A NOTE ON ERROR (b) IN THE SPLIT-PLOT DESIGN

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The ordinary split-plot design has the t whole plot treatments arranged in r replicates of a randomized complete blocks design. Each of the whole plots is subdivided in d sub-plots or split-plots. The d split-plot treatments are randomly allotted to the d split-plots within each whole plot. The analysis of variance for the ordinary split-plot design is:

Source of variation	d.f.	Expectation of mean square
Replicates	r-1	-----
Whole plot tr. = T	t-1	$\sigma_\beta^2 + d\sigma_\alpha^2 + F_1(\tau_j)$
Error (a)	(r-1)(t-1)	$\sigma_\beta^2 + d\sigma_\alpha^2$
Split-plot tr. = D	d-1	$\sigma_\beta^2 + F_2(\delta_h)$
TxD	(t-1)(d-1)	$\sigma_\beta^2 + F_3(\tau\delta_{jh})$
Error (b)	t(r-1)(d-1)	σ_β^2
Total	rtd-1	-----

The expectations of the mean squares follow from these assumptions and restrictions:

(i) the linear model for a single yield is expressible as

$$Y_{ijh} = \mu + \rho_i + \tau_j + \alpha_{ij} + \delta_h + (\tau\delta)_{jh} + \beta_{ijh}$$

(ii) the τ_j, δ_h, and $(\tau\delta)_{jh}$ are fixed effects. $F_1(\tau_j)$, $F_2(\delta_h)$, and $F_3(\tau\delta_{jh})$ represent some function of the τ_j, δ_h, and $\tau\delta_{jh}$ effects, respectively.

(iii) α_{ij} and β_{ijh} are random variables normally and independently distributed.

(iv) The effects or components of Y_{ijh} are independent.

The question arises as to whether or not it is theoretically correct (it is possible arithmetically) to partition error (b) into the two components "replicates x split-plot treatments" and "replicates x split-plot treatments x whole plot treatments." In some instances researchers have contended that the appropriate error for the D or split-plot effects is the "replicates x D" mean square. Whether or not this is a correct procedure is only indirectly discussed in the various textbooks. Most authors (Snedecor, Statistical Methods; Cochran and Cox, Experimental Designs; Yates, Design and Analysis of Factorial Experiments; Fisher, Design of Experiments) state that the mean square of "replicates x D within whole plot treatments" is the appropriate error (i.e., it contains all components of variation affecting the variation in a particular mean) for testing the D and TxD mean squares. No explicit explanation of the correctness or incorrectness of partitioning error (b) has been found in the literature although Yates (J.R.S.S., Suppl. 2:181-247) and Cochran and Cox (Experimental Designs, p. 184-5) have given material bearing on this point.

However before attempting any rigorous examination of the partitioning of error (b), let's first consider an intuitive argument against partitioning. In most cases it would be undesirable to partition error (b) since it might be argued that any "replicate x split-plot treatment" mean square is an estimate of the same parameter, σ^2_β, estimated by the "replicate x split-plot x whole plot" mean square. In addition if partitioning were followed the number of degrees of freedom associated with either mean square would be decreased.

The analysis of variance for randomized blocks experiments located at p locations or places would be:

Source of variation	d.f.	Expectation of mean square
Locations	p-1	$\sigma_\beta^2 + t\sigma_\alpha^2 + r\sigma_{\tau\gamma}^2 + rt\sigma_\gamma^2$
Replicates within locations	p(r-1)	$\sigma_\beta^2 + t\sigma_\alpha^2$
Treatments	(t-1)	$\sigma_\beta^2 + r\sigma_{\tau\gamma}^2 + rp\sigma_\tau^2$
Treatments x locations	(p-1)(t-1)	$\sigma_\beta^2 + r\sigma_\tau^2$
Treat. x reps within locations = Error (b)	p(t-1)(r-1)	σ_β^2
Total	prt-1	-----

The linear model for the above analysis is

$$Y_{ijh} = \mu + \gamma_i + \alpha_{ij} + \tau_h + (\gamma\tau)_{ih} + \beta_{ijh} ,$$

where all effects except μ are considered to be random variables.

As illustrated below, replicate 1 in location I has nothing in common with replicate 1 at location II, etc.

	Location I				Location II				...	Location p			
	rep. no.				rep. no.					rep no.			
Treatments	1	2	...	r	r+1	r+2	...	2r	...	p(r-1)	p(r-1)+1	...	rp
1													
2													
⋮													
t													

The numbering is purely arbitrary in that the r+2nd replicate at location II could just as well have been numbered r+1. Therefore the calculated "replicate x location" and "replicate" mean squares are estimates of the same quantity $\sigma_\beta^2 + t\sigma_\alpha^2$. Likewise, there is no "replicate x treatment" effect which is different from the "replicate x treatment x location" effect, i.e., they are estimates of the same parameter σ_β^2.

For a split-plot design like that just described, it would be incorrect to partition the error (b) sum of squares into the two portions "replicate x treatment" and "replicate x treatment x location" and assume that they were estimates of different quantities.

The arguments against partitioning error (b) in the above design may be employed in the discussion of error (b) from a design of the type discussed in the first part of the paper. The systematized arrangement of the design follows:

Split-plot	Replicate 1				Replicate 2					Replicate r			
	whole plot treat.				whole plot treat.					whole plot treat.			
Treat.	1	2	...	t	1	2	...	t	...	1	2	...	t
1													
2													
⋮													
d													

The items referred to as "replicates" differ for each particular set of whole plots. The "split-plot treatments x replicates" sum of squares for whole plots containing treatment number 1 is computed only from those whole plots containing treatment 1. These whole plots are called the "replicates" for this particular sum of squares. The same is true for the remaining sets of whole plots. If this argument is continued, it can be argued that the "replicate x D" and "replicate x D x T" effects are not independent. This argument does not appear to be tenable when one considers complete confounding in factorial experiments. For the dt treatment combinations the complete block or replicate is divided up into t incomplete blocks of d treatments each. The T effect is confounded with the differences among blocks while the D and DxT effects are unconfounded. If the R= replicate effect is called the third factor in the factorial then the R, T, and RxT effects are confounded with differences among the rt incomplete blocks of d plots each. Now if the rt combinations of the r levels of factor r and the t levels of factor t were allotted completely at random to the rt incomplete blocks, the D, TxD, RxD, and RxTxD effects would be unconfounded and orthogonal comparisons.

Instead of complete randomization the rt treatment combinations are allotted in such a way that the levels of r are in a systematic arrangement from r_0 to r_{r-1} and the t whole plot treatments are at random within each level of r or the replicate. The systematic arrangement of one of the factors may introduce some confounding of the other effects. This has been pointed out by Yates (J.R.S.S. Suppl. 2:181-247) and Cochran and Cox (<u>Experimental Designs</u>, p. 184-5). With the results cited here the argument for not partitioning error (b) may now be made something more than intuitive. It is not claimed that the argument is completely rigorous but that it is something approaching a rigorous explanation.

Consider the following lay-out,

	a_0		a_1	
	b_0	b_1	b_0	b_1
c_0	X_{000}	X_{010}	X_{100}	X_{110}
c_1	X_{001}	X_{011}	X_{101}	X_{111}
	1	2	3	4

where the levels of a or replicates are arranged in a systematic order, the levels of b (whole plot treatments) are randomly arranged within each level of a, and the levels of c (split-plot treatments) are randomly arranged within each of the levels of b or whole plot treatments. X_{ijh} is the yield of the ijhth plot.

The BC effect is given by the following differences

$$BC = X_{000} - X_{001} + X_{100} - X_{101} + X_{011} - X_{010} + X_{111} - X_{110}$$

$$= [X_{011} - X_{010} + X_{111} - X_{110}] - [X_{001} - X_{000} + X_{101} - X_{100}]$$

= C response in whole plots containing b_1 - C response in whole plots containing b_0.

The appropriate error mean square for the BC effect would be the interaction of blocks 2 and 4 with the levels of c plus the interaction of blocks 1 and 3 with the levels of factor c. However, these two sums of squares represent the sums of squares for AC and ABC. Thus, it appears that error (b) sums of squares should not be partitioned.

The above arguments represent the writer's reasons as to why error (b) should not be partitioned with two components. It is hoped that the reader will send along his comments. Also any other explanation than that given would be of interest to the author.