BU-173-M   Keewhan Choi   February, 1965

ABSTRACT

Reviewed in this paper are the following deterministic models of epidemics.

1. Non-recurrent Epidemics (two models).

2. Geographical Spread of Epidemics.

3. Recurrent Epidemics.

The material in this paper are excerpts from the following papers and books.

Bailey, N. T. J. (1957). The Mathematical Theory of Epidemics (pp. 25 - 29, 32 - 35).

Kendal, S. G. (1955). Deterministic and Stochastic Epidemics in Closed Population. Third Berkeley Symposium on Math. Stat. and Prob. Vol. IV (pp. 149 - 165).

Karlin, S. (1962). Unpublished lecture notes.

We consider a population of susceptible groups or people invaded by a disease of the following character: for each member who contracts the disease there exists

1. a preliminary non-infectious incubation period, followed by

2. an infectious but non-symptomatic period, leading to

3. a symptomatic period at which time the member is removed from the population of susceptibles.

Model 1. We imagine a population of n+1 members with a single person infected initially. Those infected are not removed.

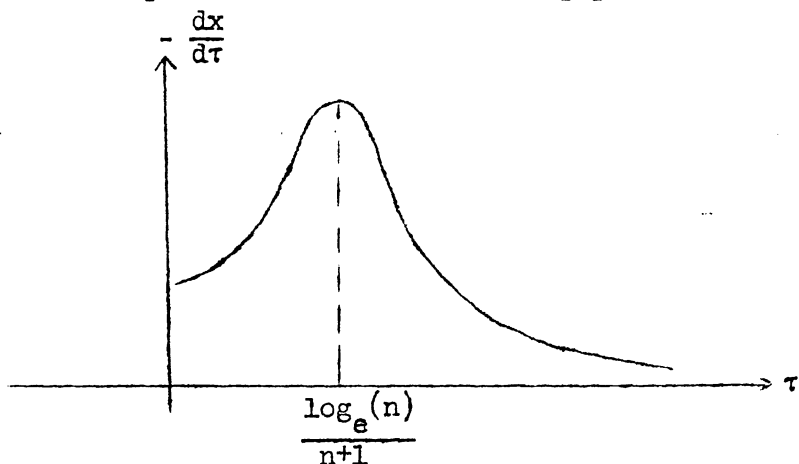$$\text{Let } x(t) = \text{number susceptibles at time } t$$
$$y(t) = \text{number infected at time } t$$

Our initial conditions are thus $x(0) = n$, $y(0) = 1$; if sick are not removed, we have $x(t) + y(t) = n + 1$ for all t. We assume the spread of the epidemic is caused by relative contact between susceptibles and the infected. Then $dx/dt = -\beta xy$, and letting $\tau = \beta t$, $dx/d\tau = -xy$. $x + y = n + 1 \implies$ $dx/d\tau = -x(n + 1 - x) = -x(n + 1) + x^2$ and we have a Bernoulli differential equation. The solution is standard, $x(\tau) = (n+1)n/(e^{(n+1)\tau} + n)$. It is common procedure to call $-dx/d\tau$ the epidemic curve, since it represents the rate of infection of susceptibles.

Then,
$$-\frac{dx}{d\tau} = (n+1)^2\, n\, e^{(n+1)\tau}/(n+e^{(n+1)\tau})^2$$

The above function has a maximum at $\tau_{max} = \dfrac{\log_e (n)}{n + 1}$.

The curve is symmetrical about τ_{max}. Since $x(\tau_{max}) = (n+1)/2$ the epidemic reaches its peak when one-half of the population is stricken.

Model 2. This model was first treated by Kermack and McKendrick in 1927 and was given an approximate solution. Kendall later gave an exact solution. We begin with the approximate solution. Let

$x(t)$ = number susceptibles in population

$y(t)$ = number infected and infectious

$z(t)$ = number removed or cured and immune.

Then, the number of susceptibles is reduced by those infected, the number of infected is increased by those stricken and decreased by those cured, with rates proportional to contact. Therefore

(1) $dx/dt = -\beta xy$

(2) $dy/dt = \beta xy - \gamma y$

(3) $dz/dt = - dx/dt - dy/dt = \gamma y$,

since $x + y + z = n$ = size of population $\Longrightarrow \frac{d(x + y + z)}{dt} = 0$ for all t. It is convenient to let $\rho = \gamma/\beta$. ρ is called the "relative removal rate" or the "threshold" constant since, from (2), $dy/dt = y(\beta x - \gamma)$, and if initially $x(0) \le \gamma/\beta = \rho$ then $dy/dt \le 0$ and no epidemic may ensue. From (1) and (3), $dx/dz = - x/\rho \Longrightarrow x = x_0 e^{-z/\rho}$ where $x_0 = x(0)$. $\therefore$ $dz/dt = \gamma(n-z-x_0 e^{-z/\rho})$ since $x + y + z = n$, for all t. The approximation now made by Kermack and McKendrick was to employ the first three terms in the expansion of $e^{-z/\rho}$.

$\therefore \frac{dz}{dt} = \gamma(n - z - x_0 \{1 - z/\rho + z^2/2\rho^2\}) = \gamma\{(n-x_0) + z(x_0/\rho - 1) - x_0 z^2/2\rho^2\}$

It is interesting to note that three terms of the expansion were required since in most cases x_0 is just above the threshold level and thus $z(x_0/\rho - 1) \sim z^2$, for z small.

We digress now to examine the solution of the above differential equation, a very common type called Ricatti's equation. Its most general form is

$$- \frac{du}{dt} = u^2 + p(t)u + q(t) \qquad (a)$$

To find the general solution, we consider the linear second order equation,

$$\frac{d^2 v}{dt^2} + p(t) \frac{dv}{dt} + vq(t) = 0 \qquad (b)$$

We claim that, if v is any general solution of (b), then $u = v'/v$ is a general solution of (a) in any region for which $v \neq 0$.

Suppose $u = v'/v$. Then

$$u' = \frac{v''v - (v')^2}{(v)^2} = \frac{v''}{v} - (\frac{v'}{v})^2$$

$$u' + u^2 = \frac{v''}{v}$$

and $u' + u^2 + p(t)u + q(t) = v''/v + p(t)\, v'/v + q(t) = 0$ (from (b)) and $u = v'/v$ is indeed a solution of (a).

Returning to our original equation, let $s = x_0 t/2\rho^2$, and we have

$$\frac{dz}{ds} = \frac{y_0\, 2\rho^2}{x_0} + z\, \frac{2\rho^2}{x_0}\, (\frac{x_0}{\rho} - 1) - z^2$$

where we have now let $z_0 = 0$ so that $x_0 + y_0 = n$.
We thus solve the related equation

$$\frac{d^2v}{dt^2} + (1 - \frac{x_0}{\rho})\, \frac{2\rho^2}{x_0}\, \frac{dv}{dt} - \frac{y_0\, 2\rho^2}{x_0} = 0$$

which has constant coefficients and is solved by utilizing a combination of exponentials.

The solution is straightforward and

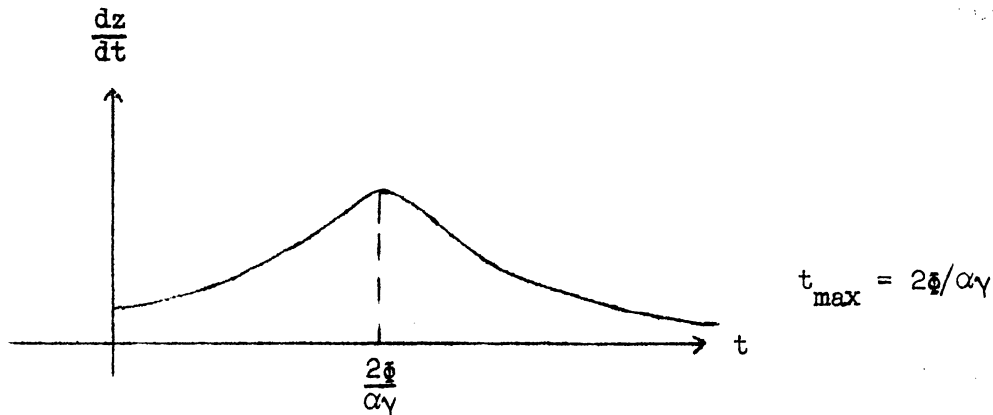$$z = \frac{\rho^2}{x_0} \left\{ (\frac{x_0}{\rho} - 1) + \alpha \tanh (\frac{1}{2} \alpha\gamma t - \phi) \right\}$$

where

$$\alpha = \left\{ (\frac{x_0}{\rho} - 1)^2 + \frac{2x_0 y_0}{\rho^2} \right\}^{1/2}$$

$$\phi = \tanh^{-1} \frac{1}{\alpha} (\frac{x_0}{\rho} - 1)$$

We now differentiate to find the "cure rate" curve.

$$\frac{dz}{dt} = \frac{1}{2} \frac{\rho^2}{x_0} \alpha^2 \gamma \operatorname{sech}^2(\frac{1}{2} \alpha\gamma t - \phi) .$$

This curve is symmetric about a maximum which coincides with the maximum infectious rate.

$$\frac{dz}{dt}$$

$$t_{max} = 2\Phi/\alpha\gamma$$

$$\frac{2\Phi}{\alpha\gamma}$$

As t $-\!\!\to \infty$, we find

$$z(\infty) = \frac{\rho^2}{x_o} \left\{ (\frac{x_o}{\rho} - 1) + \alpha \right\}$$

and we interpret this value as representing the total "size" of the epidemic; i.e., the total number infected. If we assume $y_o \ll (x_o/\rho) - 1$, then $\alpha \sim (x_o/\rho - 1)$ and

$$z(\infty) \sim \frac{2\rho^2}{x_o} (\frac{x_o}{\rho} - 1) = 2\rho(1 - \frac{\rho}{x_o})$$

Now assume x_o is a fixed value above the threshold so that $x_o = \rho + \nu$. Then

$$z(\infty) \sim 2\rho(1 - \frac{\rho}{\rho+\nu}) = 2\rho(\frac{\nu}{\rho+\nu}) \sim 2\nu \quad \text{where} \quad \nu \ll \rho$$

Thus, through the course of the epidemic, the number of susceptibles will eventually fall as far below the threshold as it was above at the outbreak.

References:

Bailey, N. T. J. (1957), The Mathematical Theory of Epidemics.

Kermack, W. O., and McKendrick, A. G. (1927-33), Contributions to the
    Mathematical Theory of Epidemics, Proc. Roy. Soc.

Kendall, D. G. (1956), Deterministic and Stochastic epidemics in Closed
    Populations, Proc. Third Berkeley Symposium on Math. Stat. and Prob.

[This material follows N. T. J. Bailey, The Mathematical Theory of Epidemics, pp. 25-29 which discusses part of a paper by D. G. Kendall, Deterministic and Stochastic Epidemics in Closed Populations, vol. IV, Third Berkeley Symposium, pp. 149-165.]

The adequacy of the approximate solution obtained by Kermack and McKendrick may be examined by considering the modified model of which it is the exact solution. As before

$$(1) \quad \begin{cases} \dfrac{dx}{dt} = -\beta xy \\[2mm] \dfrac{dy}{dt} = \beta xy - \gamma y \\[2mm] \dfrac{dz}{dt} = \gamma y \end{cases}$$

with $x(0) = x_o$, $y(0) = y_o$, $z(0) = z_o = 0$. However β is now temporarily assumed to be a function of z, $\beta(z)$. Then

$$\frac{dx}{dz} = -\frac{\beta(z)}{\gamma} y$$

from which

$$x = x_o \exp\left[-\frac{1}{\gamma} \int_o^z \beta(u)\, du\right].$$

Making the particular choice

$$(2) \quad \beta(z) = \frac{2\beta}{(1 - \frac{z}{\rho}) + (1 - \frac{z}{\rho})^{-1}}$$

with $\beta(0) = \beta$ and $\rho = \gamma/\beta$,

$$\int_o^z \beta(z)\, dz = \int_o^z \frac{2\beta(1 - \frac{u}{\rho})\, du}{(1 - \frac{u}{\rho})^2 + 1}$$

$$= -\gamma \log(1 - \frac{z}{\rho} + \frac{z^2}{2\rho^2}).$$

Thus

$$x = x_o(1 - \frac{z}{\rho} + \frac{z^2}{2\rho^2}).$$

Using $y = n - z - x$,

$$\frac{dz}{dt} = \gamma y = \gamma \left[n - z - x_0 \left(1 - \frac{z}{\rho} + \frac{z^2}{2\rho^2} \right) \right]$$

$$= \gamma \left[n - x_0 + z \left(\frac{x_0}{\rho} - 1 \right) - \frac{x_0}{2\rho^2} z^2 \right] .$$

This is exactly the approximation obtained by Kermack and McKendrick. From (2), $\beta(z) < \beta$ for $0 < z \leq \rho$ so that the approximation consistently underestimates the rate at which infection takes place and hence underestimates the total size, $z(\infty)$, of the epidemic. Further, if at any time $z > \rho$ then $\beta(z) < 0$ and the number of susceptibles is an increasing function of time.
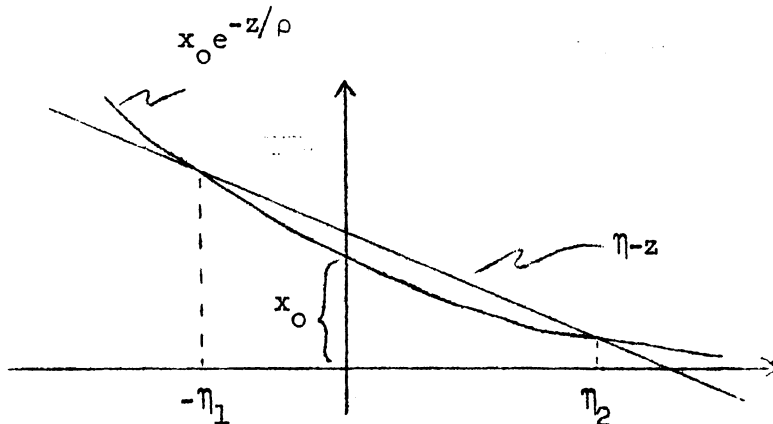
We now return to the model as given by (1) with β and γ positive constants. As before, an immediate consequence of (1) is that

$$x = x_0 \, e^{-z/\rho}$$

where $\rho = \gamma/?$, so that

$$(3) \qquad \frac{dz}{dt} = \gamma(n - z - x_0 \, e^{-z/\rho})$$

Since $x_0 \, e^{-z/\rho}$ is convex with intercept x_0 on the function axis, its graph is intersected exactly twice by the graph of $n - z$.

Letting $-\eta_1$ and η_2 be the two roots, one negative and one positive, of $dz/dt = 0$, $dz/dt > 0$ for $0 \leq z < \eta_2$. From (3)

$$dt = \frac{dz}{\gamma(n - z - x_o e^{-z/\rho})}$$

and, using $z = 0$ at $t = 0$,

(4) $$t = t(z) = \frac{1}{\gamma} \int_o^z \frac{du}{n - u - x_o e^{-u/\rho}} \; ,$$

with convergence for $0 \leq z < \eta_2$.

t(z) as given by (4) is one-to-one for $0 \leq z \leq \eta_2$ and hence has an inverse there, so that equations (3) and (4) provide a formal solution for dz/dt as a function of t, called the notifications curve by Kendall.

For $0 < z < \eta_2$,

$$t(z) = \frac{1}{\gamma} \int_o^z \frac{du}{n - u - x_o e^{-u/\rho}} \geq \frac{1}{\gamma} \int_o^z \frac{(1 - \frac{x_o}{\rho} e^{-u/\rho}) \, du}{n - u - x_o e^{-u/\rho}}$$

$$= - \frac{1}{\gamma} \log \left(\frac{n - z - x_o e^{-z/\rho}}{n - x_o} \right)$$

Therefore $\lim_{z \to \eta_2} t(z) = \infty$. Thus $z(\infty) = z_\infty = \eta_2$, and η_2 may be interpreted as the size of the epidemic.

It is usual, according to Kendall, to discuss the epidemic generated by the introduction of a trace of infectives (y_o) into a population consisting entirely of susceptibles. This motivates looking at the limits as x_o approaches n of the elapsed time t(z) until the number of removals equals z. Bailey asserts that for t(z) as given by (4) and for fixed z between 0 and η_2,

$\lim_{x_o \to \eta} t(z) = \infty$, and interprets this as showing that an infinite time elapses before the epidemic begins. Treating x_o as a continuous variable,

$$t(z) = \frac{1}{\gamma} \int_o^z \frac{du}{n - u - x_o e^{-u/\rho}} \geq \frac{1}{\gamma} \int_o^z \frac{du}{n - x_o e^{-u/\rho}}$$

$$= \frac{\rho}{n\gamma} \int_o^z \frac{\frac{n}{\rho} e^{u/\rho} du}{ne^{u/\rho} - x_o} = \frac{\rho}{n\gamma} [\log(ne^{z/\rho} - x_o) - \log(n - x_o)] .$$

Thus for any fixed z with $0 < z < \eta_2$, $\lim_{x_o \to \eta} t(z) = \infty$.

Apparently then the time required to reach any fixed positive z may be made arbitrarily large by picking x_o sufficiently close to n. [In fact, however, x_o is discrete and is at most n-1 if there is to be an epidemic, and for such x_o the integral defining t(z) converges.] Kendall (and, following him, Bailey) sees this as a difficulty which is overcome by a shift of the time origin to that time at which the notifications curve achieves its maximum. Quoting Kendall,

> "The complete change in our point of view should be
> noted; to begin with we followed the customary procedure
> of studying the development of an epidemic subsequent to its
> artificial creation by the introduction of y_o infectious
> persons into a population of x_o susceptibles. We are now
> considering an epidemic as an entity existing from t = - ∞
> to t = + ∞, and we have for convenience located our time-
> origin at the epoch of greatest activity."

Details of Kendall's analysis can be found in the paper cited above. In these notes the same results will be obtained without employing Kendall's shift in point of view.[1]/

The peak of the notifications curve, dz/dt as a function of t, occurs when

$$\frac{d}{dt} \left(\frac{dz}{dt}\right) = \gamma \frac{dy}{dt} = \gamma(\beta xy - \gamma y) = \frac{\gamma y}{\beta} (x-\rho) = 0$$

Letting $\hat{x}$, $\hat{y}$, $\hat{z}$ denote the number of susceptibles, infectives and removals at the time when dz/dt takes its maximum value, $\hat{x} = \rho$. Since $x = x_o e^{-z/\rho}$, $z = \rho \log (x_o/x)$ and $\hat{z} = \rho \log(x_o/\rho)$, and since the total population size remains constant, $\hat{y} = n - M - M \log(x_o/\rho)$.

Using Kendall's notation, let ζ_1 and ζ_2 be the number of removals before and after the peak of dz/dt, respectively. We have taken $z_o = 0$, so

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[1]/ This derivation is due to Karlin.

(5)
$$\zeta_1 = \hat{z} = \rho \log \frac{x_o}{\rho}$$

and

(6)
$$\zeta_2 = \eta_2 - \zeta_1$$

where η_2 is the unique positive root of $n - \eta - x_o e^{-\eta/\rho} = 0$. From (5) it is apparent that $\zeta_1 \geq 0$ if and only if $x_o \geq \rho$. If $x_o < \rho$ then, since x is a decreasing function of t, dz/dt does not peak but is a decreasing function for all t.

The intensity of the epidemic is given by $I = \dfrac{\zeta_1 + \zeta_2}{n} = \dfrac{\eta_2}{n}$, the ratio of the total number of removals to the population size. The ratio of the number of removals before peak activity to the total number of removals, $\zeta_1/(\zeta_1 + \zeta_2) = \zeta_1/\eta_2$, serves as a measure of the asymmetry of the epidemic. The number of infectives at the peak of the epidemic is $\hat{y} = n - M - \zeta_1$ so that the maximum height of the notifications curve is

$$(\frac{dz}{dt})_{max} = \gamma(n - \rho - \zeta_1)$$

It should be noted that for fixed n and ρ the values of ζ_1, ζ_2, I and $(dz/dt)_{max}$ are functions of x_o. In Kendall's treatment the same symbols are used for the limiting quantities as $x_o \longrightarrow n$, obtained by replacing x_o by n since the functions are continuous.

In the special case of $x_o = n$,

$$\frac{\hat{y}}{n} = 1 - \frac{\rho}{n} + \frac{\rho}{n} \log \frac{\rho}{n}$$

$$\frac{\zeta_1}{\zeta_1 + \zeta_2} = - \frac{1}{I} \frac{\rho}{n} \log \frac{\rho}{n}$$

and
$$\frac{\rho}{n} = - \frac{I}{\log(1 - I)}$$

so that the proportion of infectives in the population at the peak of the epidemic and the ratio of removals prior to peak to the total number of removals are both functions of the intensity I above. Kendall tabulates their values at various values of I, and a few are given below.

I	$\hat{y}/n$	$\zeta_1/(\zeta_1 + \zeta_2)$
20	0.6	49
40	2.5	48
60	6.8	46
80	15.5	43
90	24.2	41
95	31.9	38
98	40.3	35

I. Geographical Spread of Epidemics

References: D. G. Kendall, Journal of the Royal Statistical Society, Series A,
Vol. 120 (1957) pp. 64-67.

N. T. J. Bailey, The Mathematical Theory of Epidemics, Hafner,
1957, pp. 32-35.

The following analysis is nonrigorous, but is useful in revealing
features worthy of study in stochastic versions of the model.

Consider an infinite uniform two-dimensional population for which there
are σ individuals per unit area.

Let

$x(P,t)$ = proportion of "susceptibles" in the population at point P, at
time t.

$y(P,t)$ = proportion of "infectives" in the population at point P, at time t.

$z(P,t)$ = proportion of "removals" in the population at point P, at time t.

(1) $x(P,t) + y(P,t) + z(P,t) = 1$

for each P and t.

In a small element of area dS about P,

σdS = number of "individuals" in the area

$\sigma x(P,t)\ dS \approx$ number of "susceptibles" in the area at time t

$\sigma y(P,t)\ dS \approx$ number of "infectives" in the area at time t

$\sigma z(P,t)\ dS \approx$ number of "removals" in the area at time t

In this model the time-rate of change of the number of susceptibles in the
area is taken to be proportional to the product of the number of susceptibles
in the area and a weighted average, with weights depending upon P, of the
number of infectives in other parts of the plane. This latter factor is in-
tended to be a measure of the chance of contact of a susceptible at P with an
infective somewhere. With β as the factor of proportionality, the differ-
ential equation for the time-rate of change of the number of susceptibles in
the area is

$$\frac{d}{dt} \sigma x(P,t) \; dS = - \beta \; \sigma x(P,t) \; dS \iint \sigma y(Q,t) \; \lambda(P,Q) \; dS$$

After cancelling a σdS on each side we obtain

(2) $$\frac{d}{dt} x(P,t) = - \beta \; \sigma x(P,t) \iint y(Q,t) \; \lambda(P,Q) \; dS$$

Let

(3) $$\tilde{x}(P,t) = \iint x(Q,t) \; \lambda(P,Q) \; dS$$

$$\tilde{y}(P,t) = \iint y(Q,t) \; \lambda(P,Q) \; dS$$

$$\tilde{z}(P,t) = \iint z(Q,t) \; \lambda(P,Q) \; dS$$

where the integrals are with respect to Q over the whole plane. We assume that $\lambda(P,Q)$ is such that the integrals in (3) converge. Furthermore, we assume that $\iint \lambda(P,Q) \; dS$ is constant for each P. There is then no loss of generality in supposing that

(4) $$\iint \lambda(P,Q) \; dS = 1 \qquad \text{for all P,}$$

since this can be arranged by a suitable adjustment of the factor of pro-portionality β.

Those individuals who leave the susceptible class become infectives. Assume a removal-rate γ for infectives. Then using (2) and the second equation of (3) we have the following set of differential equations (we drop (P,t) from the notation for typographical convenience):

(5) $$\frac{dx}{dt} = - \beta \sigma x \tilde{y}$$

$$\frac{dy}{dt} = \beta \sigma x \tilde{y} - \gamma y$$

$$\frac{dz}{dt} = \gamma y$$

Multiply the third equation by $\lambda(P,Q)$ and integrate out Q to obtain

(6) $$\frac{d\tilde{z}}{dt} = \gamma \tilde{y}$$

Divide the first equation of (5) by equation (6) to obtain

$$(7) \qquad \frac{dx}{d\tilde{z}} = -\frac{\beta\sigma x}{\gamma} = -\frac{\sigma x}{\rho}$$

where

$$(8) \qquad \rho = \frac{\gamma}{\beta}$$

The differential equation (7) is easily solved by separating variables and integrating. The solution is

$$(9) \qquad x(P,t) = x(P,0) \exp[-\frac{\sigma}{\rho} \tilde{z}(P,t)]$$

We shall take $z(P,0) = 0$ for all P. This simply amounts to considering in the analysis only those individuals who are capable of entering into the epidemic. It then follows from the third equation of (3) that

$$\tilde{z}(P,0) = 0 \qquad\qquad \text{for all P.}$$

Using (1) and (9), rewrite the third equation of (5):

$$(10) \qquad \frac{dz}{dt} = \gamma y$$

$$= \gamma[1 - z - x]$$

$$= \gamma[1 - z - x(P,0) \exp\{-\frac{\sigma}{\rho} \tilde{z}(P,t)\}]$$

It follows from (5) that as $t \to \infty$, x decreases since $\frac{dx}{dt} < 0$, and z increases since $\frac{dz}{dt} > 0$. Since x and z are bounded they must tend to limits. Denote the respective limits by the omission of t; i.e.,

$$x(P,t) \dashrightarrow x(P)$$

$$z(P,t) \dashrightarrow z(P)$$

$$\tilde{z}(P,t) \dashrightarrow \tilde{z}(P) \qquad\qquad \text{etc.}$$

Since z tends to a limit we shall assume that $\frac{dz}{dt}$ tends to zero (there are examples of functions which tend to a limit without the derivative tending to zero). It then follows from the third equation of (5) that y tends to zero, i.e., $y(P) = 0$ for each P. From (1),

$$y(P) = 1 - z(P) - x(P)$$

$$= 1 - z(P) - x(P,0) \exp[- \frac{\sigma}{\rho} \tilde{z}(P)]$$

$$= 0$$

Consequently

(11) $$z(P) = 1 - x(P,0) \exp[- \frac{\sigma}{\rho} z(P)] \ .$$

From the third equation of (3) we have, at least for the case where $\lambda(P,Q)$ is of the form (13), below,

(12) $$\tilde{z}(P) = \iint z(Q) \ \lambda(P,Q) \ dS \ .$$

Denote the distance between two points P and Q by $\overline{PQ}$. Suppose that the weighting-function λ has the property that for some b > 0,

(13) $$\lambda(P,Q) = 0 \qquad\qquad \text{for } \overline{PQ} \geq b$$

$$\lambda(P,Q) > 0 \qquad\qquad \text{for } \overline{PQ} < b$$

Further, assume initial conditions

(14) $$x(P,0) = 1 - \varepsilon(P)$$

$$y(P,0) = \varepsilon(P)$$

where for some a > 0,

(15) $$\varepsilon(P) = 0 \qquad\qquad \text{for } \overline{OP} \geq a$$

$$\varepsilon(P) > 0 \qquad\qquad \text{for } \overline{OP} < a$$

and $\overline{OP}$ is the distance of P from an origin O. Then from (11), (14) and (15) we have

(16) $$z(P) = 1 - [1 - \varepsilon(P)] \exp[- \frac{\sigma}{\rho} \tilde{z}(P)] \qquad \text{for } \overline{OP} < a$$

$$z(P) = 1 - \exp[- \frac{\sigma}{\rho} \tilde{z}(P)] \qquad\qquad \text{for } \overline{OP} \geq a$$

It is clear from the first equation in (16) that z(P) > 0 for $\overline{OP}$ < a. We shall show next that z(P) > 0 for all P, i.e., the epidemic spreads everywhere.

The argument is by contradiction. Suppose there is a P_o such that $z(P_o) = 0$. This cannot occur for $\overline{OP_o} < a$. Then the second equation of (16) implies that $\tilde{z}(P_o) = 0$. But

$$\tilde{z}(P_o) = \iint z(Q) \, \lambda(P_o,Q) \, dS \qquad \text{by (12)},$$

where $\lambda(P_o,Q)$ in the integral is > 0 for $\overline{P_o Q} < b$. Consequently, we must have $z(Q) = 0$ almost everywhere for all Q such that $\overline{P_o Q} < b$. Continuing towards the origin in this way, we arrive in a finite number of steps at a contradiction of the fact that $z(P) > 0$ for all P such that $\overline{OP} < a$. Hence $z(P) > 0$ for all P, and the epidemic spreads everywhere.

Next we shall consider the severity of the epidemic, as measured by $z(P)$, as $P \to \infty$. We shall use the following additional assumptions:

(a) $z(P)$ is strictly monotone decreasing as $\overline{OP} \to \infty$.

(b) the function λ is isotropic, i.e.,

$$\lambda(P,Q) = \lambda(\overline{PQ})$$

a function of distance alone.

If $P \to \infty$ in a given direction, then $z(P)$ tends to a limit ξ, say, since $z(P)$ is bounded below. Assume that the limit is the same for all directions (clearly we have this symmetry property if the function $\mathcal{E}(P)$ depends only on $\overline{OP}$). Then from (12), (13) and (4), we have that $\tilde{z}(P) \to \xi$ as $P \to \infty$. By (16) ξ must satisfy

$$(17) \qquad \xi = 1 \exp\left(- \frac{\sigma}{\rho} \xi\right) \qquad \text{or equivalently}$$

$$1 - \xi = \exp\left(- \frac{\sigma}{\rho} \xi\right) .$$

The function $1-\xi$ equals 1 at $\xi=0$, and has constant slope -1. The convex function $\exp\left(- \frac{\sigma}{\rho} \xi\right)$ equals 1 at $\xi=0$, and has slope $- \frac{\sigma}{\rho}$ at $\xi=0$.

If $\frac{\sigma}{\rho} \leq 1$, then $\xi=0$ is the only non-negative root of (17), while if $\frac{\sigma}{\rho} > 1$, then there is a positive root between 0 and 1. See Figure 1.
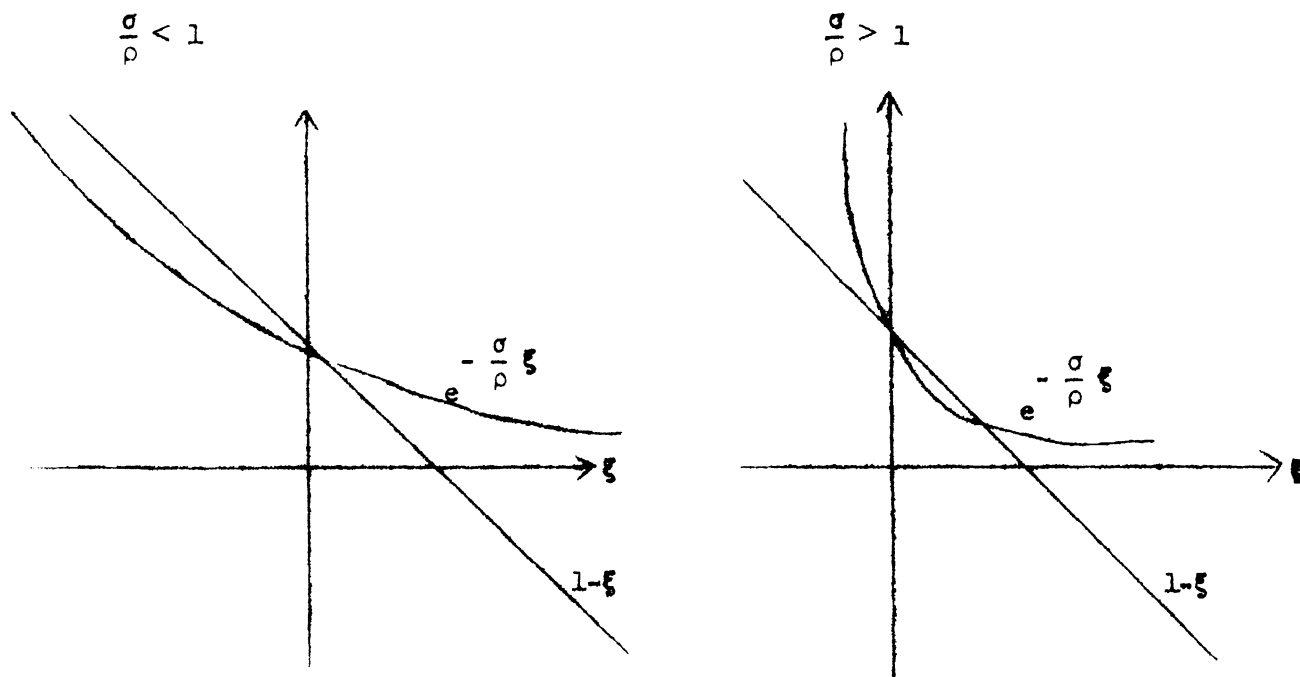
$\dfrac{\sigma}{\rho} < 1$

$\dfrac{\sigma}{\rho} > 1$

Figure 1

We shall prove the following theorem which is due to Kendall:

Theorem: Under the assumptions already stated, $z(P) \to \xi_0$ as $P \to \infty$, where $\xi_0 = 0$ for $\dfrac{\sigma}{\rho} \leq 1$ and ξ_0 is the positive root of $\xi = 1 - \exp(-\dfrac{\sigma}{\rho}\xi)$ for $\dfrac{\sigma}{\rho} > 1$.

Proof: What remains to be proved is the assertion concerning the positive root. The proof is by contradiction. For $\dfrac{\sigma}{\rho} > 1$, suppose $\xi_0 = 0$. Then for $\overline{OP}$ sufficiently large, $z(P)$ is close to zero, since we have assumed that $z(P)$ is strictly monotone decreasing as $\overline{OP} \to \infty$. If $z(P)$ is close to zero then, so is $\tilde{z}(P)$, since by (16)

$$z(P) = 1 - \exp(-\theta\, \tilde{z}(P))$$

where $\theta = \dfrac{\sigma}{\rho} > 1$. Expand $1 - \exp[-\theta\tilde{z}(P)]$, dropping powers of $\tilde{z}(P)$ higher than the first:

$$z(P) \approx \theta\tilde{z}(P)$$

Hence,

$$\frac{1}{\theta}\, z(P) \approx \tilde{z}(P) \; .$$

Then for any α satisfying $1 > \alpha > \frac{1}{8}$, we have

$$(18) \qquad \alpha \, z(P) \geq \tilde{z}(P) \qquad\qquad\qquad \text{for all } \overline{OP} > c,$$

for some sufficiently large c. Denote an n-fold iteration of the operation

$$\tilde{z}(P) = \iint z(Q) \, \lambda(P,Q) \, dS$$

by $\tilde{z}^{(n)}(P)$, i.e.,

$$\tilde{z}^{(n)}(P) = \iint \tilde{z}^{(n-1)}(Q) \, \lambda(P,Q) \, dS$$

Each $\tilde{z}^{(k)}(Q)$ depends on $\tilde{z}^{(k-1)}$ at points within a radius b of Q. Now by (18) we have

$$\alpha^2 z(P) \geq \alpha \, \tilde{z}(P) \geq \tilde{z}^{(2)}(P) \qquad\qquad \text{for } \overline{OP} > b + c \; .$$

Repeated iteration yields

$$(19) \qquad \alpha^n \, z(P) \geq \tilde{z}^{(n)}(P) \qquad\qquad \text{for } \overline{OP} > (n-1) \, b + c$$

For fixed α choose n so that $\alpha^n < \frac{1}{8}$.

With $\overline{OP} > (n-1)$ b+c, consider the two circles:

$$\Gamma_1: \quad \text{center 0, radius } \overline{OP}$$

$$\Gamma_2: \quad \text{center P, radius nb}$$
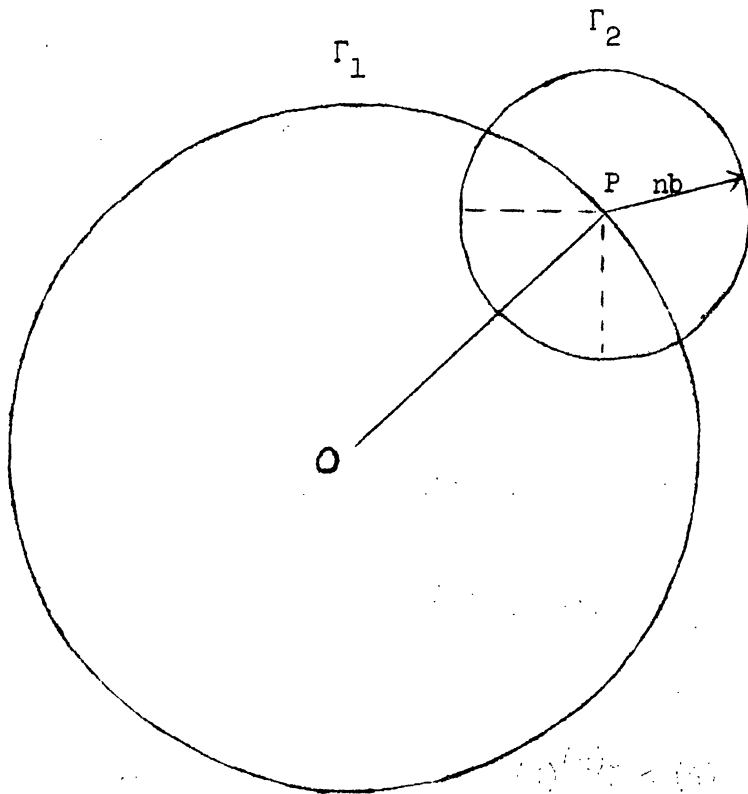
See Figure 2.

Figure 2

If $\overline{OP}$ is large enough the intersection of the two circles will be almost a semicircle of Γ_2; but in any case the intersection is at least $1/4$ of Γ_2. For Q in the intersection, $z(Q) \geq z(P)$ by the monotonicity assumption. Let $\tilde{\chi}^{(n)}(P,Q)$ be the kernel of the n-fold iteration of $\tilde{z}(P) = \iint z(Q) \, \lambda(P,Q) \, dS$, i.e.,

$$(20) \qquad \tilde{z}^{(n)}(P) = \iint z(Q) \, \tilde{\chi}^{(n)}(P,Q) \, dS \ ,$$

where integration is with respect to Q over the whole plane. Now since

$$(21) \qquad \iint_{\Gamma^2} \tilde{\chi}^{(n)}(P,Q) \, dS = 1 \ ,$$

and $\tilde{\chi}^{(n)}(P,Q)$ is isotropic, it follows that

$$\tilde{z}^{(n)}(P) \geq \int\int_{\Gamma_1 \cap \Gamma_2} z(Q) \, \tilde{\chi}^{(n)}(P,Q) \, dS$$

$$\geq z(P) \int\limits_{\Gamma_1 \cap \Gamma_2} \int \bar{\chi}^{(n)}(P,Q) \, dS$$

$$\geq \frac{1}{4} z(P)$$

But by (19), with our $\alpha^n < \frac{1}{8}$,

$$\frac{1}{8} z(P) \geq \tilde{z}^{(n)}{}_P \ .$$

This is a contradiction, so for $\frac{\sigma}{\rho} > 1$ we must have $\xi_0 > 0$.   Q E D.

In the case where $z(P) \to \xi_0 > 0$ as $P \to \infty$, we say that there is a pandemic; in this case the fraction of individuals succumbing to the disease will be at least ξ_0 even at indefinitely great distances from the origin.

II.   Recurrent Epidemics

Suppose we modify the previous model by taking into account a supply of susceptibles entering the infected area.

Let μ be the number of susceptibles entering per unit time.

Then the equations are:

$$\frac{dx}{dt} = - \beta xy + \mu; \qquad \frac{dy}{dt} = \beta xy - \gamma y$$

If

(x_0, y_0) is an equilibrium point, then

$$0 = -\beta x_0 y_0 + \mu$$

$$0 = \beta x_0 y_0 - \gamma y_0$$

$$x_0 = \frac{\gamma}{\beta}$$

$$y_0 = \frac{\mu}{\gamma}$$

To determine stability, let

$$x(t) = x_0(1 + \mu(t))$$

$$y(t) = y_0(1 + v(t))$$

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$$x_0 \frac{du}{dt} = -\beta x_0 y_0 (1+u)(1+v) + \mu$$

$$y_0 \frac{dv}{dt} = \beta x_0 y_0 (1+u)(1+v) - \gamma y_0 (1+v)$$

$$\sigma = \frac{y}{\beta_\mu} \qquad \tau = \frac{1}{\gamma}$$

$$\frac{du}{dt} = -\frac{1}{\sigma}(u+v+uv)$$

$$\frac{dv}{dt} = \frac{1}{\tau} u(1+v)$$

$$\frac{du}{dt} \cong -\frac{1}{\sigma}(u+v)$$

$$\frac{dv}{dt} \cong \frac{1}{\tau} u, \qquad\qquad \text{neglecting non-linear terms}$$

$$\frac{d^2v}{dt^2} = \frac{I}{\tau}\frac{du}{dt} = -\frac{1}{\sigma\tau}(u+v)$$

$$= \frac{-I}{\sigma\tau}(\tau \frac{dv}{dt} + v).$$

So

$$v'' + \frac{1}{\sigma}v' + \frac{1}{\sigma\tau}v = 0.$$

The solution is

$$v(t) = v_0[\exp(-\frac{t}{2\sigma})\cos\xi t]; \quad \xi = \sqrt{\frac{1}{\sigma t} - \frac{1}{4\sigma^2}}$$

provided μ is small, so $1/\sigma t > 1/4\sigma^2$.

The solution for μ can be found similarly:

$$u(t) = u_0 \frac{\sqrt{\tau}}{\sigma}\cos(t+\psi)\exp(-\frac{t}{2}); \quad \cos\psi = \frac{1}{2}\frac{\sqrt{\tau}}{\sigma}.$$

Apparently these solutions are periodic, but damped, so that as $t \to \infty$, $u(t) \to 0$, $v(t) \to 0$; i.e., $x(t) \to x_0$, $y(t) \to y_0$. Unfortunately this model does not seem to conform to observed situations.

In this model, we neglected non-linear terms in the differential equations. However, even if they are included, the solutions turn out to be periodic and damped:

$$\frac{du}{dt} = \frac{-1}{\sigma}(u+v+uv); \quad \frac{dv}{dt} = \frac{1}{\tau} u(1+v) .$$

Let

$$f(u,v) = 1+u - \log(1+u) + \frac{\tau}{\sigma}(1+v - \log(1+v)) .$$

Note that

$$u - \log(1+u) > 0 \qquad \text{if} \quad u > -1 ,$$

so that

$$f(u,v) \geq 1 + \frac{\tau}{\sigma} \qquad \text{if} \quad u, v > -1 .$$

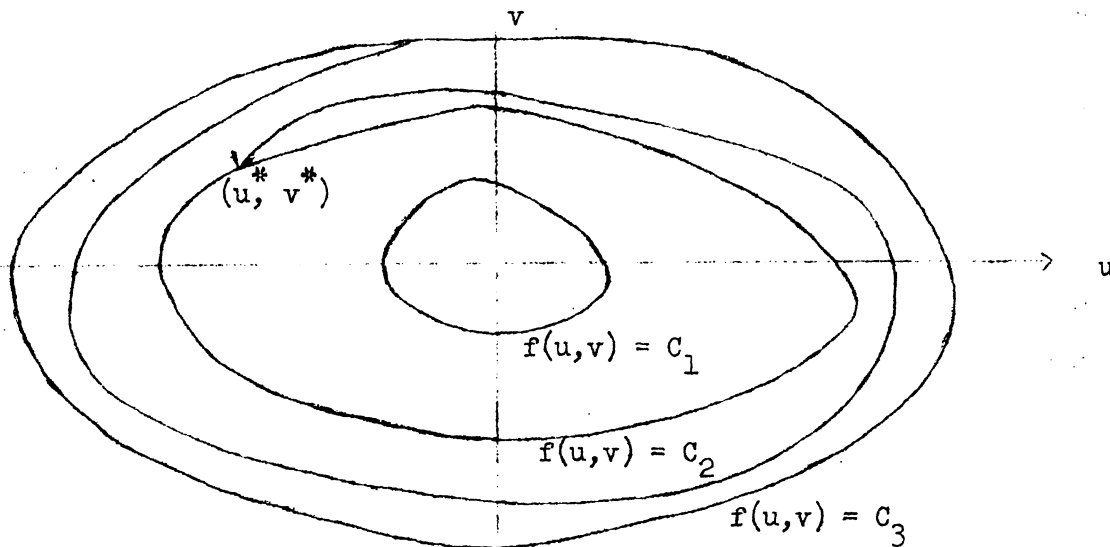Since $x(t)$, $y(t)$ must be positive, this restriction on u, v is always satisfied

$$\frac{df(u,v)}{dt} = \frac{du}{dt} [1 - \frac{1}{1+u}] + \frac{\tau}{\sigma} [\frac{dv}{dt} [1 - \frac{1}{1+v}]]$$

Substituting the expressions for du/dt, dv/dt in the equation, we get

$$\frac{df(u,v)}{dt} = - \frac{u^2}{\sigma(1+u)} \leq 0 .$$

(Strict in equality if $u \neq 0$.) Note that $f(0,0) = 1 + \frac{\tau}{\sigma}$, which is the minimum value of $f(u,v)$.

The origin correspond to $f(0, 0) = 1 + \frac{\tau}{\sigma}$. Suppose that at $t = 0$, $f(u(t), v(t)) = C_3 \neq 1 + \frac{\tau}{\sigma}$. Consider the behavior of $f(u(t), v(t))$ as $t \to \infty$. Since $[df(u, v)]/dt < 0$, since $u \neq 0$ initially, so the solution curve will move inward, staying within the curve $f(u, v) = C_3$. If it does not converge to $u = v = 0$ (the equilibrium point), then there is a sequence $\{t_n\}$ such that $f(u(t_n), v(t_n))$ converges to some value, say $C_2 < C_3$; let $u(t_n) \to u^*$, $v(t_n) \to v^*$.

Now consider $f(u, v)$ as a function of u, v. By the uniqueness of the solution of the differential equation, $f(u, v)$ has the property that any initial point $u_0 \, v_0$, for any $\mathcal{E} > 0$, and to there is a neighborhood N, of (u_0, v_0), such that if $(u'_0 \, v')\in N$, then the solution with initial value $(u'_0 \, v')$, is within ϵ of the solution with initial value (u_0, v_0) for $t \leq t_0$.

If we take u^*, v^* as initial point, then since $df/dt \leq 0$, and is at least momentarily negative, the solution must tend inward, staying within the C_2-curve. For any $\mathcal{E} > 0$. For sufficiently large n, (u/t_n), $v(t_n)$ is close enouth to (u^*, v^*) so that $f(u(t+t_n), v(t+t_n))$ is within $\mathcal{E}$ of the solution with initial point (u^*, v^*) for a finite time, t_0. But this means that ultimately, $f(u(t+t_n), v(t+t_n))$ moves inside the C_2 curve contradicting the hypothesis that u^*, v^* is the limiting point. Thus if $(u^*, v^*) \neq (0, 0)$, we arrive at a constrdicti. This proves that for any initial value, $f(u, v)$ tends to $f(0, 0)$ the equilibrium is stable.

III.

Let us consider two distinct groups within the population; for instance, they might be geographically separated.

Let $x_1(t)$, $y_1(t)$ be the numbers of susceptible and infected individuals in the first group. Let $x_2(t)$, $y_2(t)$ be the corresponding number for the second group. Then

$$\frac{dx_1}{dt} = - x_1[\beta_1 y_1 + \beta_2 y_2] + \mu + \theta(x_2 - x_1)$$

$$\frac{dy_1}{dt} = x_1[\beta_1 y_1 + \beta_2 y_2] - \gamma y_1 + \varphi(y_2 - y_1),$$

where θ, φ are coefficients representing the mutal immigration rates for the two groups. The equations for x_2, y_2 are obtained from those above by interchanging indices.

Let x_1^o, x_2^o, y_1^o, y_2^o be an equilibrium point for the system. Then

$$0 = \frac{dx_1}{dt} - \frac{dx_2}{dt} = (x_2^o - x_1^o)\,[\beta_1 y_1^o + \beta_2 y_2^o]$$

$$0 = \frac{dy_1}{dt} - \frac{dy_2}{dt} = (x_1 - x_2)(\beta_1 y_1^o + \beta_2 y_1^o) + \gamma[y_2 - y_1]\ ,$$

if $x_1^o \neq x_2^o$, then $\beta_1 y_1^o = \beta_2 y_2^o$; since $\beta_1\beta_2$, y_1^o, $y_2^o > 0$. This means $y_1^o = y_2^o = 0$, which is a trivial equilibrium. Thus the only nontrivial equilibrium point is

$$x_1^o = x_2^o$$

$$y_1^o = y_2^o$$

so

$$0 = -\,x^o\, y^o(\beta_1 + \beta_2) + \mu$$

$$0 = x^o\, y^o(\beta_1 + \beta_2) - \gamma y^o$$

$$\left.\begin{aligned} x_1^o = x_2^o = x_o = \frac{\gamma}{\beta_1 + \beta_2} \\[2mm] y_1^o = y_2^o = y_o = \frac{\mu}{\gamma} \end{aligned}\right\} \quad \text{is the only nontrivial equilibrium point}$$

let

$$x_i = x_0(1 + u_i); \qquad y_i = y_0(1 + v_i) \qquad i = 1,\ 2$$

Substituting these expressions into the differential equations, we obtain

$$x_o\,\frac{du_1}{dt} = -\,x_o y_o(1 + u_1)[\beta_1(1 + v_1) + \beta_2(1 + v_2)] + \mu + \vartheta\, x_o(u_2 - u_1)$$

$$y_o\,\frac{dv_1}{dt} = x_o y_o(1 + u_1)(\beta_1(1 + v_1) + \beta_2(1 + v_2)) - \gamma y_o(1 + v_1) + \varphi(v_2 - v_1),$$

and two other equations with indices interchanged.

Let

$$\alpha = \frac{\beta_1}{\beta_1 + \beta_2}\ ; \qquad \sigma = \frac{\gamma}{\mu(\beta_1 + \beta_2)}\ .$$

Then the equation becomes,

$$\left(\frac{d}{dt} + \frac{1}{\sigma} + \theta\right) u_1 + \frac{\alpha}{\sigma} v_1 - \theta u_2 + \frac{1 - \alpha}{\sigma} v_2 = 0$$

$$\gamma u_1 + \left[\frac{d}{dt} + \gamma(1 - \partial) + \varphi\right] v_1 - [\gamma(1 - \alpha) + \varphi] v_2 = 0$$

and two others with interchanged indices.

If we add each equation to the one with indices interchanged, we obtain

$$\left(\frac{d}{dt} + \frac{1}{\sigma}\right) (u_1 + u_2) + \frac{v_1 + v_2}{\sigma} = 0 .$$

$$\frac{d}{dt}(v_1 + v_2) - \gamma(u_1 + u_2) = 0 .$$

Let

$$U = u_1 + u_2$$

$$V = v_1 + v_2$$

then

$$\left(\frac{d}{dt} + \frac{1}{\sigma}\right) U + \frac{V}{\sigma} = 0$$

$$\frac{d}{dt} V - \gamma U = 0$$

or

$$\frac{dU}{dt} = - \frac{U}{\sigma} - \frac{V}{\sigma}$$

$$\frac{dV}{dt} = \gamma U$$

The matrix of coefficients is

$$\Delta = \begin{bmatrix} -\dfrac{1}{\sigma} & -\dfrac{1}{\sigma} \\ \gamma & 0 \end{bmatrix}$$

its trace is $-\dfrac{1}{\sigma}$, and

$$\det \Delta = \frac{\gamma}{\sigma} > 0 .$$

Thus both eigenvalues must have negative real parts.

Furthermore, if we go back and subtract corresponding equations, we obtain, letting $U' = u_1 - u_2$; $V' = v_1 - v_2$

$$\frac{dU'}{dt} = - (\frac{1}{\sigma} + 2\theta) u' + \frac{1 - 2\alpha}{\sigma} v'$$

$$\frac{dV'}{dt} = \gamma U' - [2\gamma(1 - \alpha) + 2\varphi] v'$$

The matrix of coefficients is

$$\Delta' = \begin{bmatrix} -\frac{1}{\sigma} - 2\theta & \frac{1 - 2\alpha}{\sigma} \\ \gamma & -2\gamma(1 - \alpha) - 2\varphi \end{bmatrix}$$

Its trace is negative and

$$\det \Delta' = 2(\frac{1}{\sigma} + 2\theta)(\gamma(1 - \alpha) + \varphi) - \frac{\gamma(1 - 2\alpha)}{\sigma}$$

$$\det \Delta' = 2[\frac{\gamma\alpha}{\sigma} + \frac{\varphi}{\sigma} + 2\theta[\gamma(1 - \alpha) + \varphi] > 0$$

so both eigenroots of this matrix have negative real parts.

These results together show that U, V, U', v' all tend to 0 as $t \to \infty$ and therefore, so do u_1, v_1, u_2, v_2 showing the equilibrium to be stable.

IV. <u>Spatially continuous densities of susceptibles and infectives</u>

Let $x(\xi, \eta, t)$, $y(\xi, \eta, t)$ be the two dimensional spatial frequencies of susceptibles infectives of time t.

Let (ξ, η) be the coordinates of the point P. Fix P, and a small directed interval starting at P, ΔP. Let $\partial/\partial P$ represent the directional derivative at P in the chosen direction. Then

$$x(P, t + dt) = x(P,t) - \beta x(P,t) dt[y(P,t) + \alpha \frac{\partial}{\partial P} y(P + \Delta P, t) - \alpha \frac{\partial}{\partial P} y(P,t)]$$

$$+ \mu dt + \theta[\frac{\partial x(P + \Delta P, t)}{\partial P} - \frac{\partial}{\partial P} x (P, t)] dt$$

as dt, $\Delta P \to 0$, we obtain,

$$\frac{\partial}{\partial t} x(P, t) = -\beta x(P, t)[y(P,t) + \alpha[\frac{\partial^2 y}{\partial \xi^2} + \frac{d^2 y}{d\eta^2}] + \mu + \theta[\frac{\partial^2 x}{d\xi^2} + \frac{\partial^2 x}{\partial \eta^2}]$$

or

$$\frac{\partial x}{\partial t} = -\beta x(y + \alpha W^2 y) + \mu + \theta W^2 x .$$

similarly, we obtain

$$\frac{\partial y}{\partial t} = \beta x(y + \alpha W^2 y) - \gamma y + \varphi W^2 y.$$

These equations should be compared with those of Part III, since they may be regarded as describing a model with a very large number of very small distinct groups.

Let us use these equations to examine the way a disease spreads from a newly introduced center of infection.

If

$$A = \beta x - y; \qquad B = \varphi + \alpha \beta x$$

we may write

$$\frac{dy}{dt} = Ay + BW^2 y .$$

Initially x is approximately constant, so that this equation is a standard diffusion equation. We shall solve it by means of the Fourier transform for spatial coordinates,

$$M(P, q, t) = \iint \exp(ip\xi + iq\eta) \, y(\xi, \eta, t) \, d\xi \, d\eta .$$

Note that $\frac{\partial M}{\partial t}$ is the Fourier transform of $\frac{\partial y}{\partial t}$, since we may differentiate under the integral.

To find the Fourier transform of $W^2 y$, consider

$$\iint \exp(ip\xi + iq\eta) \frac{\partial^2 y}{\partial \xi^2} \, d\xi \, d\eta$$

Integrating twice by parts, with respect to ξ, assuming that $y(P, t) \to 0$ as $P \to \infty$, we obtain

$$\iint \exp(ip\xi + iq\eta) \frac{\partial^2 y}{\partial \xi^2} \, d\xi \, d\eta = - P^2 \iint \exp(ip\xi + iq\eta) \, y \, d\xi \, d\eta = - MP^2$$

similarly

$$\iint \exp(ip\xi + iq\eta) \frac{\partial^2 y}{\partial \eta^2} \, d\xi \, d\eta = - Mq^2$$

Thus the Fourier transform of W^2y is $-M(p^2 + q^2)$. Combining these results with the equation,

$$\frac{\partial y}{\partial t} = Ay + BW^2 y$$

we obtain

$$\frac{\partial M}{\partial t} = AM - B(p^2 + q^2) \, dt$$

so

$$M = C \exp[At - B(p^2 + q^2)] \, t \; ;$$

C is an arbitrary constant.

Using the standard inversion formula for Fourier transforms, we obtain

$$y = \frac{1}{2\pi} \iint \exp[- ip\xi - iq\eta] \, M(p, \, q, \, t) \, dp \, dq$$

$$= \frac{C}{2\pi} e^{At} \iint \exp[- iq\xi - iq\eta] \exp[- B(p^2 + q^2) \, t] \, dp \, dq$$

$$= \frac{C}{2\pi} e^{At} \int \exp\left[- \left(\sqrt{B+P} + \frac{i\xi}{\sqrt{Bt}}\right)^2\right] dp$$

$$\times \int \exp\left[- \left(\sqrt{B+q} + \frac{i\eta}{\sqrt{Bt}}\right)^2\right] dq \left(\exp\left[- \frac{\xi^2 + \eta^2}{4Bt}\right]\right)$$

$$= \frac{C}{2\pi} e^{At} \cdot \sqrt{\pi} \cdot \sqrt{\pi} \cdot \frac{1}{Bt} \cdot \exp\left[- \frac{(\xi^2 + \eta^2)}{4Bt}\right]$$

$$y(\xi, \, \eta, \, t) = \frac{C}{2Bt} \exp\left(At - \frac{\xi^2 + \eta^2}{4Bt}\right)$$

where C is now determined by the initial conditions.

Let y_R be the total amount of infection outside a circle of radius R; since $y(\xi, \eta, t)$ is a density, we have

$$y_R = \iint_{\xi^2+\eta^2 \geq R^2} y \, d\xi \, d\eta = \frac{C}{2Bt} \int_{\xi^2+\eta^2 \geq R^2} \exp\left[At - \frac{\xi^2 + \eta^2}{4Bt}\right] d\xi \, d\eta \,.$$

In polar coordinates, this becomes

$$\frac{C}{2Bt} \cdot e^{At} \int_0^{2\pi} d\theta \int_R^\infty \exp(-\frac{r^2}{4Bt}) \, r \, dr$$

$$= \frac{C}{2Bt} e^{At} \cdot 2\pi \cdot (2Bt \exp[-\frac{R^2}{4Bt}])$$

$$= 2\pi C \exp[At - \frac{R^2}{4Bt}]$$

and thus

$$R = 2(AB)^{1/2} t\left\{1 - \frac{\log(y_R/2\pi C)}{At}\right\}^{1/2}$$

$R \to 2(AB)^{1/2} t$ as $t \to \infty$. For any fixed y_R, the circle of radius R, such that the total infection outside it is y_R, grows at the rate R/t; i.e., the velocity of propogation has the limiting value

$$2(AB)^{1/2} = 2[(Bx - \gamma)(\varphi + \alpha\beta x)]^{1/2} \,.$$