\mu'A\mu$$
and
$$tr(VA)^k = q . \quad \left.\right\} \quad \text{for } k = 1, 2, \ldots . \quad (12)$$

And with $V = LL'$ it is clear that the non-zero characteristic roots of L'AL must also be unity, so making L'AL idempotent. Hence, for $k \geq 2$

$$\underline{\mu}'A(VA)^{k-1}\underline{\mu} = \underline{\mu}'AL(L'AL)L'A\underline{\mu} = \mu'AVAVA\underline{\mu}.$$

Thus (12) can be replaced by the equivalent conditions

$$tr(VA)^k = q \qquad \text{for all k (L'AL idempotent)} \qquad (13)$$

$$\mu'AVA\mu = \mu'A\mu \qquad (14)$$

$$\mu'AVAVA\mu = \mu'A\mu \qquad (15)$$

3. A theorem of Rao's

Rao (1966) has stated the following theorem.

Theorem 2:(Rao) When $\underline{X}$ is $SN(\underline{\mu},V)$ necessary and sufficient conditions that $\underline{X}'A\underline{X}$ has a non-central χ^2 distribution with q degrees of freedom and non-centrality parameter $\frac{1}{2}\underline{\mu}'A\underline{\mu}$ are that

(16) L'AL be idempotent   or   (16)' VAVAV = VAV

(17) $\underline{\mu}'A\underline{\mu} = \underline{\mu}'ALL'A\underline{\mu}$ or   (17)' $\underline{\mu}'A\underline{\mu}' = \underline{\mu}'AVA\underline{\mu}$

(18) L'A$\underline{\mu}$ belong to the space of L'AL   or   (18)'VA$\underline{\mu}$ belong to the space of VAV.

We now show the equivalence of these conditions to those of Theorem 1. Obviously (16) and (17) are identical to (13) and (14). Condition (18) implies that there exists a vector $\underline{c}$ such that L'A$\underline{\mu}$ = L'AL$\underline{c}$. When this is substituted into $\underline{\mu}'AVA\underline{\mu}$, we get (15). Conversely, if (15) is true, $\underline{c}$ = L'A$\underline{\mu}$ satisfies L'A$\underline{\mu}$ = L'AL$\underline{c}$. To see this, write L'A$\underline{\mu}$ = L'ALL'A$\underline{\mu}$ + $\underline{\epsilon}$ and observe that when (15) is true, $\underline{\epsilon}$ = L'A$\underline{\mu}$ - L'ALL'A$\underline{\mu}$ has zero length, i.e., $\underline{\epsilon}$ = $\underline{0}$. Hence (15) and (18) are equivalent, and the two theorems are equivalent.

The conditions of Theorem 2 allow us to write

$$\underline{X}'A\underline{X} = (L\underline{Y} + \underline{\mu})'A(L\underline{Y} + \underline{\mu}) = (\underline{Y} + \underline{c})'L'AL(\underline{Y} + \underline{c})$$

where $\underline{c}$ satisfies L'A$\underline{\mu}$ = L'AL$\underline{c}$. Thus we have

Theorem 3: If $\underline{X}$ is $SN(\underline{\mu},V)$ and $\underline{Y}$ is $N(\underline{0},I)$ a necessary and sufficient condition

that $\underline{X}'A\underline{X}$ has a noncentral χ^2 distribution is that there exist a $\underline{c}$ for which

$$\underline{X}'A\underline{X} = (\underline{Y} + \underline{c})'L'AL(\underline{Y} + \underline{c}) . \tag{19}$$

with L'AL idempotent.

The necessary condition of theorem 3 provides an interesting geometric

characteristic of quadratic forms involving less than full rank normal variables.

If $\underline{X}$ has zero mean, $\underline{c} = \underline{0}$ so that (19) is satisfied. Otherwise, $\underline{X}$ can be regarded

as a translation of the vector $\underline{Z} = \underline{X} - \underline{\mu}$ from the origin to $\underline{\mu}$ in n-space. The

only way that the quadratic form $\underline{X}'A\underline{X} = (\underline{Z} + \underline{\mu})'A(\underline{Z} + \underline{\mu})$, in the translated

variable $\underline{Z} + \underline{\mu}$, can be χ^2 is for there to exist a corresponding translation of

$\underline{Y}$ to $\underline{Y} + \underline{c}$ in r-space such that $\underline{Y} + \underline{c}$ is the projection of $\underline{X}$ in r-space.

4. Three Corollaries

Three useful corollaries can be derived from the preceding conditions that

a quadratic form has a χ^2 distribution. The first relates to the case when V

is non-singular.

Corollary 1. If $\underline{X}$ is $N(\underline{\mu},V)$ with V non-singular, a necessary and sufficient

condition for $\underline{X}'A\underline{X}$ to be non-central chi-square with $q = tr(VA)$ d.f. and non-

centrally parameter $\frac{1}{2}\mu'A\mu$ is that VA be idempotent.

Proof: The condition is sufficient because $(VA)^2 = VA$ implies $tr(VA)^k = q$

for all k; and because V is non-singular it also implies $AVA = A$. Thus (11) is

satisfied. Necessity follows because, as shown after (12), L'AL being idempotent

is a necessary condition and therefore so is

$$L'ALL'AL = L'AL ;$$

i.e. $$LL'ALL'ALL' = LL'ALL' .$$

Thus

$$VAVAV = VAV$$

and again using the non-singularity of V this implies idempotency of VA. Thus this corollary, which is equivalent to Graybill's Theorem 4.9 (1961), is proven.

We note here a distinct difference between the non-singular and singular normal distributions. In the non-singular case (Graybill, 4.9) the necessary and sufficient condition is that VA must be idempotent, whereas in the singular case (Theorems 1 and 2) the necessary and sufficient condition is (11), or equivalently (12) or (13)-(15) or (16)-(18).

In applications V is most likely to be singular because $\underline{X}$ is of the form $\underline{X} = K\underline{X}_1$. This happens when $\underline{X}$ is a vector of n variables built up from r linearly independent variables $\underline{X}_1$ by adjoining to $\underline{X}_1$ n-r linear functions of X_1. In such cases, X_1 is $N(\underline{\mu}_1, V_{11})$ with V_{11} non-singular and

$$\underline{X} = K\underline{X}_1, \quad \underline{\mu} = K\underline{\mu}_1 \quad \text{and} \quad V = K'V_{11}K .$$

Furthermore, since $V = LL' = K'V_{11}K$ there must exist a non-singular matrix T such that $L = KT$. This property is used in proving the following corollary.

Corollary 2: When $\underline{X} = K\underline{X}_1$ as above, a necessary and sufficient condition that $\underline{X}'A\underline{X}$ has a non-central χ^2 distribution with degrees of freedom $q = tr(VA)$ and non-centrality parameter $\frac{1}{2}\underline{\mu}'A\underline{\mu}$ is that $K'AVAK = K'AK$; or equivalently that $VAVAV = VAV$.

Proof: The condition $K'AVAK = K'AK$ implies idempotency of $L'AL$ because $L = KT$, thus (16) is satisfied. In addition, with $\underline{\mu} = K\underline{\mu}_1$ (17) is satisfied; and because $\underline{\mu} = L(T^{-1}\underline{\mu}_1)$ so is (18). Hence by Theorem (2) $K'AVAK = K'AK$ is a necessary and sufficient condition. The equivalency of this condition to $VAVAV = VAV$ is seen

by noting that from the latter

$$L(L'AVAL)L' = L(L'AL)L' .$$

Therefore $\qquad L'L(L'AVAL)L'L = L'L(L'AL)L'L .$

But $(L'L)^{-1}$ exists, by the definition of $V = LL'$, and hence $L'AVAL = L'AL$. Pre- and post multiplying by $T^{-1'}$ and T^{-1} respectively and using $K = LT^{-1}$ gives $K'AVAK = K'AK$. The converse follows similarly and thus the corollary is proved.

So far we have considered conditions that are both necessary and sufficient. We now consider a simpler condition which is sufficient only. This has appeal because in application one is more likely to be interested in conditions which lead to $X'AX$ having the non-central χ^2 distribution rather than in consequences of it being so distributed. The condition is given in the form of a corollary.

Corollary 3: If X is distributed $SN(\mu,V)$ then a sufficient condition for $X'AX$ to have a non-central χ^2 distribution with non-centrality parameter $\frac{1}{2}\mu'A\mu$ and degrees of freedom $q = $ rank of VA is that $AVA = A$.

Proof: Clearly, if $AVA = A$, VA is idempotent with its rank and trace equal, and (11) is then satisfied.

If, as is done in Searle (1966), any matrix V for which $AVA = A$ is defined as a generalized inverse of A, then by this corollary we see that V being a generalized inverse of A is a sufficient condition for $X'AX$ to have a non-central χ^2-distribution.

5. A lemma by Rao

Corollary 3 brings to mind a paper by Rao (1961) in which (as Lemma 6) it is implied (by means of his Lemma 1) that if X is $SN(0,V)$ then $X'AX$ is distributed as chi-square if and only if $AVA = A$. The sufficient condition here is identical

to that developed in Corollary 3. But as a necessary condition AVA = A does not concour with (11) when μ = 0. For then the necessary condition is $\text{tr}(VA)^k = \text{tr}(VA)$ for all integer values of k, and this does not imply AVA = A.

Example: If $x' = (x_1 \ x_2 \ x_3)$ is N(0,V)

where
$$V = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 4 \end{bmatrix}, \text{ and if } A = \frac{1}{16} \begin{bmatrix} 16 & 6 & 5 \\ 6 & 4 & 3 \\ 5 & 3 & 2 \end{bmatrix}$$

then
$$x'Ax = \frac{1}{8} (8x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 5x_1x_3 + 3x_2x_3) .$$

As a consequence of V, $x_3 = -(x_1 + x_2)$ which reduces x'Ax to $x'Ax = \frac{1}{2}x_1^2$. Since

$E(x_1) = 0$ and $\text{var}(x_1) = 2$, it is clear that x'Ax is χ_1^2 . The necessary condition

$\text{tr}(VA)^k = \text{tr}(VA)$ must therefore hold. By direct multiplication we find that

$$VA = \frac{1}{16} \begin{bmatrix} 22 & 6 & 6 \\ 2 & 2 & 2 \\ -24 & -8 & -8 \end{bmatrix} \quad \text{and } (VA)^2 = \frac{1}{256} \begin{bmatrix} 352 & 96 & 96 \\ 0 & 0 & 0 \\ -352 & -96 & -96 \end{bmatrix}$$

Hence $\text{tr}(VA)^2 = \frac{1}{256}(256) = 1 = \frac{1}{16}(16) = \text{tr}(VA)$,

but

$$AVA = \frac{1}{256} \begin{bmatrix} 244 & 68 & 68 \\ 68 & 20 & 20 \\ 68 & 20 & 20 \end{bmatrix} \neq A.$$

Thus with X'AX distributed as χ_1^2 the necessary condition of (11) is upheld, but that suggested by Rao (1961), namely AVA = A, is not. We might also notice at

this point that, as mentioned earlier, in contrast to Graybill's Theorem 4.9 for non-singular V, the matrix VA is not idempotent.

6. Independence of two quadratic forms.

Consider two quadratic forms $Q_1 = \underline{X}'A\underline{X}$ and $Q_2 = \underline{X}'B\underline{X}$ where $\underline{X}$ is $SN(\underline{\mu}, V)$. There is no loss of generality in assuming A and B to be symmetric. Using (8) with tA replaced by $t_1A + t_2B$ (t_1 and t_2 being sufficiently small that the norm of $t_1A + t_2B$ is less than unity for the analogue of expansion (6) to hold) we see that the joint c.g.f. of Q_1 and Q_2 is

$$\underline{\mu}'(t_1A + t_2B)\underline{\mu} + tr(t_1VA + t_2VB)$$

$$+ \sum_{j=2}^{\infty} 2^{j-1}\left\{\underline{\mu}'(t_1A + t_2B)L[L'(t_1A + t_2B)L]^{j-2}L'(t_1A + t_2B)\underline{\mu} + tr[L'(t_1A + t_2B)L^j/j]\right\}. \quad (20)$$

On the other hand the sum of the c.g.f. for Q_1 and that for Q_2 is

$$t_1\underline{\mu}'A\underline{\mu} + t_2\underline{\mu}'B\underline{\mu} + t_1tr(VA) + t_2tr(VB)$$

$$+ \sum_{j=2}^{\infty} 2^{j-1}\left\{\underline{\mu}'[t_1AL(t_1L'AL)^{j-2}L't_1A + t_2BL(t_2L'BL)^{j-2}L't_2B]\underline{\mu} + t_1tr(L'AL)^j/j\right.$$

$$\left. + t_2tr(L'BL)^j/j\right\} \quad (21)$$

From expanding the term under the summation sign in (20) it becomes clear that if AVB = 0 (20) is identical to (21). Hence AVB = 0 is a sufficient condition for Q_1 and Q_2 to be independent (for then their joint c.g.f. reduces to the sum of the marginal c.g.f.'s). Conversely, if Q_1 and Q_2 are independent then (20) must equal (21). For j = 2 in both expressions such an equality implies

$$t_1 t_2 [\underline{\mu}'(AVB + BVA)\underline{\mu} + \tfrac{1}{2}tr(L'AVBL + L'BVAL)] = 0.$$

With A and B being non-negative definite this implies

$$\underline{\mu}'AVB\underline{\mu} = 0 \qquad \text{and} \qquad tr(L'AVBL) = 0.$$

Since A and B can be written as $A = A_1 A_1$ and $B = B_1 B_1'$ this means

$$tr(L'A_1 A_1'L)(L'B_1 B_1'L) = 0$$

i.e. $\quad tr(A_1'LL'B_1)(B_1'LL'A_1) = 0$

and hence $A_1'LL'B_1 = 0$ and so $AVB = 0$. Thus $AVB = 0$ is also a necessary condition for the independence of Q_1 and Q_2. Hence we have

Theorem 4: If $\underline{X}$ is $SN(\underline{\mu}, V)$ the non-negative quadratic forms $Q_1 = \underline{X}'A\underline{X}$ and $Q_2 = \underline{X}'B\underline{X}$ are independent if and only if $AVB = 0$.

7. Independence of linear and quadratic forms.

The method developed in the preceding section is now used to find the necessary and sufficient conditions for the independence of linear and quadratic forms. Let a series of linear forms be represented by $C\underline{X}$. Then, utilizing (1) with $\underline{\ell}' = 2t_1\underline{\mu}'AL + \underline{t}'CL$ and $\Sigma = I - 2t_1 L'AL$, where t_1 is the dummy variable corresponding to $\underline{X}'A\underline{X}$ and $\underline{t}'$ is a vector of dummy variables appropriate to $C\underline{X}$, it can be shown that the joint moment generating function of $C\underline{X}$ and $\underline{X}'A\underline{X}$ is

$$|I - 2t_1 L'AL|^{-\frac{1}{2}} \exp\{\underline{t}'C\underline{\mu} + t_1\underline{\mu}'A\underline{\mu} + \tfrac{1}{2}(2t_1\underline{\mu}'AL + \underline{t}'CL)(I-2t_1 L'AL)^{-1}(2t_1\underline{\mu}'AL+\underline{t}'CL)'\} .$$

On the other hand, the product of the moment generating functions for $C\underline{X}$ and for $\underline{X}'A\underline{X}$ is

$$|I - 2t_1 L'AL|^{-\frac{1}{2}} \exp\{\underline{t}'C\underline{\mu} + \tfrac{1}{2}\underline{t}'CVC'\underline{t} + t_1\underline{\mu}'A\underline{\mu} + 2t_1^2\underline{\mu}'AL(I - 2t_1L'AL)^{-1}L'A\underline{\mu}\}.$$

For equality of these two expressions it is necessary and sufficient that

$$\tfrac{1}{2}(2t_1\underline{\mu}'AL+\underline{t}'CL)(I-2t_1L'AL)^{-1}(2t_1\underline{\mu}'AL+\underline{t}'CL)' = \tfrac{1}{2}\underline{t}'CVC'\underline{t}+2t_1^2\underline{\mu}'AL(I-2t_1L'AL)^{-1}L'A\underline{\mu}$$

which reduces to

$$\tfrac{1}{2}\underline{t}'CL(I - 2t_1L'AL)^{-1}L'C'\underline{t} - \tfrac{1}{2}\underline{t}'CVC'\underline{t} + 2t_1\underline{\mu}'AL(I - 2t_1L'AL)^{-1}L'C'\underline{t} = 0.$$

Using (6) this becomes

$$\tfrac{1}{2}\underline{t}'CL[I + \sum_{i=1}^{\infty}(2t_1L'AL)^i]L'C'\underline{t} - \tfrac{1}{2}\underline{t}'CVC'\underline{t} + 2t_1\underline{\mu}'AL[I + \sum_{i=1}^{\infty}(2t_1L'AL)^i]L'C\underline{t} = 0 \quad (22)$$

in which every term involves the factor

$$CL(L'AL)L'C \equiv CLL'A_1A_1'LL'C = CVA_1(CVA_1)'\,.$$

Thus it is necessary that $CVA_1 = 0$ for (22) to be true. Hence $CVA = 0$ is a necessary condition for the independence of $C\underline{X}$ and $\underline{X}'A\underline{X}$; and obviously it is a sufficient condition. Thus is obtained

Theorem 5: If $\underline{X}$ is $SN(\mu,V)$ the linear forms $C\underline{X}$ and the quadratic form $\underline{X}'A\underline{X}$ are independent if and only if $CVA = 0$. This is akin to Graybill's (1961) theorem 4.17, only more general.

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