A NOTE ON DEGREES OF FREEDOM FOR THE AVERAGE EFFECTIVE ERROR VARIANCE IN TWO-DIMENSIONAL ONE-RESTRICTIONAL LATTICES

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A number of authors [Cox, Eckhardt, and Cochran (1940), Cochran (1943), Cochran and Cox (1950) and others] have suggested that the average effective error variance,

$$E_e' = E_e\left\{1 + \frac{rk\mu}{k+1}\right\} = E_e\left\{1 + \frac{r}{(r-1)(k+1)} - \frac{rE_e}{(r-1)(k+1)E_b}\right\} = \frac{r}{k+1}\left\{\frac{r}{w'+(r-1)w} + \frac{k+1-r}{rw}\right\}$$

(where E_e = intrablock error mean square, E_b = blocks adjusted for treatments mean square, k denotes the number of treatments in each of the rk incomplete blocks, r equals number of replicates = 2,3,...,k+1, and $\mu = \frac{(E_b - E_e)}{k(r-1)E_b}$) be used in making t tests of the significance of the differences between adjusted treatment means from an experiment arranged in a 2-dimensional one-restrictional lattice design with no repetition of the basic plan. If the basic plan is repeated, i.e., 2r replicates are used, slight alterations of the above values in E_e' will be necessary. In some instances (Cochran and Cox, page 285, 1950) it may be advisable to use the specific error variances rather than the average effective error variance. Regardless of which variance is used no one has bothered to determine the number of degrees of freedom associated with these error variances. It has been assumed, by some people at least, that the various error mean squares would have the same number of degrees of freedom, $(r-1)(k^2-1)-r(k-1) = f_e$ as the intrablock error mean square E_e. Rao (1947) has assumed that the average error variance has infinite degrees of freedom and consequently has used the chi-square test.

First of all, E_b and E_e are assumed to be estimates of different population parameters, i.e., σ_e^2 and $\sigma_e^2 + \frac{r-1}{r} k\sigma_b^2$. The correct degrees of freedom for E_e', which is a composite of E_e and E_b, must be some number between $[(r-1)(k^2-1) - r(k-1)] = f_e$ and $f_b = r(k-1)$ which are the degrees of freedom associated with E_e and E_b respectively.

The question of appropriate degrees of freedom may be more academic than practical since r(k-1) is quite often larger than 14 to 16 and beyond this range the 5 percent value of t changes little with an increase in degrees of freedom. There is a relatively larger change for the values of t at the one percent level if the degrees of freedom are less than 20. Beyond this point t does not change appreciably.

For smaller lattices the question of appropriate number of degrees of freedom begins to have more importance but again one might question the advisability of using small lattices without enough replication to make $f_b \geq 14$ to 16. For example suppose that one wished to use a 3x3 balanced lattice design. Two sets with r=8 would yield r(k-1) = 16 degrees of freedom for E_b. Thus the number of degrees of freedom associated with E_e' would be greater than 16 which would be suitable in light of the above considerations.

A simple method of determining the degrees of freedom for E_e' is not immediately apparent. If the mean square were of the form,

$$V = a_1E_1 + a_2E_2 \quad ,$$

where the a_i are some constants and the E_i have f_i degrees of freedom, one could find the approximate number of degrees of freedom by using the formula,

$$\text{degrees of freedom for } V \doteq \frac{(a_1E_1 + a_2E_2)^2}{\frac{(a_1E_1)^2}{f_1} + \frac{(a_2E_2)^2}{f_2}} = f_v \quad ,$$

set forth by Fairfield Smith (J. Coun. Sci. Ind. Res., 1936) and Satterthwaite (Biometrics 1946).

Another approach to determine f_v would be to make use of the formula proposed by Cochran and Cox (Experimental Designs p. 224),

$$t_\alpha = \frac{a_1E_1t_1 + a_2E_2t_2}{a_1E_1 + a_2E_2} \quad ,$$

where t_i is the t value at the α significance level and for f_i degrees of freedom, and determine t_α. One could then turn to a t table and determine the number of degrees of freedom to associate with t_α and consequently E_e'.

Despite the fact that E_e' is not a linear combination of variances it was decided to use a form of Cochran and Cox's formula, thus,

$$t_\alpha = \frac{E_e t_e}{E_e'}\left\{1 + \frac{r}{(r-1)(k+1)} - \frac{rE_e t_e}{(r-1)(k+1)E_b t_b}\right\} \quad ,$$

where t_e = t for f_e degrees of freedom and t_b = t for f_b degrees of freedom at the α significance level.

In order to observe the calculated t values and the corresponding degrees of freedom, the following values were used:

α = .05 and .01, E_e = 1, E_b = 2, 4, 8, 16, k = 3, 4, and 5, and r = 2, 3, 4, 5, and 6.

The results are presented in table 1. The tabulated t values were plotted against degrees of freedom. Using the computed values of t_α, the degrees of freedom f_α were read from a graph.

For E_b constant f_α becomes more divergent numerically from f_e as f_e increases but the effect on the different t's becomes smaller. As E_b increases f_α approaches f_e. From the data presented in table 1 it appears, if one accepts the formula for t_α, that the experimenter may regard f_e as a suitable approximation to f_α in most situations or he may use some such formula as

$$f_e = \frac{(f_e - f_b)}{k+1}$$ to approximate f_e.

Final verification of f_a will, of course, have to be deferred until one determines the distribution function for E_e'.

The above t values were approximated from Snedecor's tables of t values. For more precise work one should use the values tabulated by E. M. Baldwin in volume 33, page 362, of BIOMETRIKA.

TABLE 1. Effect on t_α and n_α for varying f_b, f_e, E_b and α.

						$E_b = 2$			$E_b = 4$			$E_b = 8$			$E_b = 16$		
α	r	f_e	f_b	t_e	t_b	E'_e	t_α	f_α	E'_e	t_α	f_α	E'_e	t_α	f_α	E'_e	t_α	f_α
									k = 3								
.05	2	4	4	2.776	2.776	5/4	2.776	4.0	11/8	2.776	4.0	23/16	2.776	4.0	47/32	2.776	4.0
.01	2	4	4	4.604	4.604	5/4	4.604	4.0	11/8	4.604	4.0	23/16	4.604	4.0	47/32	4.604	4.0
.05	3	10	6	2.228	2.447	19/16	2.260	9.1	41/32	2.243	9.5	85/64	2.235	9.8	173/128	2.232	9.9
.01	3	10	6	3.169	3.707	19/16	3.242	9.1	41/32	3.203	9.5	85/64	3.185	9.8	173/128	3.177	9.9
.05	4	16	8	2.120	2.306	7/6	2.144	14.1	5/4	2.131	15.2	31/24	2.126	15.4	21/16	2.123	15.7
.01	4	16	8	2.921	3.355	7/6	2.975	14.1	5/4	2.946	15.1	31/24	2.933	15.5	21/16	2.927	15.7
									k = 4								
.05	2	9	6	2.262	2.447	6/5	2.291	8.3	13/10	2.275	8.6	27/20	2.268	8.8	11/8	2.265	8.9
.01	2	9	6	3.250	3.707	6/5	3.317	8.3	13/10	3.281	8.6	27/20	3.265	8.8	11/8	3.257	8.9
.05	3	21	9	2.080	2.262	23/20	2.102	18	49/40	2.090	19	101/80	2.085	20	41/32	2.082	20.7
.01	3	21	9	2.831	3.250	23/20	2.879	18	49/40	2.853	19	101/80	2.842	20	41/32	2.836	20.6
.05	4	33	12	2.035	2.179	17/15	2.051	27	6/5	2.042	30	37/30	2.039	31	5/4	2.037	32
.01	4	33	12	2.734	3.055	17/15	2.768	27	6/5	2.750	30	37/30	2.742	32	5/4	2.738	32
.05	5	45	15	2.014	2.131	9/8	2.026	37	19/16	2.020	40	39/32	2.017	43	79/64	2.015	44
.01	5	45	15	2.690	2.947	9/8	2.716	37	19/16	2.702	40	39/32	2.696	43	79/64	2.693	44
									k = 5								
.05	2	16	8	2.120	2.306	7/6	2.144	14	5/4	2.131	15	31/24	2.126	15	21/16	2.123	15.7
.01	2	16	8	2.921	3.355	7/6	2.975	14	5/4	2.946	15	31/24	2.933	15	21/16	2.927	15.7
.05	3	36	12	2.028	2.179	9/8	2.044	29	19/16	2.035	33	39/32	2.032	34	79/64	2.030	35
.01	3	36	12	2.720	3.055	9/8	2.753	29	19/16	2.736	32	39/32	2.728	34	79/64	2.724	35
.05	4	56	16	2.003	2.120	10/9	2.014	45	21/18	2.008	50	43/36	2.006	52	29/24	2.004	55
.01	4	56	16	2.667	2.921	10/9	2.690	44	21/18	2.678	50	43/36	2.672	53	29/24	2.670	54

TABLE 1 (continued)

.05	5	76	20	1.992	2.086	53/48	2.000	60	37/32	1.996	66	227/192	1.994	70	459/384	1.993	72
.01	5	76	20	2.642	2.845	53/48	2.660	60	37/32	2.650	68	227/192	2.646	72	459/384	2.644	74
.05	6	96	24	1.985	2.064	11/10	1.992	74	23/20	1.988	82	47/40	1.987	86	19/16	1.986	90
.01	6	96	24	2.628	2.797	11/10	2.642	76	23/20	2.635	84	47/40	2.631	90	19/16	2.630	92