ON THE CONSTRUCTION OF FRACTIONAL REPLICATES WITH SPECIAL REFERENCE TO SATURATED DESIGNS

BU-224-M U. B. Paik and W. T. Federer July, 1966
Cornell University

ABSTRACT

A method of constructing fractional replicates from a complete factorial is developed and illustrated in the present paper. Special reference is made to the construction of saturated fractional replicates for a specified set of parameters. The method of construction involves a special ordering of the treatment observations and of the single degree of freedom parameter contrasts. Prior to presentation of the method, a generalized inverse method is used to obtain some previous results of Banerjee and Federer on the estimates of parameters and the corresponding variances. Also, a Kronecker product representation is given for the design matrix of any n-factor factorial composed of linear contrasts. Various saturated main effect plans for a 2^4 and a 3^3 factorial are presented. The efficiency of saturated main effect plans is investigated with actual plans being presented for n = 3, 4, and 5 in the 2^n factorial.

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SUMMARY

A method of constructing fractional replicates from a complete factorial is developed and illustrated in the present paper. Special reference is made to the construction of saturated fractional replicates for a specified set of parameters. The method of construction involves a special ordering of the treatment observations and of the single degree of freedom parameter contrasts. Prior to presentation of the method, a generalized inverse method is used to obtain some previous results of Banerjee and Federer on the estimates of parameters and the corresponding variances. Also, a Kronecker product representation is given for the design matrix of any n-factor factorial composed of linear contrasts. Various saturated main effect plans for a 2^4 and a 3^3 factorial are presented. The efficiency of saturated main effect plans is investigated with actual plans being presented for n = 3, 4, and 5 in the 2^n factorial.

1. INTRODUCTION

Raktoe [1966] has shown how to obtain unsaturated and saturated non-orthogonal main effect and resolution V plans---using a single replicate of a lattice design for 2^n treatments in incomplete blocks of size two. A special ordering of the 2^{n-1} incomplete blocks was used. Then, from this ordering he obtained the above

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listed fractional replicates. It is the purpose of this paper to present a method of construction of saturated and unsaturated fractional replicates for any specified set of parameters.

First we shall need to develop and define a notation. Then, some of the results of Banerjee and Federer [1963, 1964, 1966] on the estimates of parameters and their variances will be obtained using a generalized inverse procedure. This alternative development may be useful in other connections. In the next section the single degree of freedom contrast design matrix will be presented as a Kronecker product of the linear contrasts of the levels of each of the n factors. Special orderings of the observations and of the parameter contrasts are used in this Kronecker representation. With this notation developed, the method of construction of fractional replicates is then developed and illustrated with several examples. Various saturated non-orthogonal main effect plans for a 2^4 and a 3^3 factorial are given. Lastly, the efficiency of saturated fractional replicates is investigated; the most efficient (in the sense discussed by Banerjee and Federer [1963, 1966]) non-orthogonal saturated main effect plans are given for n = 3, 4, and 5 in the 2^n factorial system.

2. NOTATION

Let Y represent a column vector of N random variables $y_1, y_2, \cdots, y_N$, let B represent a column vector of N unknown parameters $b_1, b_2, \cdots, b_N$, and let the known linear orthogonal comparison matrix X (treatment design matrix) in a factorial be composed of N rows and N columns. Then, the observational equation may be represented as:

$$Y = XB + e, \tag{2.1}$$

where e is an N × 1 column vector of random error components, $e_1, e_2, \cdots, e_N$, where $E(Y) = XB$, and where $E(ee') = I\sigma^2$.

Let

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where X_1 is a $p \times N$ matrix and X_2 is an $(N-p) \times N$ matrix.

Consider the following fraction

$$Y_p = X_1 B + e_p \tag{2.2}$$

and let $X_1 = [X_{11}\ X_{12}]$. $B = \begin{bmatrix} B_p \\ B_{N-p} \end{bmatrix}$, and $X_2 = [X_{21}\ X_{22}]$, where Y_p and B_p are $p \times 1$ column vectors, X_{11} is a nonsingular $p \times p$ matrix, X_{12} and X_{21}' are $p \times (N-p)$, B_{N-p} is an $(N-p) \times 1$ column vector, and X_{22} is an $(N-p) \times (N-p)$ matrix, then

$$Y_p = [X_{11}\ X_{12}] \begin{bmatrix} B_p \\ B_{N-p} \end{bmatrix} + e_p \tag{2.3}$$

3. USE OF GENERALIZED INVERSE

Using the least squares method, the matrix expression of the normal equations for the fractional replicate given by equation (2.3) is:

$$[X_{11}\ X_{12}]'[X_{11}\ X_{12}] \begin{bmatrix} \hat{B}_p \\ \hat{B}_{N-p} \end{bmatrix} = [X_{11}\ X_{12}]'Y_p$$

$$\begin{bmatrix} X_{11}'X_{11} & X_{11}'X_{12} \\ X_{12}'X_{11} & X_{12}'X_{12} \end{bmatrix} \begin{bmatrix} \hat{B}_p \\ \hat{B}_{N-p} \end{bmatrix} = \begin{bmatrix} X_{11}' \\ X_{12}' \end{bmatrix} Y_p \tag{3.1}$$

One of the generalized inverses G of $\begin{bmatrix} X_{11}'X_{11} & X_{11}'X_{12} \\ X_{12}'X_{11} & X_{12}'X_{12} \end{bmatrix}$ is

$$G = \begin{bmatrix} (X_{11}'X_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.2}$$

for which $H = GX_1'X_1 = \begin{bmatrix} I & (X_{11}'X_{11})^{-1}X_{11}'X_{12} \\ 0 & 0 \end{bmatrix}$ (3.3)

The proof of (3.2) follows easily, i.e.,

$$\begin{bmatrix} X_{11}'X_{11} & X_{11}'X_{12} \\ X_{12}'X_{11} & X_{12}'X_{12} \end{bmatrix} G \begin{bmatrix} X_{11}'X_{11} & X_{11}'X_{12} \\ X_{12}'X_{11} & X_{12}'X_{12} \end{bmatrix} = \begin{bmatrix} X_{11}'X_{11} & X_{11}'X_{12} \\ X_{12}'X_{11}(X_{11}'X_{11})^{-1}X_{11}'X_{11} & X_{12}'X_{11}(X_{11}'X_{11})^{-1}X_{11}'X_{12} \end{bmatrix}$$

$X_{12}'X_{11}(X_{11}'X_{11})^{-1}X_{11}'X_{11} = X_{12}'X_{11}$, and since X_{11} is nonsingular, then

$$X_{11}(X_{11}'X_{11})^{-1} = X_{11}'^{-1}X_{11}'X_{11}(X_{11}'X_{11})^{-1} = X_{11}'^{-1}$$

then

$$X_{12}'X_{11}(X_{11}'X_{11})^{-1}X_{12}X_{12} = X_{12}'X_{12} .$$

Hence G is a generalized inverse of $\begin{bmatrix} X_{11}'X_{11} & X_{11}'X_{12} \\ X_{12}'X_{11} & X_{12}'X_{12} \end{bmatrix}$.

Therefore

$$\begin{bmatrix} \hat{B}_p \\ \hat{B}_{N-p} \end{bmatrix} = G \begin{bmatrix} X_{11}' \\ X_{12}' \end{bmatrix} Y_p + (H - I_{N\times N})\, Z \tag{3.4}$$

$$= \begin{bmatrix} (X_{11}'X_{11})^{-1}X_{11}' \\ 0 \end{bmatrix} Y_p + \begin{bmatrix} (X_{11}'X_{11})^{-1}X_{11}'X_{12} \\ -I_{(N-p)(N-p)} \end{bmatrix} Z \tag{3.5}$$

where Z is an (N-p) $\times$ 1 column vector of arbitrary components $z_1, z_2, \cdots, z_{N-p}$. Since Z is an arbitrary vector, we can then comsider

$$Z = -\hat{B}_{N-p} ;$$

then

$$\hat{B}_p + (X_{11}'X_{11})^{-1}X_{11}'X_{12}\hat{B}_{N-p} = (X_{11}'X_{11})^{-1}X_{11}'Y_p \tag{3.6}$$

or

$$\hat{B}_p + X_{11}^{-1}X_{12}\hat{B}_{N-p} = X_{11}^{-1}Y_p . \tag{3.7}$$

Since $X'X$ is a diagonal matrix, if X_{22}^{-1} exists, then X_{11}^{-1} exists and we can write X as follows:

$$X = \begin{bmatrix} X_{11} & X_{12} \\ \lambda' X_{11} & X_{22} \end{bmatrix}, \quad \text{where } \lambda = -X_{12}X_{22}^{-1} .$$

Let $$U_1 = \begin{bmatrix} X_{11} \\ \lambda' X_{11} \end{bmatrix}$$

then $$(U_1'U_1)^{-1}U_1'U_1 = I_{N\times N} ,$$

then $$(U_1'U_1)^{-1}(X_{11}' \;\; X_{11}'\lambda) \begin{bmatrix} X_{11} \\ \lambda' X_{11} \end{bmatrix} = I \tag{3.8}$$

$$(U_1'U_1)^{-1}(X_{11}'X_{11} + X_{11}'\lambda\,\lambda' X_{11}) = I$$

$$(U_1'U_1)^{-1}X_{11}'(I + \lambda\lambda')X_{11} = I ,$$

then $$(U_1'U_1)^{-1}X_{11}'(I + \lambda\lambda') = X_{11}^{-1} . \tag{3.9}$$

Hence, we can rewrite (3.7) as follows:

$$\begin{aligned}\hat{B}_p + (U_1'U_1)^{-1}X_{11}'(I + \lambda\lambda')X_{12}\hat{B}_{N-p} \\ = (U_1'U_1)^{-1}X_{11}'(I + \lambda\lambda')Y_p\end{aligned} \tag{3.10}$$

From S. R. Searle [1966], we note

$$\begin{aligned}\operatorname{var}\begin{bmatrix}\hat{B}_p \\ \hat{B}_{N-p}\end{bmatrix} &= G\sigma^2 \\ &= \begin{bmatrix}(X_{11}'X_{11})^{-1} & 0 \\ 0 & 0\end{bmatrix}\sigma^2 ;\end{aligned} \tag{3.11}$$

then $$\operatorname{var}(\hat{B}_p) = (X_{11}'X_{11})^{-1}\sigma^2 . \tag{3.12}$$

These results are equivalent to results of Banerjee and Federer [1963, 1964].

4. KRONECKER PRODUCT, CONSTRUCTION OF THE DESIGN MATRIX X

The ordering of the treatments in the vector Y is as follows: Set the first n-1 factors at the first level and run through all levels of the n^{th} factor consecutively; then set all levels of the first n-2 factors at the first level and set the level of the $n-1^{st}$ factor at the second level and run through all levels of the n^{th} factor in consecutive order; continue this process until all levels of the $n-1^{st}$ factor have been exhausted in consecutive order; then run through levels of the $n-2^{nd}$ factor in the manner described for the $n-1^{st}$ factor; continue this process for the $n-3^{rd}$ up to and including the first factor which exhausts all the combinations in the n-factor factorial. Consequently, if we suppose the i^{th} factor has n_i levels, then we can express the k^{th} treatment as follows:

$$(\alpha_1, \alpha_2, \cdots, \alpha_h, \cdots, \alpha_q, \cdots, \alpha_n) \tag{4.1}$$

where $$\alpha_j = \left[\frac{k_{j-1}}{\prod_{i=2}^{n} n_i}\right] \quad \text{for } j = 1, \cdots, n-1 ;$$

$$\alpha_n = k_{n-1} ,$$

where $$k_0 = k , \quad k_{j-2} \equiv k_{j-1} \ (\text{mod} \prod_{i=j}^{n} n_i) ,$$

and $[x]$ denotes the largest integer less than x or equal to x.

The treatment ordering in the examples given below follow this ordering.

The second step is to set up the linear contrast matrix for the q levels of each of the n factors; we shall denote this contrast matrix as L_{qh} where q refers to the number of levels associated with the h^{th} factor. The third step in the representation of the design matrix is to take the Kronecker product of the linear contrast matrices, i.e. $\prod_{h=1}^{n} \otimes L_{qh}$. The parameter order is such that the mean and the n^{th} factor contrasts appear first, then the first contrast of the $n\text{-}1^{st}$ factor and interactions with the n^{th} factor contrasts appear next, etc. Here, we know that the symbols of the factors are arbitrary. If we change the two factor symbols, each other in the linear contrast matrix, for example the h^{th} factor symbol and the q^{th} factor symbol, but keep the factor symbols and their order in the treatment, then, in the vector Y the k^{th} treatment (4.1) will become as follows:

$$(\alpha_1, \alpha_2, \cdots, \alpha_q, \cdots, \alpha_h, \cdots, \alpha_n) \tag{4.2}$$

The examples below illustrate the procedure.

Example 4.1, 3 X 2 factorial

Consider a 3 X 2 factorial arrangement of treatments, and suppose factor A is represented at the three levels 0, 1, and 2 and factor B at the two levels 0 and 1; then, we obtain the following coefficients for the 6 orthogonal contrasts among 6 treatments by using the Kronecker product of the two matrices L_{3A} and L_{2B} (e.g., see Robson [1959]) where

$$L_{3A} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad L_{2B} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

	M	B	A_L	A_LB	A_Q	A_QB
a_0b_0	1	-1	-1	1	1	-1
a_0b_1	1	1	-1	-1	1	1
a_1b_0	1	-1	0	0	-2	2
a_1b_1	1	1	0	0	-2	-2
a_2b_0	1	-1	1	-1	1	-1
a_2b_1	1	1	1	1	1	1

If we represent the matrix of coefficients given above by L_6, then

$$L_6 = L_{3A} \otimes L_{2B}$$

where $\otimes$ refers to the Kronecker product. L_6 is equivalent to the design matrix X of a complete 3 X 2 factorial.

Example 4.2, 2^2 factorial

$$L_4 = L_{2A} \otimes L_{2B}$$

$$\begin{array}{c} \begin{array}{cccc} \mathrm{M} & \mathrm{B} & \mathrm{A} & \mathrm{AB} \end{array} \\ = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

where the treatment order is 00, 01, 10, and 11, the factors are A and B, and the parameter order is M, B, A, and AB.

Example 4.3, 2^3 factorial

$$L_8 = L_{2A} \otimes L_4 = L_{2A} \otimes L_{2B} \otimes L_{2C}$$

$$= \begin{bmatrix} L_4 & -L_4 \\ L_4 & L_4 \end{bmatrix},$$

where the treatment order is 000, 001, 010, 011, 100, 101, 110, and 111, the factors are A, B, and C, and the parameter order is M, C, B, BC, A, AC, AB, and ABC.

Example 4.4, 3^2 factorial

$$L_9 = L_{3A} \otimes L_{3B}$$

$$= \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -2 & -1 & 0 & 2 & 1 & 0 & -2 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & -2 & 2 & -2 \\ 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 \\ 1 & 1 & 1 & 0 & 0 & 0 & -2 & -2 & -2 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 & 0 & -2 & 1 & 0 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

where the treatment order is 00, 01, 02, 10, 11, 12, 20, 21, and 22, the factors are A and B, and the parameter order is M, B_L, B_Q, A_L, A_LB_L, A_LB_Q, A_Q, A_QB_L, and A_QB_Q.

Example 4.5, 3^3 factorial

$$L_{27} = L_3 \otimes L_9 = L_{3A} \otimes L_{3B} \otimes L_{3C}$$

$$= \begin{bmatrix} L_9 & -L_9 & L_9 \\ L_9 & 0 & -2L_9 \\ L_9 & L_9 & L_9 \end{bmatrix}$$

where the treatment order is 000, 001, 002, 010, 011, 012, 020, 021, 022; 100, 101, 102, 110, 111, 112, 120, 121, 122; 200, 201, 202, 210, 211, 212, 220, 221, and 222, the factors are A, B, and C, and the parameter order is M, C_L, C_Q, B_L, B_LC_L, B_LC_Q, B_Q, B_QC_L, B_QC_Q; A_L, A_LC_L, A_LC_Q, A_LB_L, $A_LB_LC_L$, $A_LB_LC_Q$, A_LB_Q, $A_LB_QC_L$, $A_LB_QC_Q$; A_Q, A_QC_L, A_QC_Q, A_QB_L, $A_QB_LC_L$, $A_QB_LC_Q$, A_QB_Q, $A_QB_QC_L$, and $A_QB_QC_Q$.

Here, if we change the two factor symbols A and C with each other in the L_{27}, then the parameter order is M, A_L, A_Q, B_L, A_LC_L, A_QB_L, B_Q, A_LB_Q, A_QB_Q; C_L, A_LC_L, A_QC_L, B_LC_L, $A_LB_LC_L$, $A_QB_LC_L$, B_QC_L, $A_LB_QC_L$, $A_QB_QC_L$; C_Q, A_LC_Q, A_QC_Q, B_LC_Q, $A_LB_LC_Q$, $A_QB_LC_Q$, B_QC_Q, $A_LB_QC_Q$, and $A_QB_QC_Q$, and treatment order is 000, 100, 200, 010, 110, 210, 020, 120, 220; 001, 101, 201, 011, 111, 211, 021, 121, 221; 002, 102, 202, 012, 112, 212, 022, 122, and 222.

5. REARRANGING THE TREATMENT ORDER

If we recall the solution (3.7) or (3.10), we will see that we need the inverse of X_{11} or of X_{22} to obtain the solution. Also we shall see in section 6 that if the size of the fraction is less than s^{n-1} in the s^n factorial, then we can use the $s^{n-1} \times s^{n-1}$ orthogonal matrix X_{11}^* (in the sense of diagonal) instead of $s^n \times s^n$ matrix X to get a solution such as (3.7) or (3.10). Also we shall see in this case that the method of construction of a saturated fractional replicate is the problem which selects the smallest number of treatments from the treatments corresponding to the orthogonal matrix X^*_{11}. Here we also recall that, in section 4, $L_{s^{n-1}}$ is already an orthogonal matrix, then we can construct a saturated fractional replicate from the first s^{n-1} treatments in the vector Y. But, in this case, the mean effect will be confounded with the main effect A. This is the reason that we must rearrange the treatment order in the vector Y with some higher order defining contrast before constructing a fractional plan; i.e., the mean effect is required to be unconfounded with the main effects.

Now consider rearranging of the treatment order in the vector Y with some defining contrast in s^n factorial. If we use the expression (4.1) for the treatments, then the numbers α_i take on values from 0 to (s-1). The s^n-1 degrees of freedom among the s^n treatment combinations may be partitioned into $(s^n-1)/(s-1)$ sets of s-1 degrees of freedom. Each set of (s-1) degrees of freedom is given by the contrast among the s sets of s^{n-1} treatment combinations specified by the following s equations:

$$\begin{aligned} \alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n &= 0 \\ \alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n &= 1 \\ &\vdots \\ \alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n &= s-1 \end{aligned} \tag{5.1}$$

where the right-hand sides of these equations are elements of the Galois Field GF(s). The c_i's are positive integers between 0 and (s-1), not all equal to zero, and all addition and multiplication is done within the Galois Field GF(s), then the interaction $AB^{c_2} \cdots K^{c_n}$ corresponds to the equation whose left-hand side is $\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$.

Consider a defining contrast, i.e.,

$$I = AB^{c_2} \cdots K^{c_n}\,, \tag{5.2}$$

(the factor A is always included for convenience) then the identity relationships will be written as

$$\begin{aligned} I &= (AB^{c_2} \cdots K^{c_n})_0 \\ I &= (AB^{c_2} \cdots K^{c_n})_1 \\ &\ \vdots \\ I &= (AB^{c_2} \cdots K^{c_n})_{s-1} \end{aligned} \tag{5.3}$$

Let the set of treatments for fixed α_1 be $\{\alpha_1 : (\alpha_1, \alpha_2, \cdots, \alpha_n)\}$, then, from (4.3) and (4.5), we can find easily the following relationships:

$$\text{If } \boxed{\begin{array}{c} \text{in the set of} \\ \{0\colon(0,\alpha_2,\cdots,\alpha_n)\}, \\ \text{the } k^{th} \text{ treatment} \\ \text{corresponds to} \\ (AB^{c_2} \cdots K^{c_n})_0 \\ (AB^{c_2} \cdots K^{c_n})_1 \\ (AB^{c_2} \cdots K^{c_n})_2 \\ \vdots \\ (AB^{c_2} \cdots K^{c_n})_{s-1} \end{array}}\text{, then } \boxed{\begin{array}{c} \text{in the set of} \\ \{1\colon(1,\alpha_2,\cdots,\alpha_n)\}, \\ \text{the } s{+}k^{th} \text{ treatment} \\ \text{corresponds to} \\ (AB^{c_2} \cdots K^{c_n})_1 \\ (AB^{c_2} \cdots K^{c_n})_2 \\ (AB^{c_2} \cdots K^{c_n})_3 \\ \vdots \\ (AB^{c_2} \cdots K^{c_n})_0 \end{array}}, \cdots, \boxed{\begin{array}{c} \text{in the set of} \\ \{s-1\colon(s-1,\alpha_2,\cdots,\alpha_n)\}, \\ \text{the } (s-1)s{+}k^{th} \text{ treatment} \\ \text{corresponds to} \\ (AB^{c_2} \cdots K^{c_n})_{s-1} \\ (AB^{c_2} \cdots K^{c_n})_0 \\ (AB^{c_2} \cdots K^{c_n})_1 \\ \vdots \\ (AB^{c_2} \cdots K^{c_n})_{s-2} \end{array}} \tag{5.4}$$

If $s = p^m$ (p is a prime number), this pattern (5.4) will be different but will obtain similar relationships among the treatment numbers in the below.

If we denote

$$\{k_i\} \tag{5.5}$$

as the set of the treatment numbers corresponding to $(AB^{c_2} \cdots K^{c_n})_i$ in the set $\{0\colon(0,\alpha_2,\cdots,\alpha_n)\}$, then the whole set of the treatment numbers corresponding to

$(AB^{c_2} \cdots K^{c_n})_0$ in the vector Y will be

$$(\{k_0\}, \{s+k_{s-1}\}, \{2s+k_{s-2}\}, \cdots, \{(s-1)s+k_1\}). \tag{5.6}$$

If we use the notation

$$\{\underline{k}_i\} \tag{5.7}$$

as the corresponding set of the row vectors in the $L_{s^{n-1}}$, then we will easily recognize the following:

$$\begin{aligned} \{\underline{s+k}_{s-1}\} &= \{\underline{k}_{s-1}\} \\ \{\underline{2s+k}_{s-2}\} &= \{\underline{k}_{s-2}\} \\ &\vdots \\ \{\underline{(s-1)s+k}_1\} &= \{\underline{k}_1\}. \end{aligned} \tag{5.8}$$

Hence, if we rearrange the treatment order with defining contrast $I = AB^{c_2} \cdots K^{c_n}$ in the vector Y, then the first matrix $L^*_{s^{n-1}}$ corresponding to the set of $\{(AB^{c_2} \cdots K^{c_n})_0\}$ has the same row vectors as $L_{s^{n-1}}$ except ordering, i.e.,

$$L^*_{s^{n-1}} \sim L_{s^{n-1}},$$

where the notation $\sim$ means that if we rearrange the row vector order properly in the $L^*_{s^{n-1}}$, then $L^*_{s^{n-1}}$ will be the same as $L_{s^{n-1}}$.

Example 5.1, 3^3 factorial

Consider the $\frac{1}{3}$ replicate of the 3^3 factorial with the defining contrast $I = ABC^2$, then the treatment combinations which comprise the fractional replicate satisfy one of the following equations:

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \pmod 3$$

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 1 \pmod 3$$

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 2 \pmod 3$$

then the identity relationships are written as

$$I = (ABC^2)_0$$
$$I = (ABC^2)_1$$
$$I = (ABC^2)_2 \;.$$

From example (4.4) and (5.7)

$$\{\underline{k}_0\} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\{\underline{k}_1\} = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & -2 & 2 & -2 \\ 1 & 0 & -2 & 1 & 0 & -2 & 1 & 0 & -2 \end{bmatrix}$$

$$\{\underline{k}_2\} = \begin{bmatrix} 1 & 0 & -2 & -1 & 0 & 2 & 1 & 0 & -2 \\ 1 & 1 & 1 & 0 & 0 & 0 & -2 & -2 & -2 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix},$$

then, from (5.8)

$$L_9^* = \begin{bmatrix} \{\underline{k}_0\} \\ \{\underline{k}_2\} \\ \{\underline{k}_1\} \end{bmatrix} .$$

The treatment order corresponding matrix L_9^* is, from (5.6) and (5.8), as follows:

000, 011, 022, 101, 112, 120, 202, 210, 221.

6. CONSTRUCTION OF FRACTIONAL REPLICATES

We shall consider mostly the method of construction of the saturated main effect plans in s^n factorial. Although we could always construct various saturated non-orthogonal plans for any given parameter set, the general steps of the construction method may not be too instructive. The following steps, however, will be common for the construction of a fractional replicate for given parameters. The special consideration for each case will be illustrated in the following examples.

Step 1. Given the design matrix and the parameters and observation vectors $XB = Y$ in any fashion and not necessarily that of the previous section, we now rearrange the parameter matrix such that the p parameters, $p < N$, are arranged to have the p parameters of interest first and the N-p parameters not of interest to obtain B' rearranged as $(B_p^{*\prime} \quad B_{N-p}^{*\prime})$. This also rearranges the columns of X such that

$$X^*B^* = Y \tag{6.1}$$

$$\underset{N\times p}{(X_1^*} \quad \underset{N\times(N-p)}{X_2^*)} \begin{bmatrix} B_p^* \\ B_{N-p}^* \end{bmatrix} = Y \tag{6.2}$$

Step 2. Search through rows of X_1^* until there is an X_{11}, $p\times p$, which is non-singular.

Step 3. Corresponding to the rows in X_{11} will be rows in X_1^* and observations in Y. Rearrange the observations in Y into

$$\begin{bmatrix} Y_p^* \\ Y_{N-p}^* \end{bmatrix}$$

to correspond to the rows in X_{11} from X_1^*. The observations in Y_p^* yield a saturated design for the p parameters in B_p^*.

This obtained set is one of the possible sets. All possible sets are found by finding all X_{11} which have an inverse.

Example 6.1, saturated main effect plans in a 2^4 factorial

If we consider a 2^4 factorial design matrix L_{16} with the defining contrast $I = ABCD$, and if we consider only the first eight treatments after rearranging treatment order with defining contrast $I = ABCD$, then the alias scheme is as follows:

$$M \doteq ABCD, \quad A \doteq BCD, \quad B \doteq ACD, \quad C \doteq ABD, \quad D \doteq ABC$$

$$AB \doteq CD, \quad AC \doteq BD, \quad BC \doteq AD.$$

After rearranging the rows and columns under consideration of the above alias scheme and using property (5.8), we obtain the following matrix X:

$$X = \begin{bmatrix} X^*_{11} & X^*_{11} \\ X^*_{11} & -X^*_{11} \end{bmatrix} = \left[\begin{array}{rrrrrrrr rrrrrrrr}
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{array}\right] \tag{6.3}$$

where the treatment order is

$$\begin{array}{l} 0000,\ 0011,\ 0110,\ 0101,\ 1010,\ 1001,\ 1100,\ 1111; \\ 1000,\ 1011,\ 1110,\ 1101,\ 0010,\ 0001,\ 0100,\ \text{and}\ 0111, \end{array} \tag{6.4}$$

and the parameter order is

$$\begin{gathered} M,\ C,\ D,\ BCD,\ B,\ BC,\ BD,\ CD; \\ ABCD,\ ABD,\ ABC,\ A,\ ACD,\ AD,\ AC,\ AB. \end{gathered} \tag{6.5}$$

Consider the following fraction of a 2^4 factorial

$$Y_p = X_1 B + e_p\ ,\ p < 8 \tag{6.6}$$

where Y_p is a $p \times 1$ vector from the vector Y, B is a column vector of $N = 16$ unknown parameters reordered such as (6.5), X_1 is a design matrix for given Y_p and B, and e_p is a $p \times 1$ column vector of random error.

Suppose the following partition matrix of X is possible:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

$$= \left[\begin{array}{cc:c} X_{11} & X_{1211} & X_{1212} \\ X_{2111} & X_{2211} & X_{2212} \\ \hdashline X_{2121} & X_{2221} & X_{2222} \end{array}\right] \tag{6.7}$$

and

$$X^*_{11} = \begin{bmatrix} X_{11} & X_{1211} \\ X_{2111} & X_{2211} \end{bmatrix} \tag{6.8}$$

where X_{11} is a $p \times p$ ($p < 8$) nonsingular matrix, X_{2111} and X'_{1211} are each $p \times (8-p)$ matrices, X_{2111} is an $(8-p) \times (8-p)$ matrix, X_{2121} and X'_{1212} are each $8 \times p$ matrices, and X_{2221} and X'_{2212} are each $8 \times (8-p)$ matrices.

We know from (6.3) that

$$X_{1212} = [X_{11} \ \vdots\ X_{1211}], \tag{6.9}$$

and since $X_{11}^{*\prime}X_{11}^{*}$ is diagonal, if X_{11} is nonsingular, then X_{2211} is also nonsingular, and from (3.9)

$$X_{11}^{-1} = (U_{11}'U_{11})^{-1}X_{11}'(I + \lambda_1\lambda_1') \tag{6.10}$$

$$\text{where} \qquad U_{11} = \begin{bmatrix} X_{11} \\ X_{2111} \end{bmatrix} \tag{6.11}$$

$$\text{and} \qquad \lambda_1 = -X_{1211}X_{2211}^{-1} \tag{6.12}$$

then $X_{11}'(I + \lambda_1\lambda_1')X_{12} = X_{11}'(I + \lambda_1\lambda_1')\,[X_{1211} \mid X_{11} \mid X_{1211}]$

$$= [X_{11}'(I + \lambda_1\lambda_1')X_{1211} \mid 8I_{5\times5} \mid X_{11}'(I + \lambda_1\lambda_1')X_{1211}], \tag{6.13}$$

then, from (3.7) and (3.10) we will obtain the following solution for (6.6):

$$\hat{B}_p + (U_{11}'U_{11})^{-1}\,[X_{11}'(I + \lambda_1\lambda_1')X_{1211} \mid 8I \mid X_{11}'(I + \lambda_1\lambda_1')X_{1211}]\;\hat{B}_{16-p}$$

$$= (U_{11}'U_{11})^{-1}X_{11}'(I + \lambda_1\lambda_1')Y_p \tag{6.14}$$

This solution indicates clearly that the mean and main effects are not confounded with each other and that the solution depends only on λ_1. This further means that the solution depends only on X_{2211}.

Now consider the saturated main effect plans in a 2^4 factorial. Let the treatments be arranged such as (6.4) and the corresponding row vectors in X be numbered 1, 2, ..., 16 respectively, and let

$$U_{12} = \begin{bmatrix} X_{1211} \\ X_{2211} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{6.15}$$

In the matrix U_{12}, we can find easily three independent rows, i.e., the following combinations of rows make nonsingular 3×3 matrices:

$$
\begin{array}{l}
(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 2, 6), (1, 3, 4), \\
(1, 3, 5), (1, 3, 7), (1, 4, 6), (1, 4, 7), (1, 5, 6), \\
(1, 5, 7), (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 3, 8), \\
(2, 4, 8), (2, 5, 8), (2, 6, 8), (3, 4, 7), (3, 4, 8), \\
(3, 5, 7), (3, 5, 8), (3, 7, 8), (4, 6, 7), (4, 6, 8), \\
(4, 7, 8), (5, 6, 7), (5, 6, 8), (5, 7, 8), (6, 7, 8),
\end{array}
$$

where the numbers indicate the row numbers in matrix U_{12}, then the following 30 treatment combinations will be the saturated main effect plans in a 2^4 factorial:

(1) 0101 1010 1001 1100 1111	(2) 0110 1010 1001 1100 1111	(3) 0110 0101 1001 1100 1111	(4) 0110 0101 1010 1100 1111	(5) 0011 1010 1001 1100 1111
(6) 0011 0110 1010 1100 1111	(7) 0011 0110 0101 1001 1110	(8) 0000 1010 1001 1100 1111	(9) 0000 0110 1010 1001 1100	(10) 0000 0110 0101 1010 1100
(11) 0000 0011 0101 1001 1111	(12) 0000 0011 0101 1010 1001	(13) 0000 0011 0110 1010 1111	(14) 0000 0011 0110 1010 1100	(15) 0000 0011 0110 1010 1001
(16) 0000 0011 0110 0101 1111	(17) 0000 0011 0110 0101 1100	(18) 0000 0011 0110 0101 1001	(19) 0000 0011 0110 0101 1010	(20) 0011 0101 1001 1100 1111
(21) 0011 0101 1010 1001 1111	(22) 0011 0110 1010 1001 1111	(23) 0011 0110 0101 1100 1111	(24) 0000 0101 1001 1100 1111	(25) 0000 0101 1010 1100 1111
(26) 0000 0101 1010 1001 1100	(27) 0000 0110 0101 1001 1100	(28) 0000 0011 1010 1001 1111	(29) 0000 0011 1010 1001 1100	(30) 0000 0011 0101 1001 1100

(6.16)

Let $(n_1, n_2, n_3, n_4, n_5)$ be one of the above 30 plans, then by recalling (5.6), (5.7), and (5.8), we will know the following treatment combinations are also saturated main effect plans in a 2^4 factorial, i.e.,

$$(n_1 + 8,\ n_2 + 8,\ n_3 + 8,\ n_4 + 8,\ n_5 + 8) \tag{6.17}$$

Example 6.2, Raktoe's saturated main effect plan

Raktoe [1966] showed the following saturated non-orthogonal main effect plan in a 2^4 factorial:

0000
0111
1011
1101
1110

We cannot construct the above saturated main effect plan by using the method of example 6.1. If we use the general procedure, however, we could find the above Raktoe's saturated main effect plan.

In this case,

$$B^{*\prime} = (B_5^{*\prime} \vdots B_{16-5}^{*\prime}) = (M, A, B, C, D, \vdots\ AB, AC, BC, AD, BD, CD, ABC, ABD, ACD, BCD, ABCD),$$

$$X_{11} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix},$$

and

$$X_{12} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

Using (3.6) or (3.7)

$$\begin{bmatrix} \hat{M} \\ \hat{A} \\ \hat{B} \\ \hat{C} \\ \hat{D} \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -2 & -2 & -2 & -2 & 1 \\ -1 & -1 & 2 & -1 & 2 & 2 & 1 & 1 & 1 & -2 & 1 \\ -1 & 2 & -1 & 2 & -1 & 2 & 1 & 1 & -2 & 1 & 1 \\ 2 & -1 & -1 & 2 & 2 & -1 & 1 & -2 & 1 & 1 & 1 \\ 2 & 2 & 2 & -1 & -1 & -1 & -2 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \widehat{AB} \\ \widehat{AC} \\ \widehat{BC} \\ \widehat{AD} \\ \widehat{BC} \\ \widehat{CD} \\ \widehat{ABC} \\ \widehat{ABD} \\ \widehat{ACD} \\ \widehat{BCD} \\ \widehat{ABCD} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ -1 & -2 & 1 & 1 & 1 \\ -1 & 1 & -2 & 1 & 1 \\ -1 & 1 & 1 & -2 & 1 \\ -1 & 1 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_{0000} \\ y_{0111} \\ y_{1011} \\ y_{1101} \\ y_{1110} \end{bmatrix} \quad (6.18)$$

Example 6.3, saturated main effect plans and resolution IV plans having the minimum number of treatments in a 2^5 factorial

(a) Consider the defining contrasts $I \doteq ACE \doteq ABCD \doteq BDE$ in a 2^5 factorial, then the alias scheme is as follows:

$$\begin{array}{llll} M & \doteq ACE & \doteq ABCD & \doteq BDE \\ E & \doteq AC & \doteq ABCDE & \doteq BD \\ D & \doteq ACDE & \doteq ABC & \doteq BE \\ DE & \doteq ACD & \doteq ABCE & \doteq B \\ C & \doteq AE & \doteq ABD & \doteq BCDE \\ CE & \doteq A & \doteq ABDE & \doteq BCD \\ CD & \doteq ADE & \doteq AB & \doteq BCE \\ CDE & \doteq AD & \doteq ABE & \doteq BC \end{array} \quad (6.19)$$

After the procedure of changing row order such that the first set is $\{(ACE)_0, (ABCD)_0\}$, the second set is $\{(ACE)_0, (ABCD)_1\}$, the third set is $\{(ACE)_1, (ABCD)_0\}$, and the fourth set is $\{(ACE)_1, (ABCD)_1\}$ in the L_{32}, then we obtain the following matrix:

$$X = \begin{bmatrix} X^*_{11} & X^*_{12} & X^*_{13} & X^*_{14} \\ X^*_{11} & X^*_{22} & X^*_{23} & X^*_{24} \\ X^*_{11} & X^*_{32} & X^*_{33} & X^*_{34} \\ X^*_{11} & X^*_{42} & X^*_{43} & X^*_{44} \end{bmatrix} \tag{6.20}$$

where the treatment order is 00000, 00111, 01010, 01101, 10011, 10100, 11001, 11110; 01000, 01111, 00010, 00101, 11011, 11100, 10001, 10110; 11000, 11111, 10010, 10101, 01011, 01100, 00001, 00111; 10000, 10111, 11010, 11101, 00100, 00101, 01001, and 01110, and the parameter order is the same as that of L_{32}.

Using the method of the example 6.1 and considering X^*_{11}, we will obtain various saturated main effect plans in a 2^5 factorial.

(b) Consider the defining contrast I = ABCD in a 2^5 factorial, then the alias scheme is as follows:

M ≐ ABCD, E ≐ ABCDE, D ≐ ABC, DE ≐ ABCE, C ≐ ABD,
CE ≐ ABDE, CD ≐ AB, CDE ≐ ABE, B ≐ ACD, BE ≐ ACDE,
BD ≐ AC, BDE ≐ ACE, BC ≐ AD, BCE ≐ ADE, BCD ≐ A,
BCDE ≐ AE,

where the parameter order of the left side of the equal sign is followed by the parameter order in L_{32}.

In the above alias scheme, the three 3-factor interactions CDE, BDE, and BCE are not confounded with main effects or with 2-factor interactions.

If we rearrange the treatment order with the defining contrast I = ABCD and the parameter order such that the treatment order is 00000, 00001, 00110, 00111, 01010, 01011, 01100, 01101, 10010, 10011, 10100, 10101, 11000, 11001, 11110, 11111; 10000, 10001, 10110, 10111, 11010, 11011, 11100, 11101, 00010, 00011, 00100, 00101, 01000, 01001, 01110, and 01111, and the parameter order is M, E, D, DE, C, CE, CD,

B, BE, BD, BC, BCD, BCDE, CDE, BDE, BCE; ABCD, ABCDE, ABC, ABCE, ABD, ABDE, AB, ACD, ACDE, AC, AD, A, AE, ABE, ACE, and ADE, then the design matrix for a 2^5 factorial will be

$$X = \begin{bmatrix} X_{11}^* & X_{11}^* \\ X_{11}^* & -X_{11}^* \end{bmatrix} \tag{6.21}$$

Using the method of the example 6.1, we can construct a saturated plan for the parameters M, E, D, DE, C, CE, CD, B, BE, BD, BC, BCD, and BCDE $\doteq$ A, and this plan will be a resolution IV plan.

Let U_{12} be the 16 X 3 matrix which consists of the last three columns of the X_{11}^*, i.e.,

$$U_{12}' = \begin{bmatrix} -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}, \tag{6.22}$$

then we can easily find a nonsingular 3 X 3 matrix from the U_{12}', for example, the 13^{th}, 14^{th}, and 16^{th} columns of U_{12}' make a nonsingular 3 X 3 matrix, then the corresponding treatments to the remaining columns of U_{12}' are 00000, 00001, 00110, 00111, 01010, 01011, 01100, 01101, 10010, 10011, 10100, 10101, and 11110. These 13 treatments will give us the minimum number of treatments for a resolution IV plan in a 2^5 factorial. The method of the example 6.1 can be applied to get various plans.

Example 6.4, saturated main effect plans in a 3^3 factorial

In a 3^3 factorial, after the procedure of rearranging row order with defining contrast $I = ABC^2$, we will get the following matrix:

$$X^* = \begin{bmatrix} X^*_{11} & X^*_{21} & X^*_{31} \\ X^*_{11} & X^*_{22} & X^*_{32} \\ X^*_{11} & X^*_{23} & X^*_{33} \end{bmatrix}, \tag{6.23}$$

where each X^*_{ij} is a 9×9 square matrix and the treatment order is 000, 011, 022, 101, 112, 120, 202, 210, 221; 100, 111, 122, 201, 212, 220, 002, 010, 021; 200, 211, 222, 001, 012, 020, 102, 110, and 121, and the parameter order is M, C_L, C_Q, B_L, B_LC_L, B_LC_Q, B_Q, B_QC_L, B_QC_Q; A_L, A_LC_L, A_LC_Q, A_LB_L, $A_LB_LC_L$, $A_LB_LC_Q$, A_LB_Q, $A_LB_QC_L$, $A_LB_QC_Q$; A_Q, A_QC_L, A_QC_Q, A_QB_L, $A_QB_LC_L$, $A_QB_LC_Q$, A_QB_Q, $A_QB_QC_L$, and $A_QB_QC_Q$, but we could not obtain a solution such as (6.14), because the effects B_LC_L, B_LC_Q, B_QC_L, and B_QC_Q are confounded with both main effects A_L and A_Q, respectively, i.e.,

$$\begin{aligned} B_LC_L &\hat{=} -\tfrac{1}{3}A_L \hat{=} \tfrac{1}{3}A_Q \\ B_LC_Q &\hat{=} -\tfrac{1}{3}A_L \hat{=} -A_Q \\ B_QC_L &\hat{=} \tfrac{1}{3}A_L \hat{=} A_Q \\ B_QC_Q &\hat{=} -A_L \hat{=} A_Q \quad . \end{aligned}$$

However, we will find that each X^*_{ij} is a nonsingular matrix and if we rearrange the column order such that M, A_L, A_Q, B_L, B_Q, C_L, C_Q, B_LC_L, B_LC_Q, $\cdots$, and let the first 9×9 matrix of the rearranged matrix be A_{11}, then

$$A_{11} = \begin{array}{c} \begin{array}{ccccccccc} M & A_L & A_Q & B_L & B_Q & C_L & C_Q & B_LC_L & B_LC_Q \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & 0 & 0 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 & 1 & 0 & -2 & 0 & 2 \\ 1 & 0 & -2 & 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & -2 & 0 & -2 \end{bmatrix} \end{array} \tag{6.24}$$

If we use the symbols $\underline{M}$, $\underline{A}_L$, $\underline{A}_Q$, $\underline{B}_L$, $\underline{B}_Q$, $\underline{C}_L$, $\underline{C}_Q$, $\underline{B_L C_L}$, and $\underline{B_L C_Q}$ as the symbol of each corresponding column vectors respectively, then we can easily recognize that $\underline{M}$, $\underline{A}_L$, $\underline{A}_Q$, $\underline{B}_L$, $\underline{B}_Q$, $\underline{C}_L$, and $\underline{C}_Q$ are orthogonal to each other and also $\underline{M}$, $\underline{B}_L$, $\underline{B}_Q$, $\underline{C}_L$, $\underline{C}_Q$, $\underline{B_L C_L}$, and $\underline{B_L C_Q}$ are orthogonal to each other. Hence, we can say that the matrix A_{11} is nonsingular, and then, we can make $\underline{B_L C_L}$ and $\underline{B_L C_Q}$ orthogonal vectors with the first 7 column vectors.

Let such new vectors of $\underline{B_L C_L}$ and $\underline{B_L C_Q}$ be $\underline{Z}_1$ and $\underline{Z}_2$ respectively, then by using the Schmidt method of orthogonalizing the columns we obtain

$$\underline{Z}_1 = \underline{B_L C_L} - \frac{(\underline{B_L C_L} \cdot \underline{A}_L)}{\|\underline{A}_L\|^2}\underline{A}_L - \frac{(\underline{B_L C_L} \cdot \underline{A}_Q)}{\|\underline{A}_Q\|^2}\underline{A}_Q = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 1 \\ -2 \\ -2 \\ 1 \\ 1 \end{bmatrix} \qquad (6.25)$$

and

$$\underline{Z}_2 = \underline{B_L C_Q} - \frac{(\underline{B_L C_Q} \cdot \underline{A}_L)}{\|\underline{A}_L\|^2}\underline{A}_L - \frac{(\underline{B_L C_L} \cdot \underline{A}_Q)}{\|\underline{A}_Q\|^2}\underline{A}_Q - \frac{(\underline{B_L C_Q} \cdot \underline{Z}_1)}{\|\underline{B_L C_L}\|^2}\underline{Z}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}; \qquad (6.26)$$

then

$$[\underline{Z}_1 \quad \underline{Z}_2] = \begin{bmatrix} 1 & -1 \\ -2 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ -2 & 0 \\ -2 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ ignoring the common factor.} \tag{6.27}$$

Now, if we find a nonsingular 2×2 matrix from the 9×2 matrix, then we can construct a corresponding information matrix X_{11} for saturated main effect plans.

Consider partition matrix $X_{27 \times 27}$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

where X_{11} is $p \times p$ $(p < 9)$, X_{12} and X_{21}' are $p \times (27 - p)$ each, X_{22} is $(27 - p) \times (27 - p)$.

Now, consider the following fraction of a 3^3 factorial

$$Y_p = X_1 B + e_p \quad , \quad p < 9 \tag{6.28}$$

where $Y_p' = (000, 011, 022, 101, 112, 120, 202)$, then from (3.7)

$$\hat{B}_p + X_{11}^{-1} X_{12} \hat{B}_{27-p} = X_{11}^{-1} Y_p \, . \tag{6.29}$$

Now, let

$$A_{11}^{*} = \begin{array}{c} \begin{array}{cccccccc} M & A_L & A_Q & B_L & B_Q & C_L & Z_1 & Z_2 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & 0 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 & 1 & 0 & 1 & 1 \\ 1 & 0 & -2 & 0 & -2 & 1 & 1 & -1 \\ 1 & 0 & -2 & 1 & 1 & -1 & -2 & 0 \\ 1 & 1 & 1 & -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & 0 & -2 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & -1 \end{bmatrix} \end{array} = \begin{bmatrix} X_{11} & Z_{12} \\ X_{1121} & Z_{22} \end{bmatrix},$$

then $A^{*\prime}_{11}\, A^{*}_{11}$ is diagonal and $Z_{22} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is nonsingular. Hence,

$$X_{11}^{-1} = (U_{11}'U_{11})^{-1}X_{11}'(I + \mu\mu') \tag{6.30}$$

where

$$U_{11} = \begin{bmatrix} X_{11} \\ X_{1121} \end{bmatrix} , \tag{6.31}$$

$$\mu = -Z_{12}Z_{22}^{-1} \quad , \tag{6.32}$$

then (6.29) becomes as follows:

$$\begin{aligned} \hat{B}_p + (U_{11}'U_{11})^{-1}X_{11}'(I + \mu\mu')X_{12}\hat{B}_{27-p} \\ = (U_{11}'U_{11})^{-1}X_{11}(I + \mu\mu')Y_p \end{aligned} \tag{6.33}$$

The following 26 saturated main effect plans are constructed from the set $\{(ABC^2)_0\}$ in a 3^3 factorial:

(1)	(2)	(3)	(4)	(5)
022	011	011	011	011
101	101	022	022	022
112	112	101	101	101
120	120	112	112	112
202	202	202	120	120
210	210	210	210	202
221	221	221	221	221

(6)	(7)	(8)	(9)	(10)
000	000	000	000	000
101	022	022	022	022
112	112	101	101	101
120	120	120	112	112
202	202	202	120	120
210	210	210	202	202
221	221	221	221	210

(11)	(12)	(13)	(14)	(15)
000	000	000	000	000
011	011	011	011	011
101	101	101	101	022
120	112	112	112	120
202	202	120	120	202
210	210	210	202	210
221	221	221	210	221

(16)	(17)	(18)	(19)	(20)
000	000	000	000	000
011	011	011	011	011
022	022	022	022	022
112	112	112	101	101
202	120	120	202	120
210	210	202	210	210
221	221	210	221	221

(21)	(22)	(23)	(24)	(25)	(26)
000	000	000	000	000	000
011	011	011	011	011	011
022	022	022	022	022	022
101	101	101	101	101	101
120	112	112	112	112	112
202	202	202	120	120	120
221	221	210	221	210	202

7. EFFICIENCY OF THE SATURATED MAIN EFFECT PLANS IN A 2^n FACTORIAL

There are many saturated non-orthogonal main effect plans or saturated resolution IV or V plans. Some of them are easily constructed by the method of section 6. The efficiency of each plan, however, will be different. There is no guarantee which fractions constructed by the method of example 6.1 are the most efficient plans. It will be difficult sometimes to construct the most efficient non-orthogonal main effect plans by the method of example 6.1.

We cannot expect usually that $X_{11}'X_{11}$ is diagonal in saturated main effect plans, so our problem is to find a matrix X_{11} such that (Raghavarao [1959])

$$X_{11}'X_{11} = \begin{bmatrix} r & \mu & \mu & \mu & \cdots & \mu \\ \mu & r & \lambda & \lambda & \cdots & \lambda \\ \mu & \lambda & r & \lambda & \cdots & \lambda \\ \mu & \lambda & \lambda & r & \cdots & \lambda \\ & \vdots & & & & \\ \mu & \lambda & \lambda & \lambda & \cdots & r \end{bmatrix} . \tag{7.1}$$

Now consider the condition needed to obtain a matrix of the form (7.1).

Consider the following treatment design arrays:

$$\begin{array}{c} 0011 \\ 0101 \\ 0110 \\ 1001 \\ 1010 \\ 1100 \end{array} \tag{7.2}$$

If we consider only the 0 level, say 0_1, 0_2, 0_3, 0_4 to differentiate for each factor, the above treatment design arrays have complete combinations of two 0 levels, i.e., $0_1,0_2$; $0_1,0_3$; $0_1,0_4$; $0_2,0_3$; $0_2,0_4$; and $0_3,0_4$, and it will be called a balanced treatment design with respect to two 0 levels.

$$\begin{array}{c} 0001 \\ 0010 \\ 0100 \\ 1000 \end{array} \tag{7.3}$$

or

$$\begin{array}{c} 1110 \\ 1101 \\ 1011 \\ 0111 \end{array} \tag{7.4}$$

are also balanced treatment designs with respect to three 0 levels or one 0 level, respectively.

Let an $r \times p$ matrix A be the balanced treatment design arrays in a 2^n factorial, then each column in A will have the same number of elements of 0 and any $r \times 2$ submatrix in A will have the same number of row vectors (0.1) or (1.0).

Consider the relationship between the treatment design arrays and corresponding design matrix X_{11} of a main effect plan in a 2^n factorial. In an X_{11}, every element of the column that corresponds to the mean is 1 and the remaining elements in the X_{11} are -1 or 1 which depends on the element 0 or 1 in the treatment design arrays.

Let $\underline{x}_0$, $\underline{x}_1$, $\cdots$, $\underline{x}_p$ be the column vectors in an X_{11}, then, in a balanced main effect treatment design

$$\underline{x}_0 \cdot \underline{x}_i = \mu \quad \text{for } i = 1, 2, \cdots, p,$$

$$\underline{x}_i \cdot \underline{x}_j = \lambda \quad \text{for } i \neq j \text{ and } i,j = 1, 2, \cdots, p,$$

$$\underline{x}_i^2 \quad = r \quad \text{for } i = 0, 1, \cdots, p,$$

where

$\mu = r$ - number of elements which equal 0 in a column of A,

$\lambda = \mu$ - number of row vectors (0.1) or (1.0) in an $r \times 2$ submatrix in A.

Hence, we can say that the balanced main effect treatment designs such as (7.2), (7.3), and (7.4) will have the information matrices of the form (7.1).

If we add the treatments 0000 or 1111 (where each factor has all the same levels) to a balanced treatment design, we will also obtain an information matrix of the form (7.1).

The following examples are the most efficient saturated non-orthogonal main effect plans in a 2^n factorial:

(1) n=3

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix} \qquad X'_{11}X_{11} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(2) n=4

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix} \qquad X'_{11}X_{11} = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}$$

or

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad X'_{11}X_{11} = \begin{bmatrix} 5 & -1 & -1 & -1 & -1 \\ -1 & 5 & 1 & 1 & 1 \\ -1 & 1 & 5 & 1 & 1 \\ -1 & 1 & 1 & 5 & 1 \\ -1 & 1 & 1 & 1 & 5 \end{bmatrix}$$

(3) $n=5$

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{matrix} \qquad X'_{11}X_{11} = \begin{bmatrix} 6 & 2 & 2 & 2 & 2 & 2 \\ 2 & 6 & 2 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 & 2 \\ 2 & 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 2 & 6 & 2 \\ 2 & 2 & 2 & 2 & 2 & 6 \end{bmatrix}$$

or

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \qquad X'_{11}X_{11} = \begin{bmatrix} 6 & -2 & -2 & -2 & -2 & -2 \\ -2 & 6 & 2 & 2 & 2 & 2 \\ -2 & 2 & 6 & 2 & 2 & 2 \\ -2 & 2 & 2 & 6 & 2 & 2 \\ -2 & 2 & 2 & 2 & 6 & 2 \\ -2 & 2 & 2 & 2 & 2 & 6 \end{bmatrix}$$

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