

Scaling Laws and Dynamics of Sexual Activity with Interracial and Multi-Ethnic Mixing

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Abstract

A stochastic model that simulates the processes of pair formation and dissolution among interracial and multi-ethnic mixing groups is presented. Extensive simulations are carried out with the probabilities of mixing and pair-dissolution taken from published data and specific parametric families of distributions. Scaling laws associated with the distribution of partnerships, the average number of partners and its variability are identified. Connections to recent work on scale-free and small world networks are discussed.

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1 Introduction

As late as the 1950's, interracial relationships were banned in the United States. These bans continued in the South until the Supreme Court outlawed them in 1967. Since then, interracial relationships have been on the rise, although there is still a lot of discrimination and opposition to these relationships. One of the reasons for this opposition is the concern of sexually transmitted diseases (STDs) being spread from race to race [14]. Studies have shown that certain STDs are more prevalent in certain races than in others [12, 13], and the people that are most against race mixing use this as an excuse for their beliefs [14].

We are interested in understanding how STDs spread between different race groups, but first we must understand how race mixing occurs and if it has an impact on the patterns that arise in studies of sexual activity. Previous studies have shown that the number of sexual partners for individuals follows a power law distribution [see Appendix B] [6, 7]. A study conducted at Cornell University by Castillo-Chávez, Crawford, and Schwager in 1990 reproduced these findings [3] (see Figure 1). Studies have been done on mathematical models of pair formations (see Appendix) in both homogeneous and two-sex populations. Anderson and May [1] discovered scaling law patterns between the mean (μ) and the variance (σ^2), specifically $\sigma^2 \sim \mu^\beta$. Stephen P. Blythe and Carlos Castillo-Chávez [2] proposed a mathematical model that incorporates the interactions of sexual contacts in a closed homosexual population. Blythe and Castillo-Chávez were able to capture the pattern observed by Anderson and May [1] which arose in the dynamics of sexual partner formation and separation. They found that the scaling law exponent β is not universal; different values for β are plausible. Liljeros et. al. [7] observed from analysis of a Swedish survey of sexual behavior that the cumulative distribution of the number of sexual contacts in a twelve-month period decays as a power law. Liljeros et. al [7] found that $P(k) \sim k^{-\gamma}$ where k is the number of sexual partners and $\gamma > \gamma_c = 3$ for females and males. γ_c is the threshold value above which the variance of the distribution is finite. Recently, Chowell and Castillo-Chávez [5] developed a stochastic model that captures the scaling laws reported by Anderson and May [1], and Liljeros et. al. [7] which includes the basic pair formation and dissolution process for sexual partners.

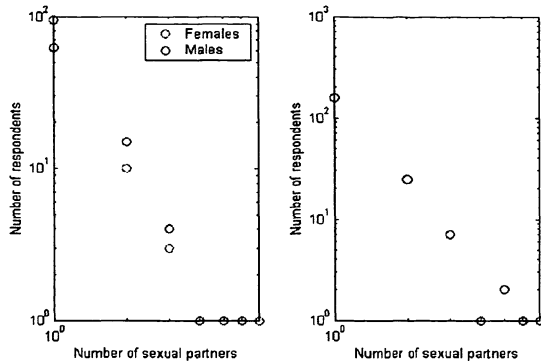


Figure 1: (left) The distributions of the number of sexual partners for females and males follow a power law of the form $P(k) \sim k^{-\gamma}$. (right) The distribution for females and males combined [3].

For our study, we plan to modify Chowell and Castillo-Chávez’s model to include race preference in pair formations. We plan to develop a stochastic model that will have four different groups, each representing a different ethnicity, and study how these different groups interact. The objective of our research is to study how patterns observed in previous studies of sexual behavior change when different race groups are introduced into the population.

Our model can be a useful tool for understanding how diseases spread between races. The fact that there exists a power law in the cumulative distribution of the number of sexual contacts indicates the possibility of “super-spreaders” being present in the population. Future modifications can be carried out on the model to include parameters like gender and age. Our model along with these modifications can simulate real world situations which have a high impact on society, such as the spread of disease and interracial relationships.

The paper is organized as follows: Section 2 explains a simple pair formation and dissolution model; Section 3 discusses our model; Section 4 discusses the numerical simulations; Section 5 presents our conclusion; and Section 6 explores future work.

2 A simple pair formation and dissolution model

Blythe and Castillo-Chàvez's [2] model describes Anderson and May's [1] observation in the approximate relationship ($\sigma^2 = a\mu^b$) between the variance and mean of the number of sexual partners taken per unit of time, where $a = 0.41$ and $b = 1.67$. In the model, s_i and f_i , are the probabilities per unit of time of the i th individual initiating a dissolution of a pair, or seeking a new partner if single, respectively.

Each individual is randomly assigned f and s values from a beta distribution. At each timestep

1. Given $\frac{s_i s_j}{\left(\frac{s_i + s_j}{2}\right)^2} < \text{rand}(0,1)$ drawn from a uniform distribution, i and j dissolve.
2. Pairs are randomly assigned.
3. Given $\frac{f_i f_j}{\left(\frac{f_i + f_j}{2}\right)^2} < \text{rand}(0,1)$ drawn from a uniform distribution, i and j form a pair (a sexual contact is made).
4. Repeat until all timesteps are completed.

race difference	number (<i>thousands</i>)	percentage %
both White	42,845	75.8
both Black	3,809	6.7
both Latino	4,739	8.4
both Other	2,059	3.6
husband White, wife Black	80	0.1
husband White, wife Latino	824	1.5
husband White, wife Other	600	1.1
husband Black, wife White	227	0.4
husband Black, wife Latino	72	0.1
husband Black, wife Other	35	0.1
husband Latino, wife White	723	1.3
husband Latino, wife Black	41	0.1
husband Latino, wife Other	35	0.1
husband Other, wife White	348	0.6
husband Other, wife Black	11	0.0
husband Other, wife Latino	35	0.1

Table 1: Interracial mixing from married couples in 2000 in the United States [11]

3 Simple pair formation and dissolution model with interracial mixing

In this section, we introduce a simple pair formation and dissolution model in a closed homosexual population with interracial mixing. We will use the following notation:

f_i : ability of the i th individual to seek a new partner

s_i : ability of the i th individual to initiate a dissolution (if paired)

$o(\Delta t)$: miniscule probability of factors other than race affecting pair-formation and dissolution

M_{ij} : race mixing probability

race	$probability(M_{ij})$
same race	.945
White, Black	.005
White, Latino	.028
White, Other	.017
Black, Latino	.002
Black, Other	.001
Latino, Other	.002

Table 2: Interracial and multi-ethnic mixing probabilities

The model will take the following form:

Given individuals i and j are single and mixing occurs, the probability that they will form a couple in the next time step is given by $P(f_i f_j) = \frac{f_i f_j}{\bar{f}_{ij}^2}$, where $\bar{f}_{ij}^2 = \frac{f_i + f_j}{2}$.

Given individuals i and j are a couple, the probability that they will dissolve in the next time step is given by $P(s_i s_j) = \frac{s_i s_j}{\bar{s}_{ij}^2}$, where $\bar{s}_{ij}^2 = \frac{s_i + s_j}{2}$. Very slight factors, other than race, are taken into account by adding a very small probability, $o(\Delta t)$. The probability that nothing will happen between individual i and j is one minus the probability of a couple forming minus the probability of a couple splitting plus the probability of other slight factors.

Our stochastic model is as follows:

$$Prob[i \text{ pairs with } j \text{ in } (t, t + \Delta t) | i \neq j, i, j \text{ are singles, and mixing occurs}] = \frac{f_i f_j}{\left(\frac{f_i + f_j}{2}\right)^2} \Delta t + o(\Delta t)$$

$$Prob[i \text{ does not pair with } j \text{ in } (t, t + \Delta t) | i \neq j \text{ and } i, j \text{ are a couple}] = \frac{s_i s_j}{\left(\frac{s_i + s_j}{2}\right)^2} \Delta t + o(\Delta t)$$

$$Prob[\text{nothing happens in } (t, t + \Delta t)] = 1 - \frac{f_i f_j}{\left(\frac{f_i + f_j}{2}\right)^2} \Delta t - \frac{s_i s_j}{\left(\frac{s_i + s_j}{2}\right)^2} \Delta t + o(\Delta t)$$

The population is separated into four ethnic groups: *White*, *Black*, *Latino*, and *Other*. The *Other* group consists of everyone that does not fit into the first three categories (i.e. Asian, Pacific Islander, multi-ethnic etc.). It is assumed that each group has the same within race probability of pairing and that the probability that individuals marry outside their race

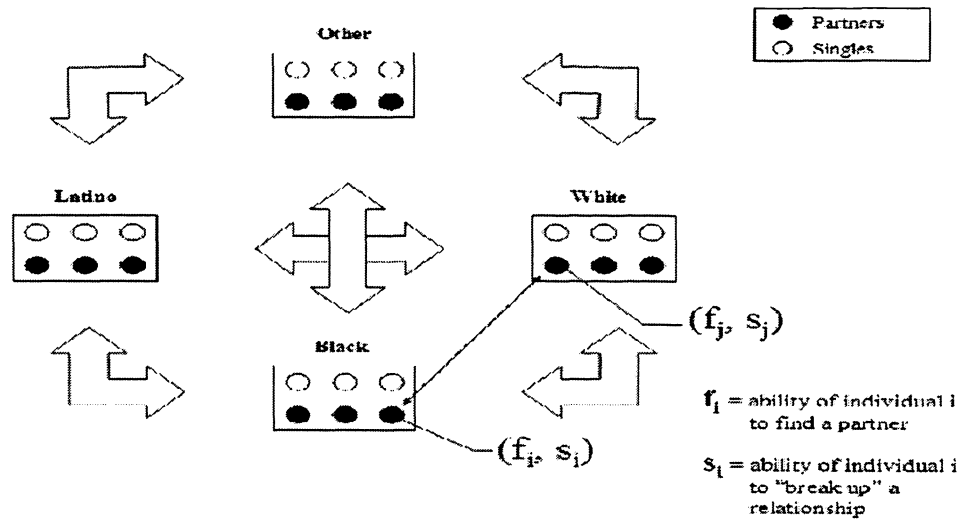


Figure 2: Schematic representation of the stochastic model for pair formation and dissolution developed in this study

is the same as the probability of having sexual contacts outside their race.¹

An algorithm was developed in order to run simulations. The algorithm consists of a loop that performs the following steps: First it takes all of the coupled individuals in the population, one couple at a time, and tests whether the probability of dissolution, $\frac{s_i s_j}{\left(\frac{s_i + s_j}{2}\right)^2}$, is less than a uniform random number with value in the interval $[0,1]$. If the probability of dissolution is less than the random number, the couple dissolves; if not, the couple remains together. Next, every unpaired individual, including those that were just "broken up", are randomly paired together. The algorithm then takes each new pair and tests whether the mixing probability (M_{ij}) is greater than a uniform random number in the interval $[0,1]$, and whether the probability of pair formation, $\frac{f_i f_j}{\left(\frac{f_i + f_j}{2}\right)^2}$, is less than a uniform random number in the interval $[0,1]$. If both of these conditions are met, that pair becomes a couple; if at least one condition is not met, the pair does not form a couple and both individuals remain single. This process is repeated for the specified number of time steps. Notice that each individual can only pair with one other individual at each time step, so people can only have

¹Appropriate data was not found for number of interracial and multi-ethnic sexual contacts, so interracial marriage data was used.

one partner at a time.

Algorithm 1 (Pair Formation and Dissolution with Racial Mixing Algorithm) *Each individual is randomly assigned f and s values from the beta distribution with $a = 1$ and $b = 5$.*

At each timestep

1. *If $(\frac{s_i s_j}{(\frac{s_i + s_j}{2})^2}) < \text{rand}(0,1)$
the couple dissolves*
2. *Single individuals (including those that are newly dissolved) are randomly paired together.*
3. *If $(M_{ij}) > \text{rand}(0,1)$ AND $\frac{f_i f_j}{(\frac{f_i + f_j}{2})^2} < \text{rand}(0,1)$
a couple forms*
4. *timestep = timestep+1*
5. *Repeat until timestep=mytimestep (where mytimestep is a specified number of timesteps)*

4 Numerical Simulations in MATLAB

Several simulations were carried out in order to observe the impact of different cases. The simulations ranged from the simplest cases, where only two groups with equal population were included, to the most complex case, where all four groups are included and have different population sizes. The probabilities of race mixing were different in each case. Some simulations had a higher probability of same race sexual contacts, and others had a higher probability of interracial sexual contacts.

4.1 Case I: Equal group size with at least one group omitted

i. Simulation 1

The population is only composed of Black and Latino individuals. For this simulation, the probability of same race sexual contacts is three times greater than the probability

Individuals	500 <i>per group</i>
timesteps	1000
realizations	30

Table 3: Initial Conditions of simulations

Race	<i>probability</i> (M_{ij})
same race	.75
Black with Latino	.25

Table 4: Conditions of simulation 1

of race mixing. This means that most of the population prefer to stay within their own race.

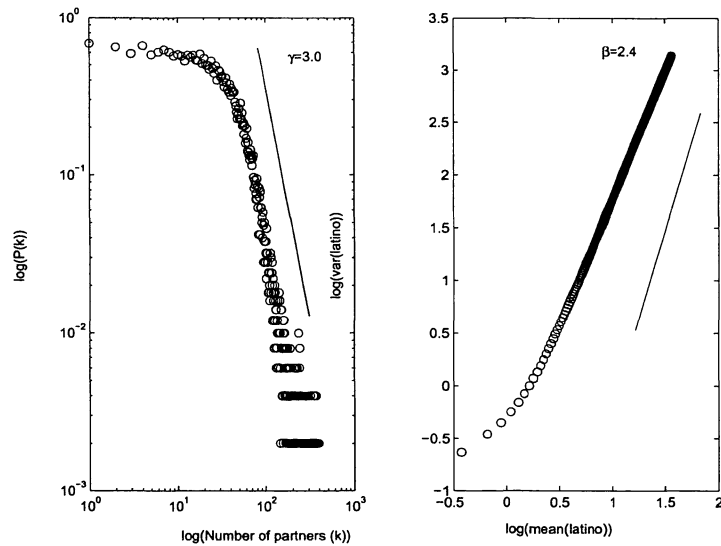


Figure 3: (left) Scaling law captured for Latino with Latino contacts by our model (right) variance vs mean of the number of sexual partners between Latino individuals.

We observed that the same race couples, regardless of race, had γ values of 3.0 (see Figure 3). For mixing between Black and Latinos, $\gamma = 3.2$. Notice that the smaller the probability of mixing, M_{ij} , the larger the γ value. The β values observed for the variance plotted against the mean were 2.4 for both groups.

ii. **Simulation 2**

Race	<i>probability</i> (M_{ij})
same race	.4
Black and Latino	.6

Table 5: Initial Conditions of simulation 2

Only two races, Black and Latino, make up the population. The probability of interracial sexual contacts is greater than the probability of same race sexual contacts. The majority of the population prefer to have sexual relations with individuals of a different race rather than with individuals from their own race.

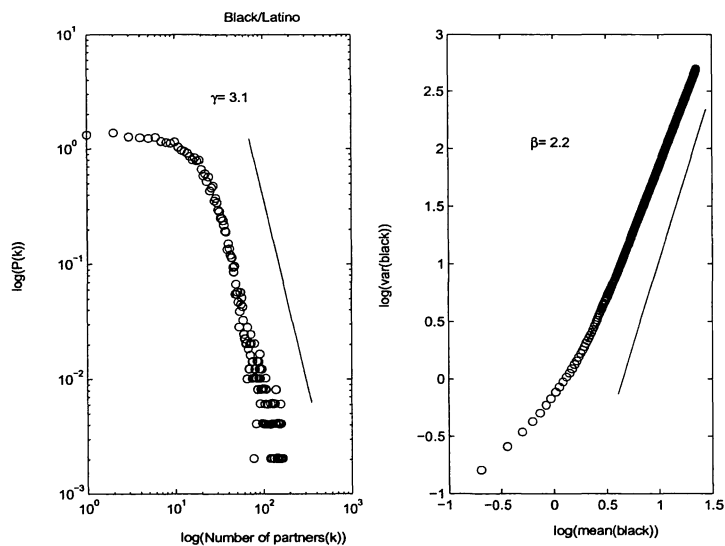


Figure 4: (left) Scaling law captured for Black with Black contacts by our model (right) variance vs mean of the number of sexual partners between Black individuals.

In this simulation we observed that the same race couples had γ values between 3.1 and 3.3, and for the mixing between Black and Latino the γ value was also 3.1. As in simulation 1, the smaller the probability of mixing, the larger the γ value (see Figure 4). The β values observed for the variance plotted against the mean were 2.2 for both groups.

iii. **Simulation 3**

Race	$probability(M_{ij})$
same race	.5
Black with Latino	.25
Black with Other	.15
Latino with Other	.1

Table 6: Initial Conditions of simulation 3

The population is broken up into three race groups: *Black*, *Latino*, and *Other*. Half of the population prefers having sexual partners within their own race. Out of the *Black* population that prefers to have sexual partners from a different race, most of them prefer *Latinos*. *Latino* and *Other* individuals are the least likely to have sexual relationships with each Other.

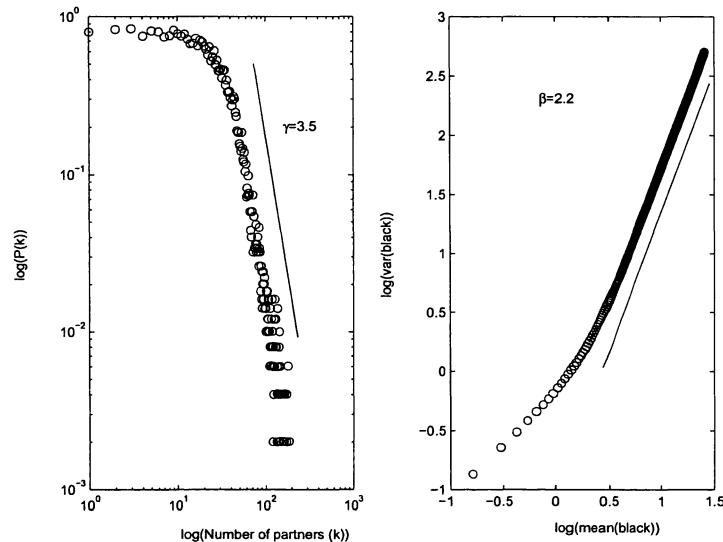


Figure 5: (left) Scaling law captured for Black with Black contacts by our model (right) variance vs mean of the number of sexual partners between Black individuals.

In this simulation we observed that the same race couples, regardless of race, had similar γ values, between 3.4 and 3.5 (see Figure 5). For the different race mixing

possibilities we obtained the following γ values.

- a. Black with Latino: $\gamma = 4.5$
- b. Black with Other: $\gamma = 4.2$
- c. There were no partnerships formed between Latinos and Other. This is probably due to the combination of a small probability, small population, and the few timesteps.

The smaller the mixing probabilities, the larger the γ value. The β values obtained from the variance plotted against the mean were 2.2 for all three groups.

4.2 Estimation of Mixing Probabilities

The probabilities of within race and between race mixing, M_{ij} are calculated using data from the 2000 U.S. Census (Table 1). The probability of same race sexual contacts are derived by adding the percentages of same race marriages. The probabilities of interracial contacts are generated by adding the percentages of different combinations of the races (Table 2). One of the difficulties that was encountered when searching for data was the lack of information on the number of interracial sexual contacts. Interracial marriage data was used as an alternative. Data on the duration of interracial marriages was also unattainable.

4.3 Case II: Equal group size with interracial probabilities derived from the Census

Individuals	500 <i>per group</i> , 4 <i>groups total</i>
timesteps	2000
realizations	100

Table 7: Initial Conditions of case II

This simulation separates the population into four race groups each having the same population. The probabilities of race mixing are those taken from Table 2. Most individuals in the population prefer to have sexual contacts with individuals from their own race.

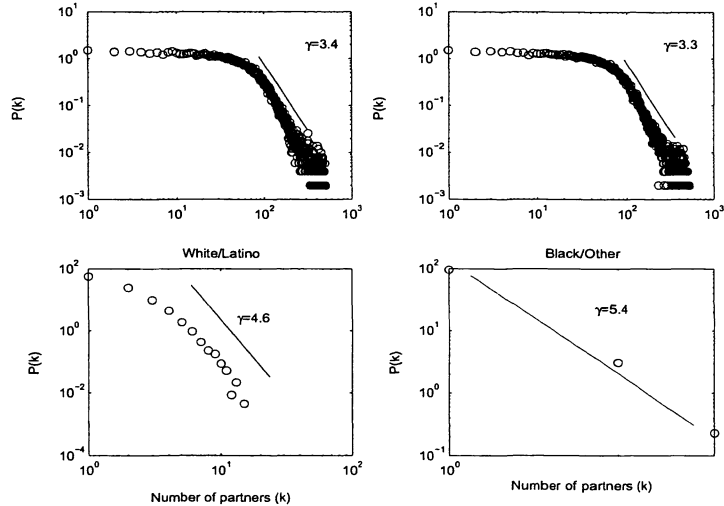


Figure 6: The distributions of the cumulative number of sexual partners between race groups generated by our stochastic model. Since there are so few data points for *Blacks* with *Others*, there was uncertainty when fitting the regression line

From this simulation we observed that the same race couples, regardless of race, had very similar γ values between 3.3 and 3.4 (see Figure 6). We expected to see this since the probabilities are the same (.945) and the population sizes are equal. An interesting observation was that the smaller the value of M_{ij} the larger the value of γ . The horizontal axis is the number of partners, so the steeper the slope the less variability in the number of partners per individual.

The variance and mean of the number of partners increases as a power law (see Figure 7), with $\beta \approx 2.4$. β remains constant regardless of the different race mixing probabilities.

4.4 Case III: Group size and interracial mixing probabilities derived from Census

This is the most complex simulation. The population is separated into four groups with population sizes proportional to the census data. The probabilities of mixing are estimated

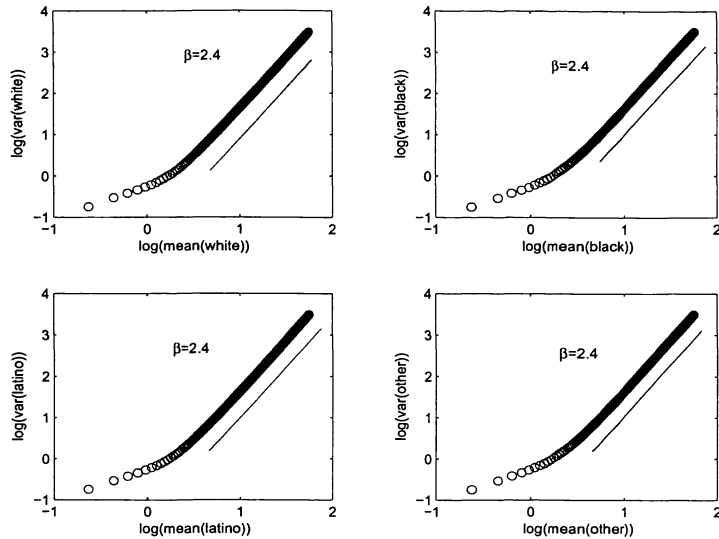


Figure 7: Variance vs. mean of the number of sexual partners within same race groups over time.

in Table 2.

We noticed some very interesting patterns in this simulation due to the differences in the population size of each group. The γ values for the same race contacts ranged from 3.4-3.8, except for the case of *Other* with *Other*, where $\gamma = 5.7$. This is most likely due to the very small population of the *Other* group. For the most part, the smaller the probability of mixing, the larger the γ value, meaning there is less variability in the number of partners per individual.

The β values for variance plotted against the mean were relatively equal (2.2-2.4). However, we noticed that the groups with the larger population sizes have the larger β values. This indicates that as population gets larger, the variability of the number of sexual contacts also increases.

5 Conclusion

Sexual contacts are well-defined networks where the nodes represent individuals and the edges represent sexual contacts. A single-sex stochastic model that simulates pair formations and

Race	<i>Individuals</i>
White	2765
Black	482
Latino	502
Other	251
timesteps	2000
realizations	47

Table 8: Initial Conditions of case III

dissolution of interracial and same race sexual contacts was proposed. Simulations were run in order to test the model. These simulations escalated from the simplest cases (two races with same population) to the most complex case (four groups each with different population sizes). The model successfully captured the scaling laws discussed by Anderson and May [1] and Liljeros et. al. [6], for each case. In each case, the smaller the value of M_{ij} , the larger the resulting γ values. This means there was less variability in the number of contacts for the groups that were least likely to form pairs. In almost every case same race couples displayed similar gamma (γ) values; the variability in the number of sexual partners was equal for same race couples. Every simulation yielded β values between 2.2 and 2.4. These values are much smaller than the β value observed by Anderson and May [1]. A possible reason for this is the fact that our population was separated into four different groups, so the populations we focused on were small. In each simulation (except for Case III), the β values were equal for each race. This means that the variability of the number of sexual contacts grows at the same rate over time regardless of race. In Case III, the groups that had the larger populations tended to have the larger β values, which shows that variability in the number of sexual contacts over time grows as population grows. The model can increase our understanding of the impact of the few individuals that have many sexual partners, and how an STD can explode in a population.

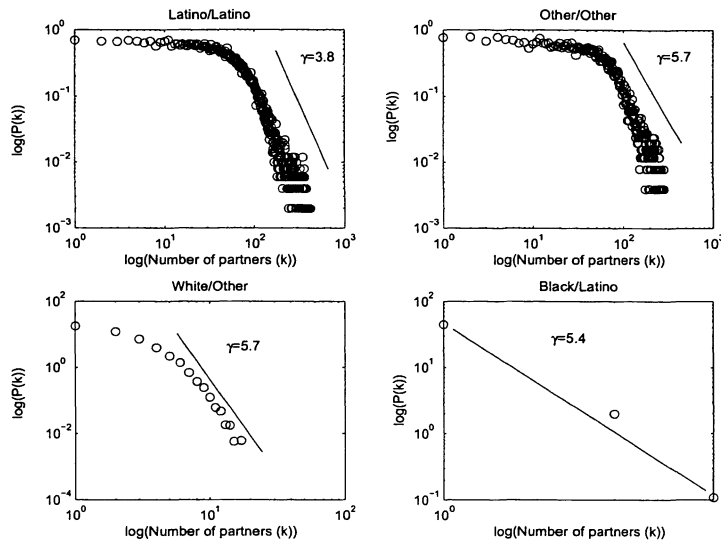


Figure 8: The distributions of the cumulative number of sexual partners between race groups generated by our stochastic model.

6 Future Work

A future goal is to add gender to the model and study the patterns that form. This can broaden the application of the model because it can be more representative of the population. Other modifications may be added such as age and demographics. This can have a large impact on society as far as racial boundaries and social policies. We would like to find a deterministic model that describes the pair formation process when these preferences are added because there are more mathematical tools to do analysis on these types of models.

7 Acknowledgments

We would first like to thank Carlos Castillo Chávez and Stephen Wirkus for giving us the opportunity to conduct this research and the advice they provided. We would also like to thank all the MTBI. faculty and graduate students for their guidance and support.

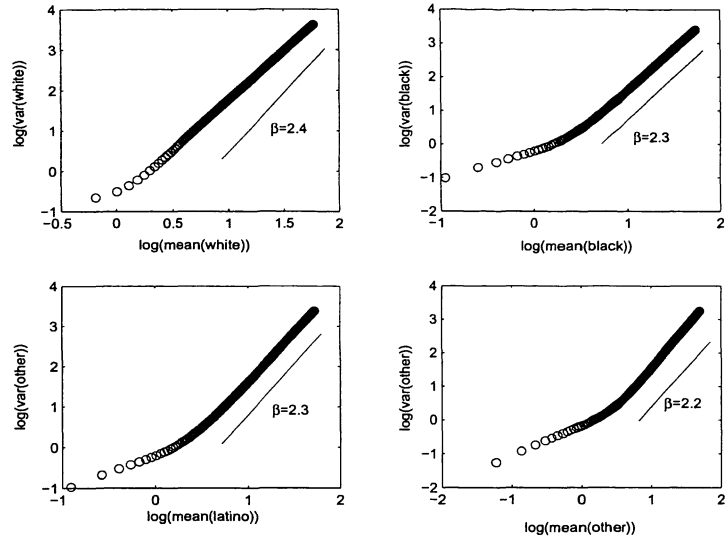


Figure 9: Variance (σ^2) vs mean (μ) of the number of sexual partners within same race groups over time.

8 Appendices

Appendix A

Beta Distribution Review

The probability density function of the beta distribution is

$$F(x) = \frac{1}{B(\alpha, \beta)} (x^{\alpha-1})(1-x)^{\beta-1}$$

where $0 < x < 1$ and $B(\alpha, \beta)$ is the beta function with formula

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

$$0 \leq x \leq 1$$

and α, β are constants > 0 .

It has mean

$$E(X) = \frac{\alpha}{\alpha + \beta}$$

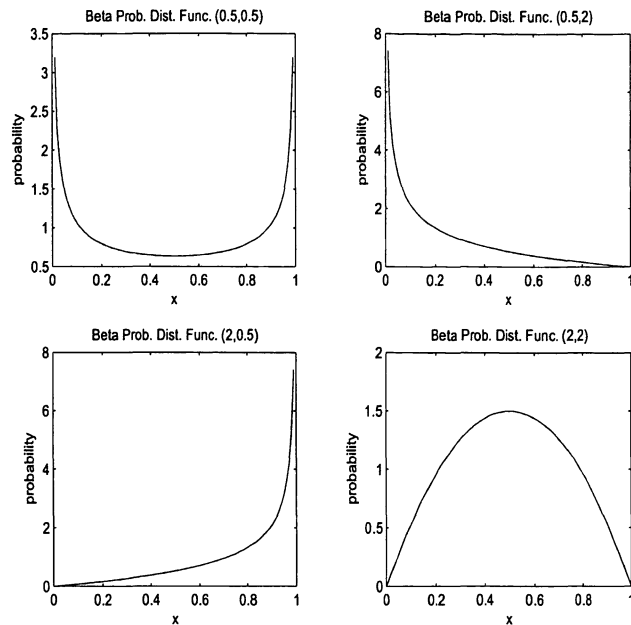


Figure 10:

Shapes of the Beta distribution with $(\alpha, \beta) = (.5, .5), (.5, 2), (2, .5), (2, 2)$

and variance

$$Var(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

α and β are known as “shape parameters” because different values of each result in different shapes for the distribution curve [8] (see Figure 3). As you increase the value of α , the distribution becomes left-tailed. The greater the value of α , the longer the tail. As you decrease α , the left tail starts to curl upward, and you end up with a U-shaped curve. As you increase the value of β , the distribution becomes right-tailed. As with α , the greater the value of β , the longer the tail becomes. As you decrease β , the tail curls upward, and you end up with a U-shaped curve.

Appendix B

Pareto Distribution Review

The probability density function of the Pareto Distribution is

$$F(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}$$

where $x \geq \alpha$, and $\alpha, \beta > 0$

[9] It has mean

$$E(x) = \frac{\beta\alpha}{\beta - 1}$$

where $\beta > 1$ and variance

$$Var(x) = \frac{\beta\alpha^2}{(\beta - 1)^2(\beta - 2)}$$

where α is known as a scale parameter and β is known as a shape parameter [10] (see Figure 4). As you increase the value of α the graph is shifted up and to the right away from the origin. The mean and the variance increase as well. As you decrease the value of α , the graph remains the same but is shifted down and to the left towards the origin, and the mean and variance decrease. As you increase the value of β the shape of the graph differs and the slope of the graph increases. For example, when β is equal to one the graph is concave up, while when β is equal to two the graph becomes concave down. Also, the mean and variance decrease. When β decreases, the graph goes from concave down to concave up and the slope of the graph decreases, and the mean and the variance increase.

The Pareto distribution is also referred to as a power-law distribution [10]. Pareto distributions are linear only in log-log plots because the function

$$F(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}$$

which can also be written as

$$F(x) = \beta\alpha^\beta x^{-(\beta+1)}$$

where $0 < x < 1$.

Since $\beta\alpha^\beta$ is a constant, let

$$c = \beta\alpha^\beta$$

and

$$\gamma = -(\beta + 1)$$

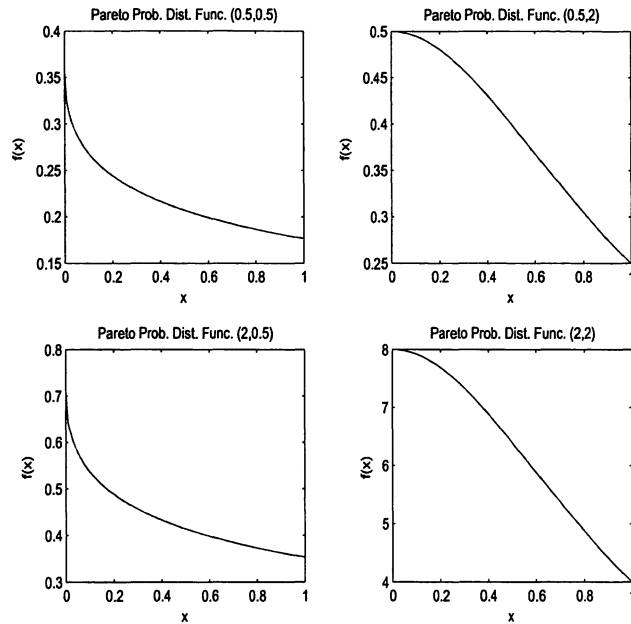


Figure 11:

Shapes of the Pareto Distribution with $(\alpha, \beta) = (.5, .5), (.5, 2), (2, .5), (2, 2)$

Taking the logarithm of both sides,

$$\log F(x) = \log c - \gamma \log x$$

and let

$$u = \log x$$

and

$$v = \log F(x)$$

So,

$$v = \log c - \gamma u$$

Which is a line with slope γ .

Appendix C

A Two-Sex Discrete Pair Formation Model

Castillo-Chávez and Yakubu [4] proposed a discrete two-sex pair formation model which is an extension of the Malthus model (1). This model includes single females ($x(t)$), single males ($y(t)$) and pairs ($p(t)$). Survival probabilities and birth rates are taken into account but are gender specific.

$$\begin{aligned} P(t+1) &= \lambda P(t) \\ P(0) &= P_0 \end{aligned} \tag{1}$$

The model takes the form:

$$\begin{aligned} x(t+1) &= \beta_x \mu_x \mu_y p(t-k) + [(1-\sigma)\mu_x \mu_y + (1-\mu_y)\mu_x]p(t) + \mu_x x(t) - \phi(x(t), y(t), p(t)) \\ y(t+1) &= \beta_y \mu_x \mu_y p(t-k) + [(1-\sigma)\mu_x \mu_y + (1-\mu_x)\mu_y]p(t) + \mu_y y(t) - \phi(x(t), y(t), p(t)) \\ p(t+1) &= \sigma \mu_x \mu_y + \phi(x(t), y(t), p(t)) \end{aligned}$$

where μ_x = The constant survival probability of females

μ_y = The constant survival probability of males

β_x = The constant per capita birth rate of females

β_y = The constant per capita birth rate of males

k = The delay constant

$(1-\sigma)$ = The divorce rate

ϕ = Marriage function

the number of single females in the next time step, $(x(t+1))$ = probability that both female and male in couple survive and give birth to a female that enters the single female population at time delay k , $(\beta_x \mu_x \mu_y p(t-k))$ + probability of divorce if both individuals in pair survive, $((1-\sigma)\mu_x \mu_y p(t))$ + probability male in couple dies and female survives, $((1-\mu_y)\mu_x p(t))$ + single surviving females $(\mu_x x(t))$ - females that are paired off, $(\phi(x(t), y(t), p(t)))$

The number of single males in the next time step is the same equation but with the males

birth rates and survival probabilities.

the number of paired individuals in the next time step $(\sigma\mu_x\mu_y)$ = the pairs that did not split up, $(\sigma\mu_x\mu_y)$ + the new pairs coming in, $(\phi(x(t), y(t), p(t)))$

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