COLLATERAL

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The theme of this dissertation is Collateral. The first chapter focuses on the meaning of collateral value. The main idea is that collateral value is the expected liquidation price conditional on the borrower's default, which is not necessarily the distressed value of the underlying asset. The second chapter compares the price movement of an asset before and after it has been used as collateral. A "V" shaped price path is documented for the asset used as collateral, suggesting a possible market instability.

BIOGRAPHICAL SKETCH

Liheng Xu received his B.S. in Statistics at Peking University. While at Peking University, he also studied actuarial science. He earned a M.A. at Indiana University in Probability and Statistics. He has been a Ph.D. student at Finance Department in Cornell University since 2007.

This Doctoral dissertation is dedicated to my wonderful family.

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For any errors or inadequacies that may remain in this work, of course, the responsibility is entirely my own.

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CHAPTER 1

AN ASSET PRICING AND BANK LENDING EQUILIBRIUM WITH COLLATERAL

1.1 Introduction

Conventional asset pricing and corporate finance theories assume a partial equilibrium lending market in which an agent or firm borrows against its own creditworthiness. Specifically, many of them assume agents can lend to each other, as in asset pricing literature, or the lending market is exogenously given and lenders have no active role other than to answer borrowing requests, as in corporate finance literature. In reality, neither of these is true. Banks perform the majority of lending, while actively managing credit risk, especially a portfolio of credit risk. The pursuant of managing the portfolio of credit risk raises serious doubts about the assumption that creditworthiness only matters at the individual level, instead, banks' portfolio view suggests that it should be determined in the aggregate economy, in the same spirit as in Modern Portfolio Theory: the risk comes from variance as well as covariance. Therefore, the lending market assumed in a partial equilibrium model no longer applies to general equilibrium.

The major theories in capital asset pricing–the CAPM–as well as those in corporate finance–the Modigliani and Miller theory, the pecking order theory and the tradeoff theory–are all based on a partial equilibrium lending market. Therefore, it may not be appropriate to apply them to explain the aggregate economy or cross section variations. What we need is a new lending market determined in general equilibrium, taking into consideration all the individuals' borrowing activities.

To achieve this, we model a lending market in which collateral plays an explicit role. Collateral is used to buffer against contractual defaults and thus is able to shift the credit risk from the single borrower to the collateral value. Depending on whether the lender will liquidate collateral or hold it till maturity, the collateral value can be determined by the liquidation price or the fundamentals of the underlying asset. On many occasions, lenders carry a cost to maintain collateral, such as in the mortgage industry, or they may fear a further loss, particularly in a crisis, so to liquidate collateral is a common action. The liquidation price is not the unconditional expected selling price, but one conditional on the borrower's default. Therefore, the liquidation price reflects the covariance between the borrower's wealth and the aggregate economy. As a result, collateral value is not just a function of the asset, but also of the borrower. A simple haircut defined as the ratio of collateral value to the market price is not able to fully reflect this information.

Moreover, by including collateral, we can study the price impact of an asset after its use as collateral. This is important because there is an increasing number of securities that can serve as collateral due to financial innovations. The recent financial crisis in 2008 has also involved the use of collateral. Chapter 2 provides a solid study of this issue.

We construct a two-period economy with three entities–entrepreneurs, investors and banks–and two endowed assets–cash and a collateralizable asset.

Entrepreneurs are distinguished from investors by their private investments. They have three ways to finance their private investments: endowed cash, proceeds from selling the endowed collateralizable asset to investors, or a loan from a bank which requires this asset as collateral. In the case of borrower default, the bank will liquidate collateral at a price dependent on all the potential buyers' cash holdings (see Oehmke 2008 for dynamically liquidating collateral). By making the borrowing capacity equal to this liquidation price, the borrowing activities for all agents are simultaneously determined in general equilibrium. We call this an endogenous lending market, in contrast to the exogenous one determined by the partial equilibrium in which no interaction exists among all the borrowing activities.

This paper proves that this endogenous lending market is indeed different from two commonly-used exogenous ones: the one with borrowing constraints (restricted lending market) and the one without (unlimited lending market). Two implications arise from this difference.

Firstly, in the endogenous lending market, the collateralizable asset price is less sensitive to the change in returns on these private investments in the endogenous lending market than in the restricted one; the reason is that the substitution effect between the asset and private investments can be partially offset by the wealth effect that exists only in the former market.

Secondly, in the endogenous lending market, banks facilitate diversification among entrepreneurs' choices of private investments. We demonstrate this by giving them two private investment options with different profitabilities. In the existing capital budgeting literature that assumes an exogenous lending market, entrepreneurs always choose the more profitable one. In contrast, in the endogenous lending market, they might choose the less profitable one in equilibrium, due to subsidized loans from banks. Banks have incentives to offer such loans to avoid dealing with entrepreneurs making the same choice, as in the idea of the Modern Portfolio Theory. In an extreme case, consider that the less profitable investment has a return negatively correlated with the aggregate economy. If an entrepreneur borrows to finance this investment, the bank can either obtain a full repayment from the borrower or sell collateral when the aggregate economy is doing well. In either case, the bank faces little risk. Therefore, the loan could be attractive enough to the borrower to compensate for the reduced profitability. In one sense, the loan from the bank subsidizes the less profitable investment, or put it another way, leverage is essential to carry a less profitable investment. This prediction is in line with the existing literature on firms' capital structure and profitability (Hail and Weiss 1967 and Gale 1972). In contrast to the pecking order theory that only considers the firm's perspective, we provide a bank-firm joint analysis.

We also show that banks value collateral the least if they face a pool of identical entrepreneurs, in the same spirit as Shleifer and Vishny (1992). These authors take a game theory approach to endogenize the collateral value. In contrast, we emphasize the role of correlations among agents' wealth in determining this value.

Our model of collateral equilibrium has its root in a series of papers by Geanakoplos, such as Geanakoplos (1997) and Geanakoplos (2003). Geanakoplos (2003) emphasizes the role of collateral in leverage and asset price crashes, whereas we focus on the role of collateral in forming an endogenous lending market. Moreover, Geanakoplos (2003) attributes the higher price of an collateralizable asset to the excessive demand by the use of leverage, on the other hand, we argue that it comes from the action of using the asset as collateral per se. As long as some entrepreneurs use the asset as collateral to borrow, the price will be higher than in the situation where all entrepreneurs crowd the market to sell. In other words, the action of using the asset as collateral is sufficient to cause a higher price, regardless of the purpose. Therefore, we broaden the scope to explain the higher price caused by this use of collateral.

Previous studies on bank lending and collateral have mainly explored issues in partial equilibrium, such as asymmetric information between borrowers and lenders (Besanko and Thakor 1987; Bester 1987), and the quality of collateral (Plaut, 1985). A key difference between studies in the partial and general equilibrium lies in the valuation of collateral. In the case of partial equilibrium, the collateral value is exogenously given. Whereas in the general equilibrium, the collateral value is endogenously determined by all the agents.

Bank lending is related to credit constraints in macroeconomics. Bernanke and Gertler (1989), Bernanke, Gertler and Gilchrist (1997) have shown that credits based on a borrower's balance sheet may have a pro-cyclical effect on the business cycle. Kiyotaki and Moore (1997) show how an exogenous shock to the economy has ripple effects across time, further amplified by the use of collateral. They all highlight the role of banks in exaggerating the business cycle across time, but in the cross-sectional view, our model indicates that banks also tend to attenuate variation by facilitating a diversified economy.

Lastly, this study is related to an extensive literature on banking. Most existing banking theories examine credit channels cross sectionally by focusing on the mechanism through which banks acquire capital from savers and transfer it to lenders. Diamond and Dybvig (1983), Diamond and Rajan (2000), Archaya, Gorton and Metric (2009) and Gale and Yorulmazer (2010) study the risks when banks operate lending with short term funding. Stiglitz and Weiss (1981) study credit rationing in an asymmetric information environment. In this study, we link the lending market to the future wealth of the aggregate economy via collateral, and examine the role of the credit channel across time.

1.2 The Model

This section provides the model structure.

1.2.1 The Agents

We construct an economy consisting of two time periods 0 and 1, and a continuum of agents, denoted by agent-i $i \in [0, 1]$. The risk free rate is $r = 1$. All the results hold with a risk free rate greater than one, $r > 1$. At time 0, for $i \in [0, 1]$, agent-i is endowed with $e_i(0)$ units of a collateralizable asset x and $e_i(1)$ units of cash, denoted by $\tilde{e}_i = (e_i(0), e_i(1))$. x can only be traded at time 0 in a competitive market where all agents are price takers. At time 1, x generates a bounded positive random payoff $x \in [x_{min}, x_{max}]$.

For a fraction of agents indexed by $[0, \delta]$, assume each also owns a private investment available only to himself, denoted by y_i for agent-i. Those agents are called entrepreneurs. The others in $(\delta, 1]$ are called investors. If entrepreneur-i spends cash c on private investment y_i, he will obtain cy_i at time 1 where $y_i \in R^+$ is a random return. All y_is are independent of x and identically distributed with a distribution function $\tilde{y}$. Consider two cases: all y_is are independent and all y_is are identical. Denote by $\Phi_{in} = \{y_i | i \in [0, \delta]$ and $y_i s$ are independent.$\}$ and

$\Phi_{id} = \{y_i | i \in [0, \delta]$ and $y_i s$ are identical.$\}$

Assume all entrepreneurs are symmetric, meaning, they have the same utility function and endowments, denoted by u and $(e(0), e(1))$, respectively. To rule out corner solutions, assume investors have a sufficient amount of cash to buy all x from entrepreneurs, that is,

Condition 1.2.1 (Sufficient Cash) $\int_\delta^1 e_i(1)di > \mu_x \int_0^\delta e_i(0)di$,

where $\mu_x = \mathrm{E}x$ is the expected payoff of x.

All agents maximize the expected utility of their final wealth at time 1. To uniform notations, define $y_i = 0$ for investors $i \in (\delta, 1]$. Denote by u_i agent-i's utility function satisfying

(i) $u' > 0$, $u'' \leq 0$,

(ii) u'' is continuous.

At time 0, agent-i chooses a wealth portfolio $w_i = (a_i, b_i, c_i)$ to maximize

$$Eu_i(a_i x + b_i + c_i y_i) \tag{1.1}$$

subject to

(i) $a_i p + b_i + c_i = e_i(0)p + e_i(1)$,

(ii) $a_i \geq 0$, $b_i \geq 0$ and $c_i \geq 0$,

where a_i is the asset x position, b_i is the cash position, c_i is the security y_i position and p is the market price of x. Short sale of asset x is not allowed, for short sale is a lability to short sellers and such liability cannot be enforced in a weak enforcement environment. The optimal demand function for x is denoted by $\bar{a}_i$. $\bar{a}_i$ is a function of the endowments $(e_i(0), e_i(1))$ and the price p. Write $\bar{a}_i$ as $\bar{a}_i(p, e_i(0), e_i(1))$.

For investors, assume as a group they are willing to hold all x if the price p is small enough:

$$\lim_{p \to 0} \int_{\delta}^{1} \bar{a}_i(p, e_i(0), e_i(1)) di > \int_0^1 e_i(0) di. \tag{1.2}$$

The definition of the market equilibrium is as follows.

Definition 1 (Market Equilibrium) *At time 0, the market for x is in equilibrium if the following conditions are satisfied:*

(1)Both entrepreneurs and investors maximize their utilities;

(2)The market for x clears.

Because entrepreneurs are symmetric, they have the same optimal demand function for x and invest the same cash in security y_is.

The primary idea of this study is to compare the different means for entrepreneurs to raise cash for their private investments: selling asset x or use it as collateral to borrow. If both entrepreneurs and investors start with sub-optimal wealth portfolios, they may still want to trade in the market just to rebalance their portfolios, even without any private investment. To focus on the entrepreneurs' trading incentives for raising cash for their private investments, the following starting equilibrium condition is imposed. The condition stipulates that there is no need for all the agents to trade in the market, if entrepreneurs have no private investments.

Condition 1.2.2 (Starting Equilibrium) *Without private investment y_is, all the agents are in equilibrium with their endowments. In other words, there exists a price p^* such that, for $i \in [0, 1]$, agent-i's endowments $(e_i(0), e_i(1))$ maximize*

$$Eu_i(ax + b) \tag{1.3}$$

subject to

(i) $ap^* + b = e_i(0)p^* + e_i(1)$*,*

(ii) $a \geq 0$*, and* $b \geq 0$*.*

The next condition is to make sure the trades flow one direction: it's the entrepreneurs that sell asset x. Now, entrepreneurs have to compare whether to sell x or use x as collateral to borrow.

Condition 1.2.3 (Capital Competing) *Entrepreneurs demand less* x *after they have* y_i*s, that is, for all* $i \in [0, \delta]$*,*

$$a_i^*(p, e_i(0), e_i(1)) > \bar{a}_i(p, e_i(0), e_i(1)), \tag{1.4}$$

where a_i^* *is the optimal holding of* x *for entrepreneur-i in condition* **2.2.1***. In a sense, the private investment* y_i*s compete with* x *for capital.*

1.2.2 The Banks

Assume banks have perfect information on each agent. Banks only offer a discount loan. This discount loan is one on which the interest is deducted from the face amount when the loan is offered. The borrower only receives the principal after the interest is deducted but must repay the full amount of the loan. Assume R is the interest rate charged by banks. Further assume that the loan is nonrecourse and the law to enforce repayment is weak. Therefore collateral is the only instrument for banks to protect against loan losses. To be specific, when a borrower defaults, the bank can only seize the collateral, but has no right to claim the borrower's wealth beyond that. Therefore, the value of collateral has to be large enough to fully cover loan losses. If the loan principle is l, the value of collateral has to be greater than or equal to l. Assume there are a sufficient number of banks in the competitive lending business and each is endowed with sufficient cash as capital. Competition implies that banks demand collateral worth the same as the loan principle. Moreover, given that the loan losses have been fully covered by collateral, banks must earn zero profits on the repayment, charging the risk free rate for the loan, that is, $R = r = 1$.

After banks seize the collateral, assume they will liquidate it immediately, instead of holing it till maturity. There are two major reasons for banks to do so: maintenance costs and further deterioration of the collateral value. The repayment date must be before the payoff of x. In addition, this timing makes it impossible for all agents to finance the purchasing of x; this confines the use of borrowed cash solely on y_i, for $i \in [0, \delta]$, and makes entrepreneurs the only borrowers.

At time 0, entrepreneurs use x as collateral to borrow from banks, and repay loans at time 1 between the realization of y and x. The sequence of time 1 events is illustrated in figure **1.1**.

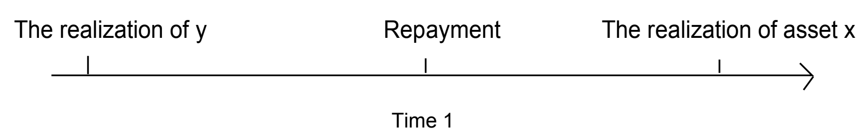

Figure 1.1: The sequence of the three events at time 1

The lending policy is summarized as the following rule.

Rule 1.2.1 (Lending Policy) *For a loan with principle l, banks require the same value*

l of asset x as collateral and charge a risk free rate $R = 1$. This is equivalent to assuming that banks have zero Value at Risk in a competitive lending business.

Since it does not matter which bank an entrepreneur borrows from, we can view all the banks together as one aggregate bank, or the bank. It is sufficient to study the behavior of this bank that operates according to the lending policy described in rule **1.2.1**.

A. Entrepreneurs' optimal borrowing

When the bank calculates the collateral value, it needs to estimate the total quantity of collateral to be liquidated in the market at time 1 when borrowers default. Therefore the bank has to study the borrowers' repayment behavior. To keep the model simple, assume there is no renegotiation between entrepreneurs and the bank once entrepreneurs default on the loan. Entrepreneurs can choose to either repay or default on the entire loan. All the result in this section hold if the model is extended to a simple type of renegotiation: partial repayment[1].

Denote by $v(\beta)$ the collateral value for β units of asset x calculated by the bank, for $\beta \in [0, 1]$. For entrepreneur-$i \in [0, \delta]$, assume he uses β_i units of x as collateral to borrow $\beta_i v(\beta_i)$ at time 0. Assume he has also used endowed cash $e(1) - b_i$ for y_i. At time 1 after the payoff of y_i, he decides whether to repay the loan, that is, he chooses 1_{F_i} to maximize

$$Eu_i((e_i(0) - \beta_i)x + (e(1) - b_i + \beta_i v(\beta_i))y_i + b_i + 1_{F_i}(\beta_i x - \beta_i v(\beta_i))) \tag{1.5}$$

[1]The partial repayment rule is:

Rule 1.2.2 (Partial Repayment Rule) *When repaying the loan, entrepreneurs can repay a fraction of it to redeem the collateral at the same ratio, that is, repay κl to redeem $\kappa\beta$ units of x, where $\kappa \in [0, 1]$.*

subject to

(i) $1_{F_i} \in \{0,1\}$, and

(ii) $(e(1) - b_i + \beta_i v(\beta_i))y_i + b_i - 1_{F_i}\beta_i v(\beta_i) \geq 0$.

$1_{F_i} = 1$ means entrepreneur-i repays the loan fully and $1_{F_i} = 0$ means he defaults on the loan completely. 1_{F_i} is a function of y_i, written as $1_{F_i}(y_i)$. By symmetry, in equilibrium, all borrowers hold the same portfolio at time 0 and hence have the same repayment function. The subscript "i" in 1_{F_i} can be dropped. Rewrite it as $1_F(y_i)$. Denote by $D = \{d | 1_F(d) = 0\}$ the set of returns for y causing the borrowers to default. In other words, the bank is to liquidate collateral when the borrowers have returns from y_is in the set D. Without further regulations on the entrepreneurs' utility function u, it may always be optimal for them to default on the loan. Two conditions regarding the positive wealth effect and the downward sloping demand curve are required to discipline entrepreneurs' repayment behaviors. Entrepreneurs' willingness to repay the loan is consistent with that in a repeated borrowing environment in which their reputation of repaying the loan is considered by a bank as an significant factor.

Condition 1.2.4 (Positive Wealth Effect) *For the demand function a_i^* in condition **2.2.1**,*

$$\frac{\partial a_i^*}{\partial e_i(1)} \geq 0 \tag{1.6}$$

holds for all $i \in [0,1]$.

Condition 1.2.5 (Downward Sloping Demand Curves) *The demand function a_i^* in condition **2.2.1** is a decreasing function of price p, $\frac{\partial a_i^*}{\partial p} \leq 0$, for all $i \in [0,1]$.*

Given these conditions, the entrepreneurs' repayment strategy is summarized as a proposition.

Proposition 1 (Optimal Repayment) *There exists a critical point y^* such that an entrepreneur is willing to repay the loan if $y \geq y^*$ and default completely if $y < y^*$, and another y^{**} such that he has the ability to repay the loan if $y \geq y^{**}$. To sum up, $D = [y_{min}, y^* \vee y^{**})$.*

Proof. Denote by β and c the amount of asset x an entrepreneur uses as collateral to borrow and hold in cash, respectively. Given two returns $y_2 > y_1$, it suffices to show that if the entrepreneur is willing to repay fully the loan with return y_1, he must be willing to do so with return y_2. Denote by $u_i = Eu(e(0)x + (e(1) - c + \beta v(\beta))y_i - \beta v(\beta))$. With return y_2, the ability to repay the loan is not an issue for he has more cash. Denote by $A(u, a)$ the minimal amount of cash together with a units of x to generate the utility u. Since the entrepreneur is willing to repay the loan with return y_1, it immediately follows that $A(u_1, e(0) - \beta) - A(u_1, e(0)) > \beta v(\beta)$. It's sufficient to show $A(u_2, e(0) - \beta) - A(u_2, e(0)) > A(u_1, e(0) - \beta) - A(u_1, e(0))$. Denote by $P(a, b)$ a price function such that the entrepreneur optimally holds a units of x and b units of cash. According to the first order condition $\frac{\partial A(u,a)}{\partial a} = -P(a, A(u, a))$,

$$\int_{e(0)-\beta}^{e(0)} P(a, A(u_1, a))da = A(u_1, e(0) - \beta) - A(u_1, e(0)) > \beta v(\beta) \qquad (1.7)$$

If $p(a, A(u_2, a)) > P(a, A(u_1, a))$ holds, then it's true that the entrepreneurs are willing to repay the loan with return y_2, according to equation (1.7). From $A(u_2, a) > A(u_1, a)$ and condition **2.2.5**, it must be $a = a^*(P(a, A(u_2, a)), a, A(u_2, a)) > a^*(P(a, A(u_2, a)), a, A(u_1, a))$. Since $a^*(P(a, A(u_1, a)), a, A(u_1, a)) = a$, it must follow $P(a, A(u_1, a)) < P(a, A(u_2, a))$ from condition **2.2.2**.

■

After solving $1_F(y_i)$, the entrepreneur-i at time 0 chooses a wealth portfolio (β_i, b_i) to maximize

$$Eu((e(0)-\beta_i)x+(e(1)-b_i+\beta_i v(\beta_i))y_i+b_i+1_F(y_i)(\beta_i x-\beta_i v(\beta_i))) \quad (1.8)$$

subject to

(i) $0 \leq \beta_i \leq e(0)$ and

(ii) $0 \leq b_i \leq e(1)$.

B. The bank's collateral valuation

This bank assesses the collateral, not by its fundamental value, but by market value. Fundamental value is the utility obtained from consuming or holding the asset. The market value is how much one receives when selling it in the market. The bank can only seize and liquidate collateral when their borrowers default at time 1. Since the market is closed at time 1, the bank sells x to all the agents over the counter. In fact, it quotes an asking price for them to purchase.

The way for the bank to compute the collateral value is "guess and verify later". The bank guesses in equilibrium, entrepreneurs in $[0,\delta_m]$ transact in the market, and those in $(\delta_m,\delta]$ borrow with collateral, in which $0 \leq \delta_m \leq \delta$. In the market, according to the symmetric maximization problem (1.1), each entrepreneur in equilibrium has the same wealth portfolio, denoted by $w_{\delta_m} = (a_{\delta_m}, b_{\delta_m}, c_{\delta_m})$ as a function of δ_m. The equilibrium price and investor-i's wealth portfolio are denoted by p_{δ_m} and $w_{i\delta_m} = (a_{i\delta_m}, b_{i\delta_m}, c_{i\delta_m})$, respectively. It can be seen that, in equilibrium, the wealth portfolios for all agents in $[0,\delta_m]\bigcup(\delta,1]$ are solely determined by the variable δ_m.

Entrepreneurs in $(\delta_m,\delta]$ choose to borrow from the bank. By symmetry, in equilibrium, they use the same quantity of x as collateral to borrow the same amount of cash from the bank. Assume each uses β units of collateral and the collateral value computed by the bank is $v(\beta)$. The valuation depends on the correlations among y_is. The difference can be seen from the following two special cases.

Case 1: all y_is are independent

Because there are a continuum of entrepreneurs in $(\delta_m, \delta]$, the measure of defaulting entrepreneurs is exactly $(\delta - \delta_m)\Pr(\tilde{y} \in D)$. At time 1, there'll be exactly $(\delta - \delta_m)\beta \Pr(\tilde{y} \in D)$ units of x to be liquidated by the bank. The wealth portfolios for all borrowers are functions of δ_m and $v(\beta)$. In the market, the equilibrium portfolios for all market participants are functions of δ_m. Define a set including all agents' wealth information at time 1, $\hbar(\delta_m, \beta, v(\beta)) = \hbar_{[0,\delta_m]} \bigcup \hbar_{(\delta_m,\delta]} \bigcup \hbar_{(\delta,1]}$ where $\hbar_{[0,\delta_m]}$, $\hbar_{(\delta_m,\delta]}$ and $\hbar_{(\delta,1]}$ are the wealth information sets for the entrepreneurs in the market, the entrepreneurs with the bank and the investors, respectively. They are

$$
\begin{aligned}
\hbar_{[0,\delta_m)} &= \{w_{1i} = (a_{\delta_m}, ((e(0) - a_{\delta_m})p_{\delta_m} + e(1) - c_{\delta_m})y_i + c_{\delta_m})| i \in [0, \delta_m)\}, \\
\hbar_{(\delta_m,\delta]} &= \{w_{1i} = (e(0) - \beta + \beta 1_F(y_i), (e(1) - b_i + \beta v(\beta))y_i \\
&\quad + b_i - \beta v(\beta) 1_F(y_i))| i \in (\delta_m, \delta]\} \; and \\
\hbar_{(\delta,1]} &= \{w_{1i} = (a_{i\delta_m}, b_{i\delta_m})| i \in (\delta, 1]\}.
\end{aligned}
$$

Seen from the continuum of entrepreneurs, both $\hbar_{[0,\delta_m)}$ and $\hbar_{(\delta_m,\delta]}$ do not vary with the random returns y_is and thus are fixed. The wealth sets for agents in the market, $\hbar_{[0,\delta_m]}$ and $\hbar_{(\delta,1]}$, are determined solely by δ_m while the set $\hbar_{(\delta_m,\delta]}$ is determined by both $v(\beta)$ and δ_m.

With the wealth information set $\hbar$ for all agents, the demand function for x can be derived. The market is closed at time 1, so the bank sells the collateral by quoting an ask price. Observing the price, all agents can decide how much to buy, but cannot sell. Specifically, assume an agent with wealth portfolio $w_1 = (a_1, b_1)$ observes the ask price p, where a_1 is x position and b_1 cash position. The agent maximizes

$$Eu(ax + b) \tag{1.9}$$

subject to

(i) $ap + b = a_1 p + b_1$, and

(ii) $a \geq a_1$.

The second constraint requires the agent to purchase only x. The optimal demand for x is denoted by $a'(p)$. This agent then buys from the bank $max(0, a'(p) - a_1)$ units of x at the quoted price p. Denote such a demand by $D_i(w_{1i}, p)$ for agent-i where w_{1i} is agent-i's wealth portfolio at time 1. The aggregated purchase from all agents is $D(\hbar, p) = \int_0^1 D_i(w_{1i}, p)$. The collateral value $v(\beta)$ solves

$$D(\hbar, v(\beta)) = (\delta - \delta_m)\beta \Pr(\tilde{y} \in D). \tag{1.10}$$

The existence of $v(\beta)$ is guaranteed by the continuity of equation (1.10). The right hand is a constant number. For the left hand, if $v(\beta)$ is sufficiently small, the total purchasing from investors alone could exceed the right hand side according to equation (1.2). As $v(\beta)$ approaches x_{max}, the total purchasing approaches zero. So, there must exist a solution for $v(\beta)$.

If there are several solutions for $v(\beta)$, in equilibrium, only the one with the highest value will be favored by borrowers. Driven by competition, the bank offers the highest value possible for $v(\beta)$ in equilibrium. Rewrite $v(\beta)$ as $v_{\delta_m}(\beta, \Phi_{in})$, emphasizing it's a function of δ_m and Φ_{in}, the fraction of entrepreneurs in the market and the private investment returns, respectively.

Case 2: all y_i are identical

Now if a borrower defaults, all borrowers default because they hold the same portfolio at time 0 and obtain the same return from securities y_is. The zero Value-at-Risk constraint stipulates that the bank evaluates the collateral in the worst scenario, that is, all y_is generate the lowest return in D, denoted by y_{min}. The amount of collateral to be liquidated by the bank is $(\delta - \delta_m)\beta$. The set of wealth information $\hbar$ is the same as before except now $y_i = y_{min}$ for all entrepreneurs. As before, write $v(\beta)$ as $v_{\delta_m}(\beta, \Phi_{id})$. It immediately follows that $v_{\delta_m}(\beta, \Phi_{in}) > v_{\delta_m}(\beta, \Phi_{id})$, for there is more collateral to be liquidated when the aggregate economy has minimum wealth.

Proposition 2 (Collateral Value Comparison) *Given $0 \leq \delta_m \leq \delta$ and $0 \leq \beta \leq e(0)$, it always holds that $v_{\delta_m}(\beta, \Phi_{in}) > v_{\delta_m}(\beta, \Phi_{id})$. The collateral value is higher in a diversified economy in which entrepreneurs have independent private investments.*

1.2.3 Summary

As the collateral value is determined by the wealth of the aggregate economy conditional on borrowers' default, the correlation among the wealth of agents is an important factor. As shown in the proposition, the collateral value tends to be higher in a diversified economy.

1.2.4 The Bank and Market Equilibrium

Now consider an economy with both the bank and a market. To finance y_i, entrepreneur-i has two choices: to sell x in the market or to use x as collateral to borrow from the bank. In a competitive market, he is only allowed to submit the demand curve to the Walrasian auctioneer who then determines the price. With this price, the entrepreneur learns the utility obtained in the market. On the other hand, the bank announces the loan terms at the beginning, namely, the collateral value function $v(\beta)$. Observing $v(\beta)$, the entrepreneur is able to compute the maximum utility he can achieve before actually borrowing from the bank.

There is no mechanism for an entrepreneur to make decisions by taking into consideration both the market and the bank at the same time, considering he does not know the exact amount of utilities he can obtain from the market. It is natural to think of his decision as a sequence. He first computes the maximum utility from the bank. This utility is his reservation utility for his next step to participate in the market. For a given price in the market, he can compute this utility. If this utility is greater than the reservation utility, he will submit the quantity together with the price to the auctioneer; otherwise, he will not reveal his demand to the auctioneer at that particular price. Therefore, for entrepreneurs, the auctioneer may only have partial demand curves. But for investors, they submit the normal continuous demand curves, for they do not have any reservation utility to participate in the market. The example of a partially revealing demand can be seen in figure **1.2**.

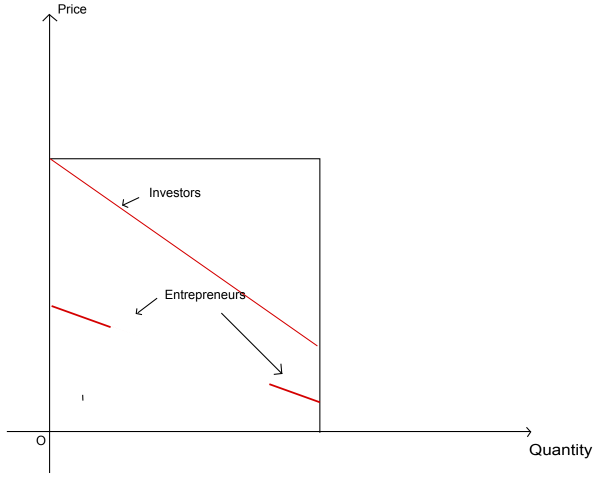

Figure 1.2: An example of demands for investors and entrepreneurs

By symmetry, all entrepreneurs have the same reservation utility from borrowing, denoted by π_b; they must submit the same demand to the auctioneer. In maximization problem (1.1). Denote by $P(\pi_b)$ the set of prices where entrepreneurs obtain a higher utility than π_b, that is,

$$P(\pi_b) = \{p|Eu(\bar{a}(p, e(0), e(1))x + e(1) + e(0)p - \bar{a}(p, e(0), e(1))p) \geq \pi_b\}. \quad (1.11)$$

Entrepreneurs submit to the auctioneer demands $\{(p, \bar{a}(p, e(0), e(1)))|p \in P(\pi_b)\}$.

Because not all demands are continuous, the auctioneer may not be able to clear the market for all entrepreneurs and investors. According to condition **2.2.3**, entrepreneurs should be sellers of x in equilibrium. If entrepreneurs obtain a utility higher than π_b for price $p_1 \in P(\pi_b)$, the prices are higher than p_1, because for the higher price, they can sell less x to raise the same amount of cash in y. Define $p(\pi_b)$ such that $p(\pi_b) = \inf P(\pi_b)$, namely, entrepreneurs prefer to sell the asset in the market for prices greater than or equal to $p(\pi_b)$.

Revisit the market with entrepreneurs in $[0, \delta_m]$ and all investors in $(\delta, 1]$, as guessed by the bank in the previous section. If the equilibrium price p_{δ_m} is greater than or equal to $p(\pi_b)$, the equilibrium will not be affected by the fraction of demands submitted by those entrepreneurs.

For those δ_m such that $p_{\delta_m} \geq p(\pi_b)$, the auctioneer is able to clear the market for entrepreneurs $[0, \delta_m]$ and investors $(\delta, 1]$. Since there are more than one ways to clear the market, the auctioneer is required to clear the market for as many entrepreneurs as possible. Specifically, the auctioneer clears the market for all investors and entrepreneurs $[0, \delta_m]$ where δ_m satisfies $p_{\delta_m} = p(\pi_b)$. If such p_{δ_m} does not exist, then $\delta_m = 0$ or $\delta_m = \delta$, depending on whether $p(\pi_b) > p_0$ or $p(\pi_b) < p_\delta$.

The mechanism is summarized as follows.

1. The bank announces the loan terms, specifically the collateral value function $v(\beta)$. Entrepreneurs calculate the optimal borrowing and its associated utility;

2. Entrepreneurs and investors submit to the Walrasian auctioneer the part of their demands which generates more utility than from borrowing as in the previous step;

3. The auctioneer sets a price to clear the market for as many entrepreneurs as possible;

4. Entrepreneurs whose demands are not accepted in the market will borrow from the bank.

The following flowchart summarizes the sequence of actions for entrepreneurs at time 0.

$$\text{Calculate the optimal borrowing from the bank} \Longrightarrow \text{Submit demands in the market}$$
$$\Longrightarrow \begin{cases} \text{Transact in the market} & \text{if demands are cleared} \\ \text{Borrow from the bank} & \text{if demands are not cleared} \end{cases}$$

The equilibrium is defined as follows.

Definition 2 (Bank and Market Equilibrium) *At time 0, the market is in equilibrium if the following conditions are satisfied:*

(1)Banks fully protect the loan loss by requiring sufficient collateral;

(2)All agents make optimal decisions;

(3)The market clears for x.

Proposition 3 (Nonempty Borrowing Equilibrium) *Given Φ_{in}, in the time 0 bank and market equilibrium, the measure of entrepreneurs borrowing from the bank is positive, that is, $\delta_m < \delta$.*

Proof. Denote by π_{δ_m} the utility for an entrepreneur in the market with entrepreneurs in $[0, \delta_m]$ and investors in $(\delta_m, \delta]$. And denote by $\pi_{b\delta_m}$ the maximum utility for a borrower when the bank offers a loan with collateral value function $v_{\delta_m}(\beta, \tilde{y})$ based on the guess that entrepreneurs $[0, \delta_m]$ transact in the market. In equilibrium, it must be $\pi_{b\delta_m} = \pi_{\delta_m}$.

We claim that π_{δ_m} is a decreasing function of δ_m. As there are more entrepreneurs in the market, each sells the same quantity of x for less price and hence receives a lower utility.

Consider $\delta_m = 0$. If $\pi_{b0} \geq \pi_0$, all entrepreneurs are optimal to borrow from the bank, and the proposition follows. Assume otherwise $\pi_{b0} < \pi_0$.

Now we compare the two utilities at $\delta_m = \delta$ and prove that $\pi_{b\delta} \geq \pi_\delta$. Assume otherwise $\pi_{b\delta} < \pi_\delta$ and it's an equilibrium for the market with all entrepreneurs. Consider a small fraction of entrepreneurs $(\delta - \epsilon, \delta]$ who now switch to borrow from the bank using a_δ units of x as collateral, the same quantity sold in the market. The value of the collateral $v_\delta(a_\delta)$ is the price the bank receives when selling $\epsilon a_\delta \Pr(D)$ units of x in the market. According to the mechanism, the bank will quote an asking price. For entrepreneurs in the market who obtain a return greater than 1, they are willing to buy $e(0) - a_{\delta-\epsilon}$ units of x at price p^*. As long as ϵ is small enough so that $(e(0) - a_{\delta-\epsilon})(\delta - \epsilon)\Pr(y \geq 1) \geq \epsilon a_\delta \Pr(D)$, the bank can sell the collateral for at least p^*. Therefore, the borrowers can obtain a larger sum of cash by using a_δ units of x as collateral. Since using a_δ units of x as collateral to borrow is not necessarily the best strategy for entrepreneurs $(\delta - \epsilon, \delta]$, they can achieve even higher utility using the banks. Thus it won't be an equilibrium for the market with all entrepreneurs. It immediately follows $\pi_{b\delta} > \pi_\delta$.

Now that both $\pi_{b0} < \pi_0$ and $\pi_{b\delta} > \pi_\delta$ hold, there must exist at least one δ_m such that $\pi_{b\delta_m} = \pi_{\delta_m}$ from continuity. Denote by $\Delta = \{\delta_m | \pi_{b\delta_m} = \pi_{\delta_m}\}$ a set consisting of all such δ_m. By continuity again, Δ is a closed set. Then there's only one equilibrium with $\delta_m^* = \min \Delta$ that generates the highest utility among all Δ, for a bank can always offer such a loan with collateral value function $v_{\delta_m^*}(\beta)$ to attract all the potential borrowers. The proof is complete. ■

This proposition distinguishes the endogenous lending market from the restricted one, for borrowing activities do exist in equilibrium for the former.

If some of the entrepreneurs leave the market to borrow from the bank, there will be less trading in the market with investors. The collateralizable asset price is under less pressure from the selling. As a result, the price can be higher when the asset can be used as collateral, in line with the prediction by Geanakoplos (2003).

Corollary 1 (Collateralizable Asset Price) *The price of an asset is higher when it can be used as collateral.*

1.3 Implications for Asset Returns

Mayers (1972) and (1973) has extended the Capital Market Pricing Model by including a nonmarketable asset for each agent in the economy. He focuses on how the expected returns of the marketable assets are affected by their correlations with nonmarketable assets. He has derived an asset pricing model in a linear form similar to the CAPM. Both the CAPM and Mayers' extended model assume an exogenous lending market. In this section, by assuming independent correlation between the marketable and nonmarketable assets, we revisit the asset pricing model by endogenizing the lending market. Specifically, we link the lending market directly to the profitability of the private investment. The private investment is very similar to the nonmarketable assets in Mayers' model, except that his nonmarketable assets pay a lump sum of money while the private investment is a production technology requiring input of cash in the beginning. Further assume all entrepreneurs possess the identical private investment. The goal here is to make bank lending less attractive so that the endogenous lending market can be distinguished from the unlimited one in which there's no borrowing constraints. The identical private investment represents high systematic risk, suggesting these entrepreneurs are in the same industry or in the same region.

To be in line with the CAPM, assume both entrepreneurs and investors have mean-variance utility functions. The conventional CAPM considers two exogenous lending markets: agents can either borrow without constraints (unlimited lending market) or cannot borrow at all (restricted lending market). With the endogenous lending market, there are three patterns of the relationship between the expected returns of the collateralizable asset and the private investment. In the mean-variance economy with exogenous lending markets, only the mean and variance play a role. For the endogenous lending market, the probability distribution of the private investment also matters, especially the value of minimum return, because a zero VaR stipulates that the bank considers the worst scenario. To demonstrate the difference among the three patterns, we construct a series of private investments with the same variance and increasing means from zero to infinity.

The difficulty is to show the difference between the endogenous and unlimited lending market. In the mean-variance economy, the collateralizable asset price is not affected by the variation of the private investments in the unlimited lending market. In the endogenous lending market, if bank lending is always a better way to finance than asset sale, there will be no asset sale or no trading in the market, leading to a price as if there is no private investment, the same result as the unlimited lending market. To demonstrate the difference, we show that asset sale does exist in certain equilibrium by using a particular form of the probability distribution for the series of private investments. With proposition

3, we show the existence of both bank lending and asset sale, distinguishing the endogenous bank lending model from the conventional exogenous restricted and unlimited lending markets.

Firstly, we derive an asset pricing model with an exogenous lending market. Secondly, we endogenize the lending market and compare the difference.

1.3.1 Exogenous Bank Lending

Assume all agents have the same mean-variance utility function with risk tolerance γ. In addition, assume they are all endowed with the same quantity of x, $e_i(0) = 1$ for $i \in [0, 1]$. Denote by y_s the series of identical private investments for entrepreneurs, $s \in [-(2q-1)t, \infty)$ in which q is arbitrarily close to one and t satisfies the following condition:

Condition 1.3.1 $\frac{\delta}{1-\delta}(t-1) > 1$.

For $s \in [0, \infty)$, let y_s be a binomial random variable such that
(i) $y_s(\omega_1) = s$ and $y_s(\omega_2) = s + t$, and
(ii) $\Pr(\omega_2) = q > \frac{1}{2}$.

And for $s \in [-(2q-1)t, 0)$, let
(I) $y_s(\omega_1) = 2qt - s$ and $y_s(\omega_2) = (2q-1)t + s$.

All the private investments y_s have the same variance $q(1-q)t^2$ and an increasing mean from $(1-q)t$ to ∞. The goal is to prove for private investment y_0, some entrepreneurs finance the private investment y_0 by selling asset x in equilibrium.

For private investments, denote $\mu_y(s) = Ey_s$ and $\sigma_y^2 = var(y_s)$ where the variance is constant and does not change with s. In addition, denote $\mu_x = Ex$ and $\sigma_x^2 = var(x)$ for the collateralizable asset x. To highlight the role of using asset x solely to finance, assume entrepreneurs have no cash, $e(1) = 0$. The results in this section can be relaxed with a positive amount of endowed cash.

For entrepreneurs in $[0, \delta]$, they maximize

$$E(a_i x + b_i + c_i y_s) - \frac{1}{\gamma} var(a_i x + b_i + c_i y_s) \tag{1.12}$$

subject to

(i) $a_i p + b_i + c_i = e_i(0)p$

(ii) $a_i \geq 0$ and $c_i \geq 0$,

where a_i is the x position, b_i is the cash position, c_i is the y_s position and p is the market price of x. In the unlimited lending market, the cash holding b_i can be either positive or negative, namely, there's no restriction. In the restricted lending market, the cash holding $b \geq 0$. For investors in $(\delta, 1]$, they maximize the same objective function (1.12) without y_s.

The goal is to find the relationship between $\frac{\mu_x}{p}$ and $\mu_y(s)$, the expected returns of x and y_s, respectively.

A. The unlimited Lending Market

The price for x satisfies the first order condition

$$\frac{\mu_x - \frac{2\sigma_x^2}{\gamma}}{p} = 1. \tag{1.13}$$

The expected return for asset x, $\frac{\mu_x}{p}$, is a constant number, regardless of the change in $\mu_y(s)$. The relation between the two expected returns $\frac{\mu_x}{p}$ and $\mu_y(s)$

is shown in figure **1.3**.

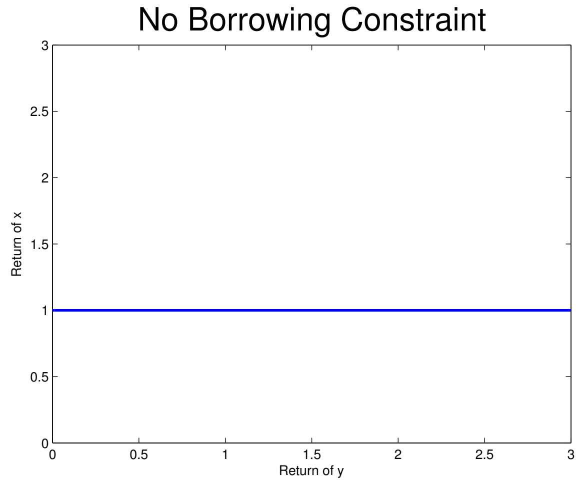

Figure 1.3: The relationship between the expected returns of x and y in an exogenous lending market

B. The Restricted Lending Market

Now consider an economy without lending. In order to raise more cash for private investment y_s, entrepreneurs have to sell x. The endowment arrangement satisfies the starting equilibrium condition **2.2.1** with price $p^* = \mu_x - \frac{2\sigma_x^2}{\gamma}$.

Additional conditions are needed to satisfy the mean-variance utility preference. Since asset x generates a positive payoff, all agents prefer more to less, at least in the range of $[0, 1]$. This requirement gives the following condition.

Condition 1.3.2

$$\frac{\gamma \mu_x}{2\sigma_x^2} \geq 1 \quad \textit{for all } i \in [0, 1] \tag{1.14}$$

In equilibrium, the market price p is determined by the investors' optimal holding of asset x

$$\bar{a}_i = \frac{\gamma(\mu_x - p)}{2\sigma_x^2}, \quad i \in (\delta, 1]. \tag{1.15}$$

Denote by $\bar{a}_i$ the optimal holding of asset x for entrepreneurs in $[0, \delta]$. By symmetry, each entrepreneur in equilibrium holds the same quantity of x and invests the same amount of cash in y_s. If the investment in y_s is nonzero, the entrepreneur must have sold some x to investors to raise the needed cash, and hence holds less x than investors do. The marginal utility of holding x for entrepreneurs in $i \in [0, \delta)$ is $\mu_x - \frac{2\bar{a}_i\sigma_x^2}{\gamma}$, which is greater than $\mu_x - \frac{2\bar{a}_j\sigma_x^2}{\gamma} = 1$ for investors in $j \in (\delta, 1]$ due to $\bar{a}_i < \bar{a}_j$. Therefore, it's optimal for entrepreneurs to hold no cash. For them, the marginal utility of x and y_s must be the same

$$\frac{1}{p}(\mu_x - \frac{2\bar{a}_i}{\gamma}\sigma_x^2) = \mu_y(s) - \frac{2\bar{c}_i}{\gamma}\sigma_y^2. \tag{1.16}$$

Solving it, this equation yields the optimal holdings of x

$$\bar{a}_i = \frac{\frac{2}{\gamma}\sigma_y^2 p + \mu_x - p\mu_y(s)}{\frac{2}{\gamma}(\sigma_x^2 + p\sigma_y^2)}, \quad i \in [0, \delta]. \tag{1.17}$$

Given price p, $\bar{a}_i$ is seen to be a decreasing function of $\mu_y(s)$. But the equilibrium price p of asset x is also a function of $\mu_y(s)$. To consider the full effect of $\mu_y(s)$ on the asset's optimal holding $\bar{a}_i$, take the derivative of $\bar{a}_i$ with respect to p to obtain $\frac{\frac{2}{\gamma}\sigma_x^2\sigma_y^2 - \mu_y\sigma_x^2 - \mu_x\sigma_x^2}{(\frac{2}{\gamma}(\sigma_x^2 + p\sigma_y^2))^2}$, which is less than zero according to equation (1.14). For entrepreneurs, $\bar{a}_i$ is therefore a decreasing function of both p and μ_y. Seen from equation (1.15), the investors' optimal holding of asset x, $\bar{a}_i$, is a decreasing function of p. It follows that the equilibrium price p is a decreasing function of $\mu_y(s)$. Otherwise, the optimal holdings of x for all the agents $\bar{a}$ decrease as $\mu_y(s)$ increases and the market cannot clear. The relationship between the two expected returns of x and y is shown in figure **1.4**.

Proposition 4 *In equilibrium, the price of asset x is a decreasing function of μ_y, the expected return of asset y.*

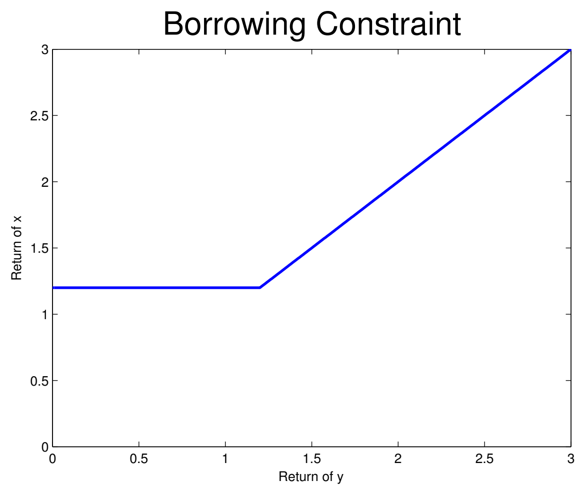

Figure 1.4: The relationship between the expected returns of x and y in an exogenous lending market

1.3.2 Endogenous Bank Lending

Intuitively, when $\mu_y(s)$ is low, the expected return from x should also be low, for entrepreneurs have little incentive to sell x in exchange for cash to invest in y_s. On the other hand, as $\mu_(s)$ increases, entrepreneurs become willing to sell more x to raise cash, which could potentially push down the price. Meanwhile, the loan from the bank becomes increasingly attractive as the collateral value appreciates with the increasing return of y_s. As a result, more entrepreneurs leave the market for the bank. Under less selling pressures, the market price of x remains high, causing a low expected return. To sum up, the price of x is the same as in equation (1.13) when $\mu_y(s)$ is extremely low or high, that is, when $s = -(2q-1)t$ or $s > 1$, respectively. The remaining riddle is for the values of $\mu_y(s)$ in the middle range. In the rest of this section, we show that the price of asset x is indeed lower than equation (1.13) for private ivestment y_0, distinguishing the endogenous lending market from the unlimited one.

Now assume $\beta v(\beta)$ is an increasing function of β for any collateral value function v. Borrowers fail to repay the loan only when ω_1 happens, that is, $y_0(\omega_1) = 0$. In this scenario, no entrepreneurs own any cash after suffering a bad return on the private investment. The bank has to liquidate the collateral, and hence the liquidation price is $\mu_x - \frac{\delta\beta+1-\delta}{1-\delta}\frac{2\sigma_x^2}{\gamma}$ for an amount of $\beta\delta$ units of x to be liquidated. Take the derivative of $\beta v(\beta) = \beta(\mu_x - \frac{\delta\beta+1-\delta}{1-\delta}\frac{2\sigma_x^2}{\gamma})$ with respect to β and let it be greater than zero to obtain the following condition.

Condition 1.3.3 (Monotonicity) $p^* - \frac{4\delta\sigma_x^2}{(1-\delta)\gamma} > 0.$

Under condition **1.3.3** and **1.3.1**, the participants in the market will be nonempty.

Proposition 5 (Nonempty Market) *For private investment y_0, some entrepreneurs trade in the market in equilibrium for a certain value of the probability q.*

Proof. Assume all entrepreneurs prefer to borrow from the bank. Denote by β the quantity of x in equilibrium used as collateral to borrow. Since all entrepreneurs default at the same time, the collateral value $v(\beta) = \mu_x - \frac{\beta\delta+1-\delta}{1-\delta}\frac{2\sigma^2}{\gamma}$ is the price when the bank sells $\beta\delta$ units of x to the investors. Denote by U_b the utility for the entrepreneurs borrowing from the bank. It's important to note that the collateral value $v(\beta)$ does not change as the probability of obtaining good return q increases, as long as $q < 1$. Therefore it follows that $\lim_{q\to 1^-} U_b = \mu_x + \beta v(\beta)(t-1) - \frac{\sigma_x^2}{\gamma} \le \mu_x + v(1)(t-1) - \frac{\sigma_x^2}{\gamma}$.

If a very small fraction of entrepreneurs $[0, \epsilon_1]$ switch to transact in the market, the market equilibrium price p_{ϵ_1} satisfies $\lim_{\epsilon_1 \to 0} = p^*$. Denote by $U_m(\epsilon_1)$ the utility for the entrepreneurs in the market. Then it follows $\lim_{q \to 1^-} \lim_{\epsilon_1 \to 0} U_m(\epsilon_1) \geq a\mu_x + (1-a)p^*t - \frac{a^2\sigma_x^2}{\gamma}$, for any $a \in [0, 1]$. Particularly, let $a = 0$ and obtain $\lim_{q \to 1^-} \lim_{\epsilon_1 \to 0} U_m(\epsilon_1) \geq p^*t$.

According to condition **1.3.1**, it holds that $p^*t > \mu_x + v(1)(t-1) - \frac{\sigma_x^2}{\gamma}$. Therefore by continuity, for a q arbitrarily close to 1, it's optimal for a small fraction of entrepreneurs to transact in the market. ■

Corollary 2 *From both propositions **5** and **3**, the endogenous lending market is indeed different from the two commonly used exogenous lending markets: the restricted and unlimited lending markets. For a certain return y of the private project, in the endogenous lending market, ex ante homogeneous entrepreneurs choose different optimal strategies, whereas in the two endogenous lending markets, they choose the same optimal one.*

For such a series of private investments y_s in the proposition, the relationship between the expected returns of x and y_s is shown in figure **1.5**.

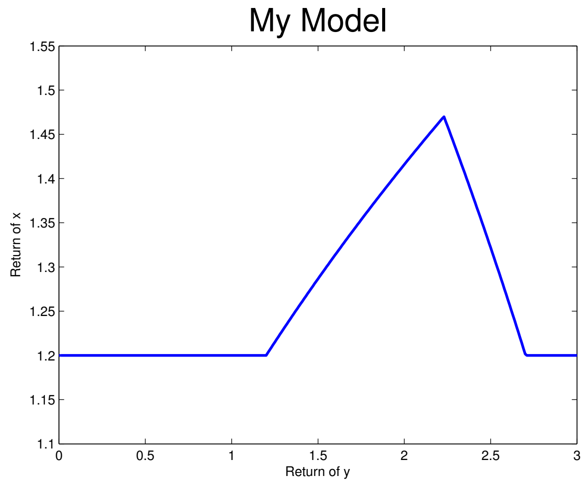

Figure 1.5: The relationship between the expected returns of x and y in an endogenous lending market

1.3.3 Summary

The private investments have both the substitution and wealth effects on the collateralizable asset's return, because on the one hand, they compete with the collateralizable assets for the entrepreneurs' limited wealth, and on the other hand, they define the entrepreneurs' borrowing limits. According to the starting equilibrium condition, both entrepreneurs and investors already hold the optimal wealth portfolios without the private investments. The price of the collateralizable asset declines only when entrepreneurs sell it to investors in order to raise cash. The more profitable the private investments, the more entrepreneurs want to sell the collateralizable asset in order to invest. This is known as the substitution effect. The wealth effect derives from the way the endogenous lending market works. The borrowing capacity for entrepreneurs is the collateralizable asset's value used as collateral, that is, the value in the future after entrepreneurs receive the payoff from their private investments. As these private investments become more profitable, entrepreneurs' wealth increases. This increased wealth bids up the collateral value and hence the borrowing capacity. This in turn reduces the need to raise cash by selling the collateralizable asset. As a result, the price of the collateralizable asset remains the same as when there is no private investment.

1.4 Banks Facilitate Diversification

So far in this economy, each entrepreneur has a given private investment. They make decisions on how to allocate capital between the collateralizable asset x and their private investments. In this section, we study how entrepreneurs choose among different private investments in equilibrium. Traditional capital budgeting theories study it mainly from the firms' (entrepreneurs') perspective and is confined to a partial equilibrium. In this section, however, we focus on the effect of bank loans on the entrepreneurs' choices. As the lending market is endogenously determined in the aggregate economy, all firms' decisions are made in a general equilibrium.

A. An example with two choices for entrepreneurs

To keep the model solvable without losing insights, assume two perfectly hedgeable securities y_A and y_B with two states ω_1 and ω_2 such that

$$y_A(\omega_1) = \rho + \epsilon, y_A(\omega_2) = 0,$$

$$y_B(\omega_1) = 0, y_B(\omega_2) = \rho,$$

$$\Pr(\omega_1) = \Pr(\omega_2) = \tfrac{1}{2}$$

where $\epsilon > 0$ is an arbitrarily small number and $\rho > 2$. In each state, there's only one security generating a positive return. Furthermore, assume both y_A and y_B are independent of x. Each entrepreneur can choose either y_A or y_B as his personal investment at time 0 to maximize

$$Eu_i(a_i x + b_i + c_i y_i) \tag{1.18}$$

subject to

(i) $a_i p + b_i + c_i = e_i(0)p + e_i(1)$

(ii) $y_i \in \{y_A, y_B\}$.

For convenience, define a type-A(type-B) entrepreneur as one choosing $y_A(y_B)$. The objective function for investors are the same as before without $\{y_A, y_B\}$.

In both scenarios: without bank lending or for exogenous bank lending, both type-A and type-B entrepreneurs obtain the same loan contracts. As a result, they always prefer y_A to y_B when maximizing equation (1.18). In equilibrium, all entrepreneurs select y_A.

With endogenous bank lending, however, bank loans are no longer the same for both the type-A and type-B entrepreneurs. In other words, when type-A and type-B entrepreneurs use the same quantity of asset x as collateral to borrow, the collateral value, and hence the amount of the loan, will be different. The greater the number of the same type of entrepreneurs in the economy, the less each can borrow, because banks liquidate the collateral at a more distressed time, in the same spirit as Shleifer and Vishny (1992). In the extreme case when all entrepreneurs choose y_A, the collateral value for a potential type-B entrepreneur is at the largest. The timing when the type-B entrepreneur defaults is associated with a good return for all type-A entrepreneurs. When compensated by the better loan from banks, some entrepreneurs are expected to switch from y_A to y_B. Overall, banks prefer those entrepreneurs whose wealth is negatively correlated with the aggregate economy. The formal proof is in the following proposition.

Proposition 6 (Diversification) *The measure of the set of entrepreneurs choosing the inferior investment y_B is positive if ϵ is small enough.*

Proof. Assume no entrepreneurs choose y_B. In equilibrium, let the entrepreneurs in $[0, \delta_m]$ transact in the market and those in $(\delta_m, \delta]$ borrow from banks. The proof is for $\delta_m < \delta$ only. A similar argument applies to $\delta_m = \delta$.

Denote by $L(z)$ a price enabling investors to buy z units of x. It immediately follows that the market equilibrium price $p_{\delta_m} = L(a_{\delta_m}\delta_m)$ and $L(0) = p^*$. The collateral value function for A-type entrepreneurs is $v_A(\beta) = L(a_{\delta_m}\delta_m + \beta(\delta - \delta_m))$. This is because banks can only sell collateral to investors when a type-A entrepreneur defaults. Given there's no B-type entrepreneurs, the collateral value function for them is $v_B(\beta) \geq p^* - \epsilon_1$ for an arbitrarily small number $\epsilon_1 > 0$, because the bank only liquidates x when A-type entrepreneurs obtain good return $y_A = \rho$. It must hold that $v_B(\beta) > v_A(\beta)$.

Now, if a fraction of borrowers switch to security y_B and use β_{δ_m} units of x as collateral for a loan, he can obtain $\beta_{\delta_m} v_B(\beta_m)$ cash from the bank. Let ϵ be small enough so that $\beta_{\delta_m} v_B(\beta_m) r > \beta_{\delta_m} v_A(\beta_m)$. Then the borrower achieves a higher utility than before. Yet using β_{δ_m} is not necessarily the optimal strategy with collateral value function $v_B(\beta)$. The maximum utility with $v_B(\beta)$ is greater than that with $v_A(\beta)$. Therefore, all borrowers should switch to security y_B and the equilibrium breaks down. The claim that no entrepreneurs choose security y_B is false and the proposition follows. ■

In the proof of the proposition, we show that the inferior investment can only be financed by borrowing. If an entrepreneur does not borrow, he must choose the more profitable investment. This prediction is in line with the existing empirical findings on the negative correlation between leverage and profitability.

Lemma 1 (Subsidized Loan) *Leverage is essential for less profitable investments.*

1.4.1 Summary

Our model predicts that banks can facilitate diversification among the entrepreneurs. Usually, banks minimize credit risk by lending to a diversified group of entrepreneurs. With collateral, it is easier for banks to achieve this, because the collateral value is already based on the aggregate economy. The more diversified the aggregate economy, the higher the collateral value (proposition **2**). In a sense, even if a bank only lends to one entrepreneur, it faces the same minimum risk as when it lends to many. Therefore, collateral lending makes it easier for banks to diversify their risk.

1.5 Conclusion

In this chaper, we study an endogenous lending market that considers the interaction among borrowers. This interaction effect is absent in much of the existing financial literature. In this endogenous lending market, we are able to show both the wealth and substitution effects in the relationship between returns of two assets. The wealth effect, however, does not exist in the conventional asset pricing literature that assumes an exogenous lending market.

In addition, we also examine capital budgeting issues with a bank-firm joint analysis by granting two investment options to entrepreneurs. Traditional capital budgeting theories often rank investments according to profitability. Using a bank-firm joint analysis, we show that profitability is no longer the sole criterion. Since banks manage a portfolio of credit risks, the correlation between the two investments also plays a role. Less profitable investments could exist in equilibrium due to subsidized loans from banks, if these investments are negatively correlated with the aggregate economy. Moreover, the fact that they must rely on loans is in line with empirical findings about the negative correlation between leverage and profitability. Indeed, more profitable investments can be executed without leverage in the equilibrium.

With the assumption of perfect information in the model, we mainly focus on the benefits of using collateral to borrow, such as the reduced asset price volatility and diversification in an economy. The next chapter indicates, however, that the use of collateral can cause two severe consequences: 1) market instability and 2) contagion in an imperfect information environment where banks have little information about entrepreneurs who are not their customers. This study suggests that high quality information is essential in an economy that uses collateral to borrow, more so than one that does not. As we mentioned in the introduction, the use of collateral connects each individual's borrowing capacity to the aggregate economy, whereas, without collateral, this borrowing capacity is determined at the individual level. In one sense, the use of collateral helps create a more closely-linked economy. To fully exploit the advantage of such an economy, high quality information is essential.

CHAPTER 2

ASSET PRICE CYCLES AND BANK LENDING

2.1 Introduction

Previous studies on the relation between an asset's price and bank lending have explored the effect bank credit has on the pirce (Brunnermeier and Pedersen, 2001; Allen and Gale, 2000; He and Krishnamurthy, 2008). Unlike these papers, we focus on assets that can be used as collateral for borrowing. It is important to understand the impact of collateral on asset price paths because there are an increasing number of assets that can serve as collateral due to securitization (Rajan 2006; Gorton and Souleles, 2006; Coval, Jorek and Stafford, 2009; Gorton and Metric, 2009), and because banks that hold collateral need to extract information from the market price. When an asset can be used as collateral the market price may not reflect all investors' information because not everyone participates in the market–some choose to deal with banks. Banks need not disclose information on their collateral holdings nor on their borrowers, even though they are experts in monitoring (Dibvig, 1984). This lack of transparency hinders banks from knowing each other's information, causing collateral to be misvalued, especially when liquidating an asset in distressed times, the value is likely to be constrained by the available buyers' wealth. Shleifer and Vishny (1991), for example, have shown that the sale price of an asset in a distressed industry is less than its fundamental value because potential buyers of the same distressed industry are often under financial pressures, too. In contrast to banks, the market gathers public and private information via the equilibrium mechanism (Grossman and Stiglitz 1980), even though the market has less private information on individuals than banks.

In this paper, we show how these different approaches to information revelation affect the price of a collateralizable asset. To do so, we construct an economy involving two entities: entrepreneurs and banks, and two assets: cash and a collateralizable asset. Endowed with cash and the asset, each entrepreneur has its own production opportunity. Each production opportunity requires cash as an input. Because entrepreneurs' wealth depends on production returns, so will the price path of the asset. To show how the price path is affected by the financial structure, we consider two kinds of economies–one relies solely on the capital market (a market economy) to raise cash, while the other consists of both the market and commercial banks (a banking economy). If entrepreneurs choose to borrow from banks, they have to post this asset as collateral.

In the market economy, the price of the asset is low in the beginning when entrepreneurs sell the asset to raise cash. As production returns realize, entrepreneurs repurchase the assets in the market, making the price increase through time.

In the banking economy, banks are the dominant means for raising cash due to their higher valuation of collateral. This is because a bank evaluates its borrower's collateral at the price at which it can be sold, constrained by the bank's limited information on other banks and entrepreneurs. This structure is justified by the fact that banks don't report their collateral holdings. In the beginning, the price of the asset is at a higher level than that in the market economy because there is no selling pressure. As production returns are realized, entrepreneurs with good returns repay the loans to redeem the collateral. These transactions are between entrepreneurs and banks, and therefore it has little effect on the asset's price. Entrepreneurs with bad returns may default on the loans and consequently the lenders liquidate the collateral to protect themselves. The liquidation occurs because banks hold collateral only to protect their loans. The default of a entrepreneur and the resulting collateral liquidation forces other banks that hold the same asset as collateral to revise its value downward, demanding more collateral from the entrepreneurs. When entrepreneurs cannot fulfill this collateral call, they have to default and forfeit their collateral to the lending banks. As a result, all banks that hold the same asset as collateral liquidate simultaneously, driving down the asset's price. As entrepreneurs' production continues after the panic liquidation period, entrepreneurs repurchase the asset from the market. Aided by this repurchasing wave, the price recovers from its nadir.

Our model of collateral equilibrium has its root in a series of papers by Geanakoplos (1997), Geanakoplos and Zame (1998), and Geanakoplos (2003). In Geanakoplos (2003), he argues that asset price drops sharply when the wealth transfers from the optimistic to the pessimistic following bad news on the asset's fundamentals. But bad news also causes the asset's volatility to hike, which by itself can justify the price crash. To avoid the multiple effects of bad news, we keep the asset's fundamentals unchanged. Instead of optimistic/pessimistic owners, we consider long term versus short term holders. In their model, all parties are final consumers of the asset–but with different views regarding its value. In contrast, we have two distinctive parties, banks and entrepreneurs, which hold the asset for completely different reasons. Entrepreneurs consume the asset in the final period, but banks hold it only as collateral for lending. The value of collateral in their model depends on its fundamental value. The value of collateral in our model depends on the estimated liquidation price that can differ from its fundamental value.

Our emphasis on financial structure relates our paper to the extensive literature on banking. This literature can be divided into two groups, one studying the risk of banking, and the other studying banks and social welfare. Examples of the first group are Diamond and Dybvig (1983), Rajan (2005), Archaya, Gale and Yorulmazer (2009), Shleifer and Vishny (2010). Examples of the second group are Dybvig (1984), Allen and Gale (1997), diamond and Rajan (2001). Like Diamond and Dybvig (1983) and Allen and Gale (2004), this paper shows both the benefit and cost of banking. Banks increase social welfare, but banks inhibit the transmission of information via the private dealings with their customers. The role of information and asset prices began with Grossman and Stiglitz (1980) and has grown to an important branch of research–market microstructure. O'Hara (2003) provides a review.

Bank lending is related to credit constraints in macroeconomics. Bernanke and Gertler (1989), Bernanke, Gertler and Gilchrist (1997) have shown that credits based on a borrowers' balance sheet may have a pro-cyclical effect on the business cycle. Moore and Kiyotaki (1997) show how an exogenous shock to the economy has ripple effects across time further amplified by the use of collateral. In their model, the collateralizable asset itself is input to production. In our model, however, the collateralizable asset plays no role in production except to facilitate financing. While they emphasize the role of the leverage; we focus on the role of bank liquidation in the market price.

Finally, our paper is also related to the literature on contagion. Contagion can happen across assets or agents, via direct or indirect links. One example of a direct link is that assets prices are correlated, as shown in King and Wadhwani (1990), Calvo (1999) and Kodres and Pritsker (2002). Indirect contagion links arise from the asset-agent web. Assets that are owned by the same agent can have contagion due to asset-agent-asset, a contagion spread by the common owner who reduces his portfolio holdings after suffering a loss from one asset, as in Kyle and Wei (2001). Similarly, we can have contagion in the form of agent-asset-agent, as shown by Allen and Gale (2000b). This paper, however, contains a contagion due to entrepreneur-bank-entrepreneur. One entrepreneur's default forces banks to demand more collateral from the remaining entrepreneurs, thus triggering more entrepreneurs to default.

2.2 The Benchmark Model

This section presents the benchmark model.

2.2.1 The Entrepreneurs

We construct an economy consisting of two time periods 0 and 1, and N entrepreneurs, denoted by entrepreneur-i $i = 1, 2, ..., N$. The risk free rate is $r \geq 1$. At time 0, for $i = 1, 2, ..., N$, entrepreneur-i is endowed with $e_i(0)$ units of risky asset x and $e_i(1)$ units of cash, denoted by $\tilde{e}_i = (e_i(0), e_i(1))$. x can be traded at time 0 in a competitive market where all entrepreneurs are price takers. At time 1, x will generate a random payoff $x \in R^+$.

Next, we introduce entrepreneurs' productions. The production is private information and only known by its owner. It is characterized as constant returns to scale and denoted by y_i for entrepreneur-i, $i = 1, 2, ..., N$. Production y_i requires cash at time 0 and matures at time 1 before the payoff of x, generating a random return $y_i \in R^+$ per unit of input. For example, if c units of cash are spent on production y_i, the output will be cy_i units of cash at time 1.

All entrepreneurs maximize the expected utility of their final wealth. Assume entrepreneur-i has utility function u_i, and at time 0, chooses a wealth portfolio $w_i = (a_i, b_i, c_i)$ to maximize:

$$Eu_i(a_i x + b_i r + c_i y) \tag{2.1}$$

subject to

(i) $a_i p + b_i + c_i = e_i(0)p + e_i(1)$,

(ii) $a_i \geq 0$, $b_i \geq 0$ and $c_i \geq 0$.

where a is the position on asset x, b is the cash holding, c is the investment in production y, and p is the market price of x. The optimal demand function for x is denoted by a_i. a_i is a function of the endowments $(e_i(0), e_i(1))$, risk free rate r and the price p. In this section, we focus on the change of the asset price p and cash wealth $e_i(1)$, so we often write a_i as $a_i(p, e_i(1))$.

The formal definition of the market equilibrium is as follows.

Definition 3 (Market Equilibrium) *At time 0, the market is in equilibrium if the following two conditions are satisfied:*
(1)Each entrepreneur chooses an optimal portfolio (a, b, c)*;*
(2)The market for x *clears.*

2.2.2 The Banks

Now we add banks to the model. Banks only offer nonrecourse loans and thus require collateral for protection. For the nonrecourse loan, if a borrower defaults, the bank can seize the collateral, but has no right to claim the borrower's wealth beyond the collateral. At time 0, entrepreneurs use x as collateral to borrow from banks to finance their productions, and repay the loan at time 1 between the payoff of y and x. We choose this timing for two reasons. First, it enables banks to avoid the uncertainty caused by x's random payoff. Second, it allow banks to focus on the market value of x rather than its fundamental value. This is consistent with banks choosing to liquidate collateral in practice. Assume further that the loan is of a discount type: entrepreneurs borrow $\frac{l}{R}$ and repay l, where R is the discount rate charged by the banks.

Assume there are a sufficient number of banks in a competitive lending environment. To preserve the stability of the economy, we situate banks in a minimal risk environment, requiring them to perform safe lending without having a loss in any state. To achieve this, collateral plays an essential role, as it protects banks from a borrower's default. In case default happens, the banks are assumed to liquidate the collateral at time 1 before the payoff of x. Since the market is closed at time 1, banks sell the collateral via over the counter. To protect the bank from the loss, we impose the following condition.

Rule 2.2.1 (Safe Lending) *To fully protect the loan, banks require from the borrowers equal value or more of asset x as collateral.*

A. The Banks' Pricing Model of Collateral

As assumed before, banks evaluate the collateral, not by its fundamental value, but by the market value. Fundamental value is the utility obtained from consuming or holding the asset. The market value is how much one receives when selling it in the market. Since banks can only sell it in the future when the borrowers default, at present, they have to use models to estimate the market value. To do so, banks rely on their information about the potential buyers. We assume banks know all the entrepreneurs' endowments and utility functions, but are unaware of the entrepreneurs' production opportunities. A bank learns this production information only if a entrepreneur asks for a loan. The bank is unable to infer the other entrepreneurs' productions nor does it disclose its lending activity. Besides the private information on their customers, banks observe the public information, the price of x in the market.

The assumption that banks do not know the production information is not essential to the model as long as banks do not have perfect information. The results of the paper hold if banks overvalue the collateral. Allowing banks to estimate the production information simply adds a probability layer to the results in the paper, for sometimes banks overvalue the collateral and other time undervalue. Since the paper is primarily concerned with the scenario of banks' overvaluing collateral, we simplify the model by assuming banks have no production information. While banks can never hope to have the perfect information on entrepreneurs, they rely on what the market price can reveal. As will be shown later, the market price fails to inform banks because banks themselves distort the market price.

According to the bank's information, banks think entrepreneurs other than his borrower at time 1 will maximize:

$$Eu_i(a_i x + b_i) \tag{2.2}$$

subject to

(i) $a_i p + b_i = e_i(0)p + e_i(1)r$.

The cash wealth grows to $e_i(1)r$ in the budget constraint because the bank knows the cash $e_i(1)$ will earn the risk free rate r from time 0 to time 1. Assume the demand function for x is $a_i^*(p, e_i(1))$. This demand ignores the production opportunities. If the bank uses model $v(\beta)$ to estimate the selling price for β units of x, the model price $v(\beta)$ has to satisfy

$$\sum_{j \in M} a_j^*(v(\beta), e_j(1)) = \beta + \sum_{j \in M} e_j(0), \tag{2.3}$$

where M is the set of entrepreneurs excluding the borrower.

To loan out l safely, banks require β units of collateral such that $\Pr(\beta v(\beta) \geq l) = 1$, equivalent to $\beta v(\beta) \geq l$. Since they are in a competitive lending market, they can only demand an equal value of the collateral $\beta v(\beta) = l$.

Since the loan loss is already protected by the collateral in the default state, the profits from repayment are zero due to competition: $l - \frac{l}{R} r = 0$, that is, $R = r$.

Lemma 2 (Lending Policy) *If the banks perform safe lending, they demand β units of x as collateral to lend l, such that $\beta v(\beta) = l$ and charge the risk free rate r.*

B. The entrepreneurs' optimal borrowing

At time 1, after the outcome of y, the entrepreneurs will decide how to repay the loan. We first introduce a partial repayment option.

Definition 4 (Partial Repayment Option) *When repaying the loan, entrepreneurs are allowed to repay a fraction of the loan to redeem the collateral at the same ratio, that is, repay κl to redeem $\kappa\beta$ units of x.*

Banks are not concerned about the partial repayment option because the loan is fully secured by the collateral, as reflected in the pricing model v. But this option allows entrepreneurs to utilize more strategies, such as project financing, thus entrepreneurs are better off with the option than without.

We now solve the entrepreneurs' optimal repayment strategy. Assume entrepreneur-i at time 0 chooses strategy $\tilde{w}_i = (\beta_i, b_i)$, where β_i is the amount of x used as collateral to borrow $\frac{\beta_i v(\beta_i)}{r}$and b_i is the position on cash. His wealth status before repaying the loan is $(e_i(0)-\beta_i)x+b_i r+(e_i(1)-b_i+\frac{\beta v(\beta_i)}{r})y_i$. After y_i realizes, assume for the entrepreneur it is optimal to repay $\kappa_i\beta_i v(\beta_i)$ to redeem $\kappa_i\beta_i$ units of x (Without the partial repayment option, κ_i can only be 0 or 1), then κ_i solves

$$\max_{\kappa_i} Eu((e_i(0)-\beta_i)x+b_i r+(\frac{\beta_i v(\beta_i)}{r}+e_i(1)-b_i)y_i+\kappa_i\beta_i x-\kappa_i\beta_i v(\beta_i)), \quad (2.4)$$

subject to

(i) $0 \leq \kappa_i \leq 1$.

The optimal κ_i is a function of $e_i(0)$, $e_i(1)$, r_i, b_i, β_i, and y_i. We only focus on y_i and write κ as $\kappa(y_i)$. In an example of a project financing loan for y_i, κ_i would look like

$$\kappa_i = 1_{(\beta_i v(\beta_i)+e_i(1)-b_i)y_i \geq \beta_i v(\beta_i)} + \frac{(\beta_i v(\beta_i) + e_i(1) - b_i)y_i}{\beta_i v(\beta_i)} 1_{(\beta_i v(\beta_i)+e_i(1)-b_i)y_i < \beta_i v(\beta_i)}. \tag{2.5}$$

This type of repayment cannot be enforced by banks because the loan is assumed to be nonrecourse.

After solving κ, the entrepreneur at time 0 chooses a wealth portfolio $\tilde{w}_i = (\beta_i, b_i)$ to maximize

$$Eu_i((e_i(0) - \beta_i)x + b_i r + (\frac{\beta_i v(\beta_i)}{r} + e_i(1) - b_i)y_i + \kappa_i \beta_i (x - v(\beta_i))) \tag{2.6}$$

subject to

(i) $0 \leq b_i \leq e_i(1)$, and

(ii) $0 \leq \beta_i \leq e_i(0)$.

2.2.3 The Equilibrium with both Banks and the Market

In this section, we consider an economy with both banks and the market. The combined markets work as follows:

1. Entrepreneurs ask the loan terms from a bank, and then calculate the optimal borrowing and its associated utility;
2. Entrepreneurs then submit to the market auctioneer a fraction of their demands where there's more utility than from borrowing as in the previous

step;

3. Entrepreneurs whose demands are not accepted in the market will return to the bank to borrow

$$\text{Ask loans} \Longrightarrow \text{Submit demands} \Longrightarrow \begin{cases} \text{Transact in the market} & \text{if demands are cleared} \\ \text{Return to banks} & \text{if demands are not cleared} \end{cases}$$

The equilibrium is defined as follows.

Definition 5 (Bank and Market Equilibrium) *At time 0, the economy is in equilibrium if the following three conditions are satisfied:*
(1)The loan contracts are consistent with the banks' information and revealed in the market;
(2)Entrepreneurs' strategies are optimal;
(3)The market clears for x.

Although there is more than one mechanism for bank lending and the market, the one we use narrows the entrepreneurs' choices so that they choose only one bank and one method of financing: either selling x in the market or borrowing from banks. The reason for using one bank is to prevent the leakage of information on the entrepreneur's production. A entrepreneur will obtain the same loan contract no matter which bank he chooses. Different entrepreneurs may obtain different loans, however. The reason for one financing is to keep the model solvable.

Since entrepreneurs are submitting demands that are not continuous as shown in Figure **2.1**, the market auctioneer may be unable to set a price satisfying all the entrepreneurs' demands. Some of the entrepreneurs' demands have to be excluded to clear the market. Viewing the market auctioneer as analogous to a limit order book, he'll ignore all bid orders at low prices and ask orders at high prices.

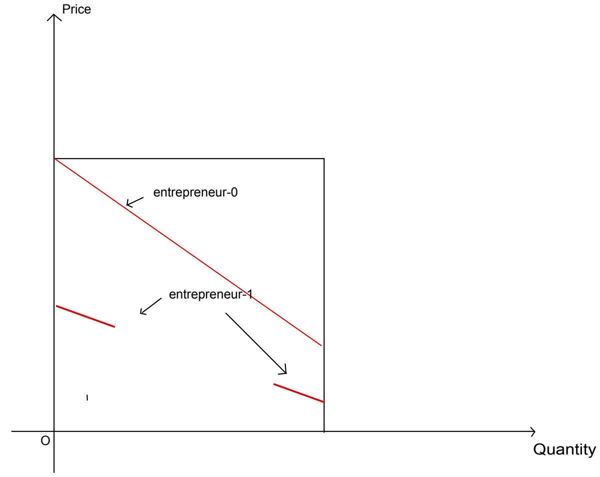

Figure 2.1: Entrepreneur-1's partially revealing demand curve for x in the existence of banks

Since entrepreneurs are not obliged to repay the loan, they can default voluntarily, in a sense, their purpose for borrowing is to sell asset x to the bank. To prevent this behavior, we suppose that the entrepreneurs start in equilibrium. Moreover, if all entrepreneurs are in equilibrium, so is any subgroup. The market price remains the same when some are absent. Simply by looking at the price, banks find no contradiction of their model.

Condition 2.2.1 (Starting Equilibrium) *Without production, all entrepreneurs start in equilibrium. In other words, there exists a price $\tilde{p}$ such that each entrepreneur's* *endowment* $(e_i(0), e_i(1))$ *maximizes*

$$Eu_i(ax + br) \tag{2.7}$$

subject to

(i) $a\tilde{p} + b = e_i(0)\tilde{p} + e_i(1)$,

(ii) $a \geq 0$, *and* $b \geq 0$.

Assume the optimal demand for x to the problem above is $\tilde{a}_i(p, e_i(1))$. We require entrepreneurs to have a downward sloping demand function of price p in the above maximization problem.

Condition 2.2.2 (Downward Demands) *The demand function* $\tilde{a}_i$ *is a decreasing function of price* p, $\frac{\partial \tilde{a}_i}{\partial p} \leq 0$, *for all* $i \in N$.

Recall that a_i is the entrepreneur's optimal demand function to the original maximization problem (2.1). We compare a_i and $\tilde{a}_i$ and impose the following condition.

Condition 2.2.3 (Capital Competing) *Entrepreneurs will demand less* x *if they have the production opportunities, that is, for all* $i \in N$, $\tilde{a}_i(p, e_i(1)) > a_i(p, e_i(1))$, *ceteris paribus. In a sense, production competes with* x *for capital.*

This condition excludes production that can be a hedge against x. A portfolio of hedged assets can generate high returns with little risk, making the combination more attractive than individually. Finally, we have to make sure the market is not empty.

Condition 2.2.4 (Nonempty Market) *At least one of the entrepreneurs has no production.*

Under the four conditions, we prove the following no asset sale equilibrium.

Proposition 7 (No Asset Sale Equilibrium) *In the time 0 bank and market equilibrium, no entrepreneurs sell x in the market regardless of the partial repayment option. The market price p_0 is equal to $\tilde{p}$ in condition* **2.2.1**. *This price confirms the banks' belief that no entrepreneurs have productions other than their customers.*

Proof. We prove by contradiction. To keep the notation simple, in this proof, we drop the cash wealth argument in all the demand functions, and write them as a function of only price. Assume that there exists such an entrepreneur j that sells $\beta_j > 0$ units of x in the market. Denote the market price by p_0 and the remaining entrepreneurs in the market by a set M. At current price p_0, the market clearing condition requires $\sum_{i \in M} a_i(p_0) = \sum_{i \in M} e_i(0)$. From the capital competing condition **2.2.3**, we have for every $i \in M$, $\tilde{a}_i(p_0) \geq a_i(p_0)$. It thus follows that $\sum_{i \in M} \tilde{a}_i(p_0) \geq \sum_{i \in M} e_i(0)$. According to the starting equilibrium condition **2.2.1** $\tilde{a}_i(p) = e_i(0)$, we obtain $\sum_{i \in M} \tilde{a}_i(p_0) \geq \sum_{i \in M} \tilde{a}_i(p)$. From the downward demand condition **2.2.2**, it immediately follows $p_0 \leq p$. From the following inequalities:

$$\sum_{i \in M \setminus j} \tilde{a}_i(p_0) \geq \beta_j + \sum_{i \in M \setminus j} e_i(0) \tag{2.8}$$

$$\tilde{a}_i(p_0) = e_i(0), \quad \text{for all } i \in N \setminus M, \tag{2.9}$$

sum them up and obtain $\sum_{i \in N \setminus j} \tilde{a}_i(p_0) \geq \beta_i + \sum_{i \in N \setminus j} e_i(0)$.

Next, we compute the bank's modelling value $v(\beta_j)$ if β_j units are used as collateral. According to the bank's pricing model (2.3), β_j will solve

$\sum_{i\in N\setminus j} a_i^*(v(\beta_j)) = \beta_j + \sum_{i\in N\setminus j} e_i(0)$. Now we want to show that $v(\beta_j) \geq rp_0$. It suffices to prove that $a_i^*(p) = \tilde{a}_i(rp)$. If we multiple the budget constraint of maximization problem (2.1) by r, we will have the same budget constraint as in problem (2.3), except with the new price rp. So the two maximization problems (2.3) and (2.7) are essentially the same. Finally, according to $\sum_{i\in N\setminus j} a_i^*(v(\beta_j)) = \beta_j + \sum_{i\in N\setminus j} e_i(0) \leq \sum_{i\in N\setminus j} \tilde{a}_i(p_0)$, we have $v(\beta_j) \geq rp_0$.

Assume entrepreneur-j's optimal cash holding is b_j. His wealth portfolio is $w_j = (e_j(0) - \beta_j, b_j, e_j(1) - b_j + \beta_j p_0)$. The utility from w_j is $u_{mkt} = Eu_j((e_j(0) - \beta_j)x + b_j r + (e_j(1) - b_j + \beta_j p_0)y_j)$. If instead, he uses β_j units of collateral x to borrow from the bank, and chooses not to repay the loan regardless ofthe outcome of y_i, that is $\kappa(e_i(0), e_i(1), r, \beta_j, b_i, y_i) = 0$ for all $y_i \in R^+$. In addition, this repayment strategy doesn't require the partial repayment option. For this borrowing strategy, he achieves $u_{bank} = Eu_j((e_j(0) - \beta_j)x + b_j r + (e_j(1) - b_j + \frac{\beta_j v(\beta_j)}{r})y_j)$, which is greater than u_{mkt} because of $\frac{v(\beta_j)}{r} \geq p_0$. Yet, u_{bank} is not the maximal utility he can gain from borrowing. So the point (p_0, β_j) does not belong to the demands he submits to the market auctioneer. This is a contradiction. We thus establish that in equilibrium, no entrepreneurs sell x. If there's no seller, there should be no buyer. For those entrepreneurs that stay at the market, they optimally hold their endowment. Since there's at least one entrepreneur in the market, the price is set at $p_0 = \tilde{p}$. ■

At time 1, when entrepreneurs decide how to repay the loan, we have the following optimal repayment proposition. But we need one more condition.

Condition 2.2.5 (Positive Income Effect) $\frac{\partial \tilde{a}_i}{\partial e_i(1)} \geq 0$, *that is, without production, entrepreneurs will demand more* x *as their cash wealth increases.*

Proposition 8 (Optimal Repayment) *$\kappa(y)$ is a nondecreasing function of y. With the partial repayment option, we have $\kappa(0) = 0$ and $\kappa(r) = 1$. Without the option, we can find a number $0 \leq y^* \leq 1$ such that $\kappa(y) = 1_{y>y^*}$. Simply stated, entrepreneurs fully repay the loan if $y \geq r$ and default completely if $y = 0$.*

Proof. We prove with the partial repayment option only. A similar argument applies to the case without the option. When the entrepreneurs decide how to repay the loan, they essentially solve problem (2.4). We now redefine the problem so that the entrepreneur maximizes:

$$Eu_i(ax + b) \tag{2.10}$$

subject to

(i) $av(\beta_i) + b = (e_i(0) - \beta_i)v(\beta_i) + b_i r + (e_i(1) - b_i + \frac{\beta_i v(\beta_i)}{r})y_i$, and

(ii) $e_i(0) - \beta_i \leq a \leq e_i(0)$.

In the budget constraint, $v(\beta_i)$ becomes the price, as it is the unit cost for the entrepreneur to acquire x from the bank. The optimal demand for x depends on y_i. We denote it by $A(y_i) = min(max(a_i^*, e_i(0)), e_i(0) - \beta_i)$, where $a_i^* = a_i^*(v(\beta_i), (b_i r + (e_i(1) - b_i + \frac{\beta_i v(\beta_i)}{r})y_i))$ as derived from problem (2.3). Since $a_i^*(pr, e_i r) = \tilde{a}_i(p, e_i)$, it follows that a_i^* is also a nondecreasing function of cash wealth and $A(y_i)$ is a nondecreasing function of y_i. Now that $\kappa(y_i) = \frac{A(y_i) - (e_i(0) - \beta_i)}{\beta_i}$, $\kappa(y_i)$ is also a nondecreasing function of y_i.

Finally, we need to show $\kappa(1) = 1$ and $\kappa(0) = 0$. If $y \geq r$, we can find a price p^* such that $a_i^*(p^*, b_i r + (e_i(1) - b_i)y_i + \beta_i v(\beta_i)(\frac{y_i}{r} - 1)) = e_i(0)$. Since $b_i r + (e_i(1) - b_i)y_i + \beta_i v(\beta_i)(\frac{y_i}{r} - 1) \geq e_i(1)r$ and by the starting equilibrium condition **2.2.1**, $a_i^*(\tilde{p}r, e_i(1)r) = e_i(0)$, it follows that $p^* \geq \tilde{p}r \geq v(\beta_j)$. The last inequality holds because

$$\sum_{j\in N\setminus i} a^*(r\tilde{p}, re_j(1)) = \sum_{j\in N\setminus i} e_j(1) \leq \beta_i + \sum_{j\in N\setminus i} e_j(1) = \sum_{j\in N\setminus i} a^*(v(\beta_j), re_j(1)) \quad (2.11)$$

At price p^*, $(e_i(0)-\beta_i)p^*+b_ir+(e_i(1)-b_i+\frac{\beta_i v(\beta_i)}{r})y_i \leq e_i(0)p^*+b_ir+(e_i(1)-b_i)y_i+\beta_i v(\beta_i)(\frac{y_i}{r}-1)$, so $(e_i(0)-\beta_i, b_ir+(e_i(1)-b_i+\frac{\beta_i v(\beta_i)}{r})y_i)$ meets the same budget constraint as the optimal portfolio $(e_i(0), b_ir+(e_i(1)-b_i+\frac{\beta_i v(\beta_i)}{r})y_i-\beta_i v(\beta_i))$ and therefore is inferior to the optimal.

Next, we prove $\kappa = 0$ if $y = 0$. If $b_i = 0$, then $\kappa = 0$ is self evident for entrepreneurs have no cash. For $b_i > 0$, we prove by contradiction. Assume the entrepreneur repays $g \leq b_ir$ to the bank to redeem $h = \frac{g}{v(\beta_i)}$ units of x and the optimal repayment strategy is $\kappa(y_i)$. Now we construct a new borrowing strategy $(\beta', b_i - \frac{g}{r})$ with a new repayment function $k'(y_i) = \frac{k(y_i)\beta_i - h}{\beta_i - h}$ where $\beta' v(\beta') = (\beta_i - h)v(\beta_i)$. It immediately follows that $\beta' \leq \beta_i - h$. In this new strategy, the time 1 wealth after repaying the loan is

$$(e_i(0)-\beta')x+(b_i-\frac{g}{r})r+(e_i(1)-b_i+\frac{g}{r}+\frac{\beta' v(\beta')}{r})y_i+k'(y_i)((\beta')x-\beta' v(\beta')). \quad (2.12)$$

The cash generated from production is $(e_i(1)-b_i+\frac{g}{r}+\frac{\beta' v(\beta')}{r})y_i = (e_i(1)-b_i+\frac{\beta v(\beta)}{r})y_i$, the same as the old strategy. This is also true for the second part of the cash component $(b_ir-g)-k'(y_i)\beta' v(\beta') = b_r - k(y_i)\beta_i v(\beta_i)$. Now it suffices to show the position in x is more in the new strategy than in the old, and the contradiction is found. Indeed, we have

$$e_i(0) - \beta' + \kappa'\beta' = e_i(0) - \beta'(\frac{\kappa\beta_i - \beta_i}{\beta_i - h}) \geq e_i(0) + \kappa\beta_i - \beta_i. \quad (2.13)$$

■

2.2.4 Summary

At time 0, when banks overvalue the collateral, they attract entrepreneurs that need cash to produce away. The market price thus reflects information only from entrepreneurs that do not have production. Seen from such a price, banks find a confirmation of their beliefs that other entrepreneurs don't have production. The fact that banks can distort the market price cautions against the use of mark-to-market accounting.

At time 1, if there are entrepreneurs that default, banks will liquidate the collateral. The price they actually obtain could be lower than their model price. This discrepancy can be decomposed into two parts. The first is the banks' limited information about entrepreneurs. Entrepreneurs may have less wealth than the banks previous thought. The second is the banks' limited information on each other. Even if a bank knows every entrepreneur's wealth and is able to calculate correctly the selling price for his own liquidation, he still faces price uncertainty because other banks may liquidate at the same time.

One reason the market price fails to remedy the limited information sharing among banks is that there is no trading at such price.

2.3 Implications for the Asset's Price

2.3.1 A Multi-period Model

Based on the insight of the benchmark model, we study two implications: the asset price path of x over time and how contagion spreads among entrepreneurs via banks. In order to show this, we build a model simple enough to have a closed-form solution. The new economy consists of four time periods 0, 1, 2 and 3, and three entrepreneurs: entrepreneur-0, entrepreneur-1 and entrepreneur-2. The risk free rate is assumed to be r. At time 0, each entrepreneur is endowed with $\frac{1}{3}$ unit of asset x and 1 unit of cash, as denoted by $e_i(0) = (\frac{1}{3}, 1)$, for $i = 0, 1, 2$. The number in the parentheses indicates the time. This rule applies to all notation in this section. So, the aggregate economy has 1 unit of asset x and 3 units of cash. Asset x generates a random payoff x at the final time 3 with mean 1 and second moment σ^2 (not variance). All entrepreneurs consume their wealth only at time 3. For convenience, we denote by $N = \{0, 1, 2\}$ the index set of entrepreneurs. In this economy, each entrepreneur has enough cash to buy all asset x, because the price of x can be shown to be less than 1 in an economy with risk averse entrepreneurs. As a result, the entrepreneurs have no wealth constraint when trading x.

The productions y_1 and y_2 are assumed to be binary random variables owned by entrepreneur-1 and entrepreneur-2 only. While both require cash at time 0, they mature at different times: y_i in time i, $i = 1, 2$. We define y_i as follows:

$$y_i = \begin{cases} \theta, & \text{with probability } q_i \\ 0, & \text{with probability } 1 - q_i \end{cases}, \tag{2.14}$$

where $\theta > 1$. The expected return is $q_i\theta$. To unify notation, we assign a no return production y_0 to entrepreneur-0 such that $Pr(y_0 = 0) = 1$. Moreover, we assume that x, y_1 and y_2 are independent.

The only difference between the two returns is the probability of obtaining a good outcome. For simplicity, we assume that $\frac{q_1}{r} > \frac{q_2}{r^2}$, that is, entrepreneur-1 has a better production technology than entrepreneur-2.

Because of the agency problem (Bernanke and Gertler, 1989), we assume that entrepreneurs behave myopically when maximizing the present value of the future cash flows.

Condition 2.3.1 (Myopia) *All entrepreneurs are myopic.*

The entrepreneurs' production decisions proceed as follows. At time 0, entrepreneur-i chooses a wealth portfolio $w_i(0) = (a_i(0), b_i(0), c_i(0))$ to maximize the time 0 present value of future cash flows, where $a_i(0)$ is the amount of x, $b_i(0)$, the amount of cash, and $c_i(0)$, the amount of capital spent on the production. Given a wealth portfolio $w_i(0) = (a_i(0), b_i(0), c_i(0))$, the time 0 present value is defined as $P(w_i(0), 0) = \frac{a_i(0)x - \frac{(a_i(0)x)^2}{\gamma}}{r^3} + b_i(0) + \frac{c_i(0)\theta 1_{y_i}}{r^i}$. Here we use 1_{y_i} as the indicator function of the good outcome of production y_i. The present value of asset x is assumed to be $\frac{E(a_i(0)x - \frac{(a_i(0)x)^2}{\gamma})}{r^3}$ where the risk aversion is reflected in the risk compensated expected payoff, not the discount rate r, and r^3 in the denominator indicates the payoff is at time 3. That of production is assumed to be $\frac{Ec_i(0)\theta 1_{y_i}}{r^i}$ where the denominator r^i indicates the production yields at time i and the cash $b_i(0)$ remains $b_i(0)$. There's no explicit risk adjustment for production because the return θ is risk adjusted. P is the function used by all entrepreneurs to discount future cash flows.

We denote the market price of x at time t by p_t, for $t = 0, 1, 2$. All entrepreneurs act as price taker in a competitive market. Short positions are excluded for they can lead to negative wealth.

The time 0 decision is summarized as follows:

Problem 1 (Time 0 Market) *At time 0, entrepreneur-i, for* $i \in N$*, chooses a wealth portfolio* $w_i(0) = (a_i(0), b_i(0), c_i(0))$ *to maximize:*

$$P(w_i(0)) = \frac{E(a_i(0)x) - \frac{1}{\gamma}E(a_i(0)x)^2}{r^3} + b_i(0) + \frac{c_i(0)q_i\theta}{r^i}, \tag{2.15}$$

subject to

(i) $a_i(0)p_0 + b_i(0) + c_i(0) = \frac{1}{3}p_0 + 1$*, and*

(ii) $a_i(0) \leq 1$*,* $b_i(0) \geq 0$ *and* $c_i(0) \geq 0$*.*

2.3.2 Market Equilibrium at t=0

For problem **1**, we obtain

$$c_i(0) = 0, \qquad \text{if } \tfrac{q_i\theta}{r^i} \leq 1; \tag{2.16}$$

$$b_i(0) = 0, \qquad \text{if } \tfrac{q_i\theta}{r^i} > 1. \tag{2.17}$$

$\frac{q_i\theta}{r^i} \leq 1$ is a trivial case in the model, so we focus on the case $\frac{q_i\theta}{r^i} > 1$, that is, investing in production is better than merely holding cash. We make this as a condition in the model.

Condition 2.3.2 (Profitable Production) *Productions* y_1 *and* y_2 *are preferred to cash holding, that is,* $\frac{q_1\theta}{r} > 1$ *and* $\frac{q_2\theta}{r^2} > 1$*.*

For two productive entrepreneurs, the demand function for x is

$$
\begin{aligned}
a_i(0) &= 0, &\quad \text{if } p_0 \geq \tfrac{1}{q_i\theta r^{3-i}} \\
a_i(0) &= \tfrac{\gamma(1-p_0 q_i\theta r^{3-i})}{2\sigma^2}, &\quad \text{if } \tfrac{1}{q_i\theta r^{3-i}} > p_0 > \tfrac{1-\frac{2\sigma^2}{\gamma}}{q_i\theta r^{3-i}} \\
a_i(0) &= 1, &\quad \text{if } p_0 \leq \tfrac{1-\frac{2\sigma^2}{\gamma}}{q_i\theta r^{3-i}}.
\end{aligned} \tag{2.18}
$$

This set of equations means that, as the expected return of production increases, the entrepreneur is willing to hold less x and sell more. The price of x thus depends on the opportunity cost of the cash. Since entrepreneurs don't hold cash, $b_i(0) = 1$, they put the rest of their wealth in production $c_i(0) = 1 + (\frac{1}{3} - a_i(0))p_0$, for $i = 1, 2$.

For the unproductive entrepreneur-0, he has no production, so it follows that $c_0(0) = 0$, and

$$
\begin{aligned}
a_0(0) &= 0, &\quad \text{if } p_0 \geq r^{-3} \\
a_0(0) &= \tfrac{\gamma(1-p_0 r^3)}{2\sigma^2}, &\quad \text{if } r^{-3}(1 - \tfrac{2\sigma^2}{\gamma}) < p_0 < r^{-3} \\
a_0(0) &= 1, &\quad \text{if } p_0 \leq r^{-3}(1 - \tfrac{2\sigma^2}{\gamma}),
\end{aligned} \tag{2.19}
$$

and $b_0(0) = 1 + \frac{1}{3}p_0 - a_0(0)p_0$.

Definition 6 (Market Equilibrium) *In the x market at time 0, entrepreneur-i submits to the market auctioneer the demand function $a_i(0)$ according to equations (2.18) or (2.19), $i \in N$. The market auctioneer will then set a price p_0. In equilibrium,*
(i)The Entrepreneurs' holdings are optimal, and
(ii)The market clears at the price p_0, $a_1(0) + a_2(0) + a_0(0) = 1$.

In this economy entrepreneur-1 is the most willing to sell x, followed by entrepreneur-2 and entrepreneur-0, according to the rank of the production

Figure 2.2: All entrepreneurs hold x

profitability. The asset price of x is thus driven by the entrepreneurs' production decisions in addition to its risk.

CASE 1: Every entrepreneur holds a nonzero amount of x

In this case, we have $a_0(0) = \frac{\gamma(1-p_0r^3)}{2\sigma^2}$, $a_1(0) = \frac{\gamma(1-p_0q_1\theta r^2)}{2\sigma^2}$ and $a_2(0) = \frac{\gamma(1-p_0q_2\theta r)}{2\sigma^2}$. The market clearing condition is

$$\frac{\gamma(1-p_0r^3)}{2\sigma^2} + \frac{\gamma(1-p_0q_1\theta r^2)}{2\sigma^2} + \frac{\gamma(1-p_0q_2\theta r)}{2\sigma^2} = 1. \tag{2.20}$$

Solving gives the price $p_0 = \frac{3-2\frac{\sigma^2}{\gamma}}{r^3+q_1\theta r^2+q_2\theta r}$. From the condition of entrepreneur-1's nonzero x holding $a_1(0) = \frac{\gamma(1-p_0q_1\theta r^2)}{2\sigma^2} > 0$, we require $p_0 < \frac{1}{q_1\theta r^2}$, or $\frac{3-2\frac{\sigma^2}{\gamma}}{r^3+q_1\theta r^2+q_2\theta r} < \frac{1}{q_1\theta r^2}$. Figure 2.2 illustrates this case.

CASE 2: Entrepreneur-2 and entrepreneur-0 hold a nonzero amount of x

In this case, we have $a_0(0) = \frac{\gamma(1-p_0r^3)}{2\sigma^2}$, $a_1(0) = 0$ and $a_2(0) = \frac{\gamma(1-p_0q_2\theta r)}{2\sigma^2}$. The market clearing condition is then

$$\frac{\gamma(1-p_0q_2\theta r)}{2\sigma^2} + \frac{\gamma(1-p_0r^3)}{2\sigma^2} = 1. \tag{2.21}$$

Solving gives the price $p_0 = \frac{2-2\frac{\sigma^2}{\gamma}}{r^3+q_2\theta r}$. From the x holding conditions of entrepreneur-1 and entrepreneur-2, we require $\frac{1}{q_2\theta r} > p_0 \geq \frac{1}{q_1\theta r^2}$. Figure 2.3 illustrates this case.

Figure 2.3: Only entrepreneur-0 and entrepreneur-2 hold x

CASE 3: Only entrepreneur-0 holds x

In this case, we have $a_0(0) = 1$ and $a_1(0) = a_2(0) = 0$. The market clearing condition is then

$$\frac{\gamma(1 - p_0 r^3)}{2\sigma^2} = 1. \tag{2.22}$$

Solving gives the price $p_0 = r^{-3}(1 - \frac{2\sigma^2}{\gamma})$. We require $p_0 \geq \frac{1}{q_2\theta r}$ here. Figure 2.4 illustrates this case.

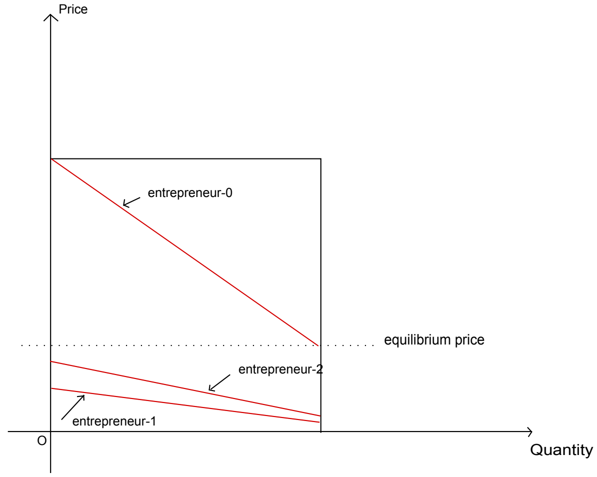

Figure 2.4: Only entrepreneur-0 holds x

These results can be summarized as follows.

Result 1 *In market equilibrium we have*

$$
\begin{aligned}
p_0 &= \frac{3 - 2\frac{\sigma^2}{\gamma}}{r^3 + q_1\theta r^2 + q_2\theta r}, && \textit{if } 2 - 2\tfrac{\sigma^2}{\gamma} < \tfrac{r^3 + q_2\theta r}{q_1\theta r^2} \\
p_0 &= \frac{2 - 2\frac{\sigma^2}{\gamma}}{r^3 + q_2\theta r}, && \textit{if } \tfrac{r^3 + q_2\theta r}{q_1\theta r^2} \leq 2 - 2\tfrac{\sigma^2}{\gamma} \leq \tfrac{r^3 + q_2\theta r}{q_2\theta r} \\
p_0 &= r^{-3}(1 - 2\frac{\sigma^2}{\gamma}), && \textit{if } 2 - 2\tfrac{\sigma^2}{\gamma} > \tfrac{r^3 + q_2\theta r}{q_2\theta r}
\end{aligned}
\tag{2.23}
$$

The discount rate in the price includes the opportunity cost of cash: $q_1\theta r^2$ *and* $q_2\theta r$*, which is often greater than the risk free rate.*

2.3.3 Bank Lending

As in the benchmark model, both of the two productive entrepreneurs can borrow from banks at time 0 and repay the loan after the production pays off. So entrepreneur-1 will repay the loan at time 1 and entrepreneur-2 at time 2. Assume bank-i lends to entrepreneur-i using the model v_i for the collateral value. The loan is also a discount type. With collateral value l, entrepreneur-1 can borrow $\frac{l}{R}$ and entrepreneur-2 $\frac{l}{R^2}$, where R is the loan discount rate charged by banks. Now denote the entrepreneurs' wealth portfolio by $\tilde{w}_i = (\frac{1}{3} - \beta_i(0), b_i(0))$ where $\beta_i(0)$ is the amount of x used as collateral to borrow $R^{-i}\beta_i(0)v_i(0, \beta_i(0))$ from bank-i, and $b_i(0)$ is the cash holding. The "0" in $v_i(\beta_i, 0)$ indicates the time bank-i models the value of x, for banks need to update the valuation every period. This wealth portfolio states that the entrepreneur-i holds $\frac{1}{3} - \beta_i$ units of x and $b_i(0)$ cash, spends $1 - b_i(0) + R^{-i}\beta_i(0)v_i(0, \beta_i(0))$ on production y_i, and posts $\beta_i(0)$ units of x to bank-i as collateral.

First, we need to compute the collateral value $v_i(\beta_i, 0)$ for bank-i and formulate the bank's lending policy.

Lemma 3 (Lending Policy) *For $i = 1, 2$, bank-i lends to entrepreneur-i at time 0 and entrepreneur-i will repay it at time i. If entrepreneur-i at time 0 gives $\beta_i(0)$ units of x to bank-i as collateral, bank-i will lend $R^{-i}\beta_i(0)v_i(0, \beta_i(0))$ to entrepreneur-i. $v_i()$ can be calculated as $v_i(0, \beta_i(0)) = r^{i-3}(1 - \frac{2\sigma^2}{3\gamma} - \frac{\sigma^2}{\gamma}\beta_i(0))$. Since banks are protected from loss in each state, the loan discount rate will be $R = r$, due to a competitive lending market.*

Proof. Upon an entrepreneur-i's default, bank-i sells $\beta_i(0)$ units of x in the market. According to its limited information, bank-i assumes that the other two entrepreneurs are both of the unproductive type with the same demand function $\alpha(p) = \frac{\gamma(1-r^{3-i}p)}{2\sigma^2}$. Therefore, for them to hold $\frac{2}{3} + \beta_i(0)$, the selling price has to be $\frac{\gamma(1-r^{3-i}v_i(0,\beta_i(0)))}{2\sigma^2} = \frac{1}{3} + \frac{\beta_i(0)}{2}$. Solve to obtain $v_i(0, \beta_i(0)) = r^{i-3}(1 - \frac{2\sigma^2}{3\gamma} - \frac{\sigma^2}{\gamma}\beta_i(0))$. ∎

The assumption of linking collateral value to the future price follows Kiyotaki and Moore (1994). As more x is used as collateral to borrow, the value per unit of collateral will decline. This fact can also be seen directly from the equation for v. To simplify the model, we assume βv is an increasing function of β. The following condition assures this.

Condition 2.3.3

$$\frac{\sigma^2}{\gamma} < \frac{1}{2} \tag{2.24}$$

Entrepreneurs can also use their endowed cash to invest in production. For the three sources of funding for production, we can establish a pecking order: internal cash is better than a bank loan and a bank loan is better than a direct sale in the market. The second claim has already been proved in the benchmark model. Now we only need to show internal cash is better than a bank loan. Since entrepreneur-0 doesn't need to borrow from the bank, we focus only on entrepreneur-1 and entrepreneur-2.

Result 2 (Cash Optimality) *To finance y_i, entrepreneur-i always prefers cash to the loan from bank-i, for $i = 1, 2$.*

Proof. We only prove it with the partial repayment option. A similar argument applies where the option is not allowed. We prove this by contradiction. Assume the optimal wealth portfolio for entrepreneur-i is $\widetilde{w}_i(0) = (\frac{1}{3} - \beta_i(0), b_i(0))$ where $\beta_i(0) > 0$ and $b_i(0) > 0$. Then at time i, the entrepreneur will have $\frac{1}{3} - \beta_i(0)$ units of x on hand, $b_i(0) + (1 - b_i(0) + r^{-i}\beta_i(0)v_i(0, \beta_i(0)))y_i$ in cash, and a loan contract "borrowing $\beta_i(0)v_i(0, \beta_i(0))$ from bank with collateral $\beta_i(0)$ unit of x". The cash component consists of the retained cash "$b_i(0)$" from time 0 and the payoff $(1 - b_i(0) + r^{-i}\beta_i(0)v_i(0, \beta_i(0)))y_i$ from production. When repaying the loan at time i, entrepreneur-i will maximize

$$r^{3-i}E(Ax - \frac{Ax}{\gamma}) + B \tag{2.25}$$

subject to

(i)$Av_i(0, \beta_i(0)) + B = (\frac{1}{3} - \beta_i)v_i(0, \beta_i(0)) + b_0(i)r^i + (1 - b_i(0) + r^{-i}\beta_i(0)v_i(0, \beta_i(0)))y_i$,

(ii) $\frac{1}{3} - \beta_i \leq A \leq \frac{1}{3}$.

We know that at price $p = r^{3-i}(1 - \frac{2\sigma^2}{3\gamma})$ the entrepreneur optimally holds $\frac{1}{3}$ units of x, if he has no wealth constraint. For price $v_i(0, \beta_i(0)) \leq p$, the entrepreneurs will demand more than $\frac{1}{3}$ if possible. This strong demand for x implies the entrepreneur will repay as much of the loan as he can. Whether the entrepreneur can repay the loan depends on the outcome of production at time i. We thus consider two cases separately and, for each case, construct a new wealth portfolio dominating the current one.

Case one: $b_i(0)r^i \geq \beta_i(0)v_i(0, \beta_i(0))$

In this case, entrepreneur-i can repay the loan regardless of the return of y_i. After repaying the loan, the entrepreneur will have $\frac{1}{3}$ unit of x and $b_i(0)r^i - \beta_i(0)v_i(0, \beta_i(0)) + (1 - b_i(0) + r^{-i}\beta_i(0)v_i(0, \beta_i(0)))y_i$ in cash. But a new wealth portfolio without borrowing at time 0 $\widetilde{w}_1^*(0) = (\frac{1}{3}, b_i(0) - r^{-i}\beta_i(0)v_i(0, \beta_i(0)))$ can exactly replicate this time 1 payoff.

Case two: $b_i(0)r^i < \beta_i(0)v_i(0, \beta_i(0))$

In this case, entrepreneur-i can no longer repay the full amount of the loan if production yields zero. Using cash $b_i(0)r^i$, the entrepreneur will be able to redeem $\frac{b_i(0)r^i}{v_i(0,\beta_i(0))}$ units of x. His financial status at time 1 in the two different states of y_i's return are

$$\begin{cases} (\frac{1}{3} - \beta_i(0) + \frac{b_i(0)r^i}{v_i(0,\beta_i(0))}) \text{ units of } x \text{ and zero in cash}, & \text{if } y_i = 0 \\ \frac{1}{3} \text{ units of } x \text{ and } (1 - b_i(0) + r^{-i}\beta_i(0)v_i(0,\beta_i(0)))\theta & \\ +b_i(0)r^i - \beta_i(0)v_i(0,\beta_i(0)) \text{ in cash}, & \text{if } y_i = \theta \end{cases} \tag{2.26}$$

We construct a new wealth portfolio $\widetilde{w}_i'(0) = (\frac{1}{3} - \beta_i'(0), 0)$, such that

$$1 + r^{-i}\beta_i'(0)v_i(0,\beta_i'(0)) = 1 - b_i(0) + r^{-i}\beta_i(0)v_i(0,\beta_i(0)). \tag{2.27}$$

In this new portfolio, entrepreneur-i maintains the same spending on y_i by using cash first and then financing the difference with borrowing. The wealth portfolio at time 0 changes to

$$\begin{cases} \frac{1}{3} - \beta_i'(0) \text{ units of } x \text{ and zero in cash}, & \text{if } y_i = 0 \\ \frac{1}{3} \text{ units of } x \text{ and } (1 + r^{-i}\beta_i'(0)v_i(0,\beta_i'(0)))\theta - \beta_i'(0)v_i(0,\beta_i'(0)) \text{ in cash}, & \text{if } y_i = \theta \end{cases} \tag{2.28}$$

Equation (2.27) guarantees that the two strategies generate the same wealth under the good return of y_i at time 1. The difference between the two strategies, however, is the position of x after the bad return of y_i. We claim $\frac{1}{3} - \beta_i'(0) > \frac{1}{3} - \beta_i(0) + \frac{b_i(0)r^i}{v_i(0,\beta_i(0))}$. From equation (2.27), we have $\beta_i'(0)v_i(0,\beta_i'(0)) = \beta_i(0)v_i(0,\beta_i(0)) - b_i(0)r^i$. It follows that $\beta_i'(0)v_i(0,\beta_i(0)) > \beta_i(0)v_i(0,\beta_i(0)) - b_i(0)r^i$, for $v_i(0,\cdot)$ is a decreasing function and $\beta_i'(0) < \beta_i(0)$. Therefore the new strategy is better than the original one, which contradicts the optimality of the original. The proposition follows. ■

Remark 4 *The result is stronger than that in the optimal repayment proposition 8. In that proposition, entrepreneurs will not necessarily use cash first, instead, the optimal capital structure has the combination of both cash and loans. The result on cash optimality is derived from a more structured model.*

Now we compute the entrepreneur-i's optimal borrowing strategy, for $i = 1, 2$. The entrepreneurs maximize

$$\max_{\beta_i(0), b_i(0)} \quad \{r^{-3}[E(\frac{1}{3} - \beta_i(0)1_{D_i})x - \frac{1}{\gamma}E((\frac{1}{3} - \beta_i(0)1_{D_i})x)^2] + b_i(0) + q_i\theta r^{-i}(1 - b_i(0)) + q_i\theta r^{-i}(r^{-i}\beta_i(0)v_i(0, \beta_i(0))) - r^{-i}q_i\beta_i(0)v_i(0, \beta_i(0))\}, \tag{2.29}$$

subject to

(i)$0 \leq \beta_i(0) \leq \frac{1}{3}$, and

(ii)$0 \leq b_i(0) \leq 1$.

where 1_{D_i} is the indicator function regarding default.

The Lagrange equation is

$$\begin{aligned} L = \quad & r^{-3}[E(\frac{1}{3} - \beta_i(0)1_{D_i})x - \frac{1}{\gamma}E((\frac{1}{3} - \beta_i(0)1_{D_i})x)^2] \\ & + b_i(0) + r^{-i}q_i\theta(1 - b_i(0) + r^{-i}\beta_i(0)v_i(0, \beta_i(0))) - r^{-i}q_i\beta_i(0)v_i(0, \beta_i(0)) \\ & + \lambda_1(\frac{1}{3} - \beta_i(0)) + \lambda_2\beta_i(0) + \lambda_3(1 - b_i(0)) + \lambda_4 b_i(0) \end{aligned} \tag{2.30}$$

with Lagrange multipliers $\lambda_j \geq 0$, for $j \in [1, 4]$, and complementary slackness conditions:

$$\lambda_1(\beta_i(0) - \frac{1}{3}) = 0 \tag{2.31}$$

$$\lambda_2\beta_i(0) = 0 \tag{2.32}$$

$$\lambda_3(b_i(0) - 1) = 0 \tag{2.33}$$

$$\lambda_4 b_i(0) = 0. \tag{2.34}$$

From proposition **2**, we see that if the entrepreneur hasn't spent all endowed cash on production, namely, $b_i(0) > 0$, then there's no borrowing, thus no risk of default, $\beta_i(0) = 0$ and $1_D = 0$. In the slackness condition, we have $\lambda_4 = 0$. Taking derivatives with respect to $b_i(0)$ in the Lagrange equation generates $1 - q_i\theta r^{-i} - \lambda_3 = 0$. Since $1 - q_i\theta r^{-i} < 0$, we must have $\lambda_3 < 0$, which is a contradiction. So it's impossible for entrepreneurs to retain cash if the production is profitable, $q_i\theta r^{-i} > 1$. In other words, we must have $b_i(0) = 0$.

After the entrepreneurs spend all of their endowed cash on production, they can still borrow from banks to produce more. Because of the "all or nothing" characteristics of production, entrepreneurs will default on the loan completely if they suffer a bad return from production. It thus follows that $1_{D_i} = 1 - 1_{y_i}$. Recall that bank-i uses the pricing model $v_i(0, \beta_i(0)) = r^{i-3}(1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - \frac{\sigma^2}{\gamma}\beta_i(0))$. Taking derivatives with respect to $\beta_i(0)$, we have

$$(\beta_i(0)) \qquad -r^{-3}[(1-q_i) + \frac{(1-q_i)\sigma^2(2\beta_i(0) - \frac{2}{3})}{\gamma}] \\ +r^{-6}[q_i(r^{-i}\theta - 1)(1 - \frac{2\sigma^2}{3\gamma} - \frac{2\sigma^2}{\gamma}\beta_i(0))] - \lambda_1 + \lambda_2 = 0 \qquad (2.35)$$

The solution is

$$\beta_i(0) = \frac{(q_i\theta r^{-3-i} - q_i r^{-3} - 1 + q_i)(1 - \frac{2}{3}\frac{\sigma^2}{\gamma})}{\frac{2\sigma^2}{\gamma}(q_i\theta r^{-3-i} - q_i r^{-3} - q_i + 1)}, \text{if } \frac{q_i\theta r^{-3-i} - q_i r^{-3} - 1 + q_i}{q_i r^{-3}(\theta r^{-i} - 1)} < \frac{4\sigma^2}{3\gamma}$$

$$\beta_i(0) = \frac{1}{3}, \text{if } \frac{q_i\theta r^{-3-i} - q_i r^{-3} - 1 + q_i}{q_i r^{-3}(\theta r^{-i} - 1)} \geq \frac{4\sigma^2}{3\gamma}. \qquad (2.36)$$

Now with both banks and the market, we will obtain the following equilibrium.

Result 3 (Market and Banking Equilibrium) *According to proposition **7**, both* *entrepreneur-1 and entrepreneur-2 optimally borrow from banks and entrepreneur-0 stays in the market with market price* $p_0 = r^{-3}(1 - \frac{2\sigma^2}{3\gamma})$.

2.4 The Price Path of x in the Two Economies

In this section, we calculate the price paths of x in the two different economies– the banking economy and the market economy, and document a contagion effect at time 1. Even without any outside impact on the price, the price will grow at the market interest rate. To focus on the effect from the financial structures, we set the gross interest rate to be one, $r = 1$.

2.4.1 The Price Paths of x in Banking Economy

At Time 0

At time 0, denote the optimal borrowing for entrepreneur-1 and entrepreneur-2 by $\beta_1(0)v_1(0, \beta_1(0))$ and $\beta_2(0)v_2(0, \beta_1(0))$, respectively, where $\beta_1(0)$ and $\beta_2(0)$ are the required amount of collateral x. In the market, entrepreneur-0 submits his demand curve $a_0(0) = \frac{\gamma(1-p_0)}{2\sigma^2}$, and the market sets a price p_0 so that $a_0(0) = \frac{1}{3}$. The solution yields the market price at time 0, $p_0 = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma}$.

At Time 1

At time 1, if entrepreneur-1 has a good return from production y_1, he repays the loan to redeem the collateral. After that, he is holding $\frac{1}{3}$ units of x and

$(\beta_1(0)v_1(0,\beta_1(0))+1)\theta - \beta_1(0)v_1(0,\beta_1(0))$ in cash. Entrepreneur-0 at this time holds $\frac{1}{3}$ units of x and 1 cash. Entrepreneur-2 holds $\frac{1}{3}-\beta_2(0)$ units of x, but has no cash, instead, he has ongoing production y_2 and a loan contract with bank-2.

In the market, entrepreneur-i, for $i = 0, 1$, maximizies

$$Ea_i(1)x - E(a_i(1)x)^2 + b_i(1) \tag{2.37}$$

subject to

(i)$a_0(1)p_1 + b_0(1) = \frac{1}{3}p_1 + 1$, and

(ii)$a_1(1)p_1 + b_1(1) = \frac{1}{3}p_1 + (\beta_1(0)v_1(0,\beta_1(0)) + 1)\theta$.

Entrepreneur-2 maximizes a different objective function:

$$\begin{aligned} & E(a_2(1)-\beta_2(0)1_{D_2})x - \frac{1}{\gamma}E(a_2(1)-\beta_2(0)1_{D_2})^2x^2 \\ & + b_2(1) + q_2\theta(\beta_2(0)v_2(0,\beta_1(0))+1) - q_2\beta_2(0)v_2(0,\beta_1(0)) \end{aligned} \tag{2.38}$$

subject to

(i)$(a_2(1)-\beta_2(0))p_1 + b_2(1) = (\frac{1}{3}-\beta_2(0))p_1$, and

(ii)$\frac{1}{3} \geq a_2(1) \geq \beta_2(0)$.

Entrepreneur-2 can only sell but not buy x for he has no cash.

Solving these problems yields the demand functions of x for all three entrepreneurs at time 1: $a_0(1)(p_1) = \frac{\gamma(1-p_1)}{2\sigma^2}$, $a_1(1)(p_1) = \frac{\gamma(1-p_1)}{2\sigma^2}$, and $a_2(1)(p_1) = \frac{\gamma(1-p_1)}{2\sigma^2} + \beta_2(0)q_2$. Given any price p_1, entrepreneur-2 is demanding more x than the other two. Given that their initial positions of x are the same before the market opens, entrepreneur-2 must be a buyer, which is impossible because entrepreneur-2 doesn't have cash. The market can only clear for entrepreneur-0 and entrepreneur-1. The market price is then determined by

$a_0(1)(p_1) + a_1(1)(p_1) = \frac{2}{3}$. Solve the market price to obtain $p_1 = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} = p_0$. The price is the same as time 0 after a good return of y_1.

On the other hand, if entrepreneur-1 suffers a bad return and obtains nothing from his production y_1, he is left with $\frac{1}{3} - \beta_1(0)$ units of x without cash. He defaults on his loan and the bank sells the collateral to the market, hoping to fetch the price $v_1(0, \beta_1(0)) = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - \beta_1(0)\frac{\sigma^2}{\gamma}$. This price is received as long as the other two entrepreneurs, entrepreneur-2 and entrepreneur-0, are able to buy. Since entrepreneur-2 at time 0 has already used his cash for production, he is not able to purchase from the market. As a result, entrepreneur-0 will be the only buyer. Moreover, the default of entrepreneur-1 is public information, which also affects bank-2's pricing model $v_2(1, \beta)$ on the collateral x. Recall that the lending bank evaluates the collateral by assuming it can be sold to both entrepreneurs: entrepreneur-1 and entrepreneur-0. Now that entrepreneur-1 defaults, bank-2 will update the pricing model at time 1 as $v_2(1, \beta_2(0)) = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - \beta_2(0)\frac{2\sigma^2}{\gamma}$ to reflect the fact that $\beta_2(0)$ units of x will be sold to a single entrepreneur, entrepreneur-0. Entrepreneur-2 will get a margin call $v_2(0, \beta_2(0)) - v_2(1, \beta_2(0))$ from bank-2.

There're two issues concerning entrepreneur-2's response: his ability and his willingness. We first find conditions on which entrepreneur-2 is able to satisfy the marginal call. Assume now entrepreneur-2 needs to post additional $\beta_2(1) - \beta_2(0)$ units of x to the bank, where $\beta_2(1) \leq \frac{1}{3}$. The bank holds a total of $\beta_2(1)$ units of x and values it as $\beta_2(1)v_2(1, \beta_2(1)) = \beta_2(1)(1 - \frac{2\sigma^2}{3\gamma} - \beta_2(1)\frac{2\sigma^2}{\gamma})$. Setting this value equal to the value of the loan, we have $\beta_2(0)v_2(0, \beta_2(0)) = \beta_2(1)v_2(1, \beta_2(1))$. Since the right side is an increasing function from condition [2.3.3], entrepreneur-2 satisfies the marginal call if and only if $\beta_2(0)v_2(0, \beta_2(0)) \leq \frac{1}{3}v_2(1, \frac{1}{3})$, the maximal amount of loan backed by $\frac{1}{3}$ unit of collateral. We formalize this as follows

Result 4 (Contagion) *Entrepreneur-2 is able to satisfy the marginal call if and only if* $\beta_2(0)v_2(0,\beta_2(0)) \leq \frac{1}{3}v_2(1,\frac{1}{3})$, *that is,* $\beta_2(0)(1-\frac{2\sigma^2}{3\gamma}-\beta_2(0)\frac{\sigma^2}{\gamma}) < \frac{1}{3}(1-\frac{4\sigma^2}{3\gamma})$.

Given entrepreneur-2's ability, we now examine his two options: fulfilling the bank's collateral call or defaulting. To simplify the notation, we denote by d the loan from the bank, such that $d = \beta_2(0)v_2(0,\beta_2(0))$. If entrepreneur-2 chooses to post additional collateral $\beta_2(1)-\beta_2(0)$ ($\beta_2(1)$ is determined by $\beta_2(1)v_2(1,\beta_2(1)) = d$) to the bank, his time-2 expected utility is

$$U_2 = E(1-\beta_2(1)1_D)x - \frac{1}{\gamma}E(1-\beta_2(1)1_D)^2x^2 - q_2[\theta(1+d)-d]. \quad (2.39)$$

On the other hand, if he chooses to default voluntarily, his time-2 expected utility is

$$U_2^* = E(1-\beta_2(0))x - \frac{1}{\gamma}E(1-\beta_2(0))^2x^2 + q_2[\theta(1+d)]. \quad (2.40)$$

By calculation, we obtain $U_2 - U_2^* = \frac{\sigma^2}{\gamma}(2\beta_2(0)^2 - 3(1-q_2)\beta_2(1)^2)$. Entrepreneur-2 will choose to default if and only if $2\beta_2(0)^2 - 3(1-q_2)\beta_2(1)^2 < 0$.

Result 5 *Given entrepreneur-2's ability to meet the margin call, he defaults if and only if* $2\beta_2(0)^2 - 3(1-q_2)\beta_2(1)^2 < 0$.

The situation for the price of x is worse if entrepreneur-2 is optimal to default, for both bank-1 and bank-2 are to sell x. We'll show this worse scenario as an example in this paper and assume the condition in proposition [5] holds.

Now according to the entrepreneur-0's demand function $a_0(1)(p_1) = \frac{\gamma(1-p_1)}{2\sigma^2}$, we have $a_0(1)(p_1) = \frac{1}{3} + \beta_1(0) + \beta_2(0)$. Derive the price from the equation $p_1 =$

$1-\frac{2}{3}\frac{\sigma^2}{\gamma}-(\beta_1(0)+\beta_2(0))\frac{2\sigma^2}{\gamma}$. Finally, we need to verify that at this price, it's optimal for both entrepreneur-1 and entrepreneur-2 not to sell. According to the two entrepreneurs' demand function $a_1(1)(p) = \frac{\gamma(1-p)}{2\sigma^2}$ and $a_2(1)(p) = \frac{\gamma(1-p)}{2\sigma^2}+q_2\beta_2(0)$, they both want to purchase at such low price p_1, but cannot, because they don't have cash. So the economy is in equilibrium with price p_1.

To sum up, the price of x at time 1 depends on the outcome of y_1, which is illustrated below:

$$\begin{cases} p_1 = p_0 = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma}, & \text{if } y_1 = r; \\ p_1 = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - (\beta_1(0) + \beta_2(0))\frac{2\sigma^2}{\gamma} < p_0, & \text{if } y_1 = 0. \end{cases} \tag{2.41}$$

At time 2

y_2 realizes at time 2. All entrepreneurs now maximize the same objective function:

$$Ea_i(2)x - \frac{1}{\gamma}E(a_i(2)x)^2 + b_i(2) \tag{2.42}$$

for $i = 0, 1, 2$, but with different wealth constraints depending on (y_1, y_2). This same objective function implies the same demand function of x $a_i(2)(p_2) = \frac{\gamma(1-p_2)}{2\sigma^2}$ for $i \in N$. We discuss all possible paths.

For path $(y_1, y_2) = (0, 0)$

For a path with two consecutive bad returns, the wealth constraint for entrepreneur-0 is $a_0(2)p_2 + b_0(2) = (\frac{1}{3} + \beta_1(0) + \beta_2(0))p_2 + (1 - (\beta_1(0) + \beta_2(0))p_1)$, for entrepreneur-1 $a_1(2)p_2 + b_1(2) = (\frac{1}{3} - \beta_1(0))p_2$, and for entrepreneur-2 $a_2(2)p_2 + b_2(2) = (\frac{1}{3} - \beta_2(0))p_2$. As argued before, the new equilibrium price is $p_2 = p_1 = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - (\beta_1(0) + \beta_2(0))\frac{\sigma^2}{\gamma}$ determined by entrepreneur-0's demand function. Both entrepreneur-1 and entrepreneur-2 want to purchase more x at this low price but cannot do so because of their wealth constraints.

For path $(y_1, y_2) = (0, r)$

For a good return of y_2, the wealth constraint for entrepreneur-0 is $a_0(2)p_2 + b_0(2) = (\frac{1}{3} + \beta_1(0) + \beta_2(0))p_2 + (1 - (\beta_1(0) + \beta_2(0))p_1)$, for entrepreneur-1 $a_1(2)p_2 + b_1(2) = (\frac{1}{3} - \beta_1(0))p_2$, and for entrepreneur-2 $a_2(2)p_2 + b_2(2) = (\frac{1}{3} - \beta_1(0))p_2 + (\beta_2(0)v_2(0, \beta_1(0)) + 1)\theta$. Entrepreneur-2 and entrepreneur-0, unlike entrepreneur-0, have no wealth constraints to purchase x. The market price p_2 is determined by $a_0(2)(p_2) + a_2(2)(p_2) = \frac{2}{3} + \beta_1(0)$. Solve the equation to obtain the price $p_2 = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - (\beta_1(0))\frac{\sigma^2}{\gamma}$. At this price, entrepreneur-1 wants to buy but cannot do so due to insufficient wealth. So, the economy is in equilibrium with price $p_2 > p_1$.

For path $(y_1, y_2) = (r, 0)$

Entrepreneur-2 defaults at time 2 due to the bad outcome from production. Bank-2 is now selling $\beta_2(0)$ units of x to the market. The wealth constraint for entrepreneur-0 is $a_0(2)p_2 + b_0(2) = \frac{1}{3}p_2 + 1$, for entrepreneur-1 $a_1(2)p_2 + b_1(2) = \frac{1}{3}p_2 + (1 + \beta_1(0)v_1(0, \beta_1(0)))\theta - \beta_1(0)v_1(0, \beta_1(0))$, and for entrepreneur-2 $a_2(2)p_2 + b_2(2) = (\frac{1}{3} - \beta_2(2))p_2$. The price p_2 is determined by $a_0(2)(p_2) + a_1(2)(p_2) = \frac{2}{3} + \beta_2(0)$. Solve the equation to obtain the price $p_2 = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - \beta_2(0)\frac{\sigma^2}{\gamma}$. Again, we can verify that at this price entrepreneur-2 is willing to buy but cannot do so due to the wealth constraint. So the market is in equilibrium at price $p_2 < p_1$. It should be noted that bank-2 sold x at its model price $p_2 = v_2(0, \beta_1(0))$, so it doesn't suffer any losses.

For path $(y_1, y_2) = (r, r)$

This is the best economy of the four paths. Entrepreneur-2 is able to repay the loan and redeem the collateral. The wealth constraint for entrepreneur-0 is $a_0(2)p_2 + b_0(2) = \frac{1}{3}p_2 + 1$, for entrepreneur-1 $a_1(2)p_2 + b_1(2) = \frac{1}{3}p_2 + (1 + \beta_1(0)v_1(0, \beta_1(0)))\theta - \beta_1(0)v_1(0, \beta_1(0))$, and for entrepreneur-2 $a_2(2)p_2 + b_2(2) = \frac{1}{3}p_2 + (1 + \beta_2(0)v_2(0, \beta_1(0)))\theta - \beta_2(0)v_2(0, \beta_1(0))$. All entrepreneurs have enough cash and the price p_2 is determined by $a_0(2)(p_2) + a_1(2)(p_2) + a_2(2)(p_2) = 1$. Solve the equation to obtain the price $p_2 = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma}$.

The price paths are summarized as follows:

$$\text{if } (y_1, y_2) = (0, 0), \qquad p_0 > p_1 = p_2 \tag{2.43}$$

$$\text{if } (y_1, y_2) = (0, r), \qquad p_0 > p_1 < p_2 \tag{2.44}$$

$$\text{if } (y_1, y_2) = (r, 0), \qquad p_0 = p_1 > p_2 \tag{2.45}$$

$$\text{if } (y_1, y_2) = (r, r), \qquad p_0 = p_1 = p_2. \tag{2.46}$$

2.4.2 The Price Evolution for x in the Market Economy

In this section, entrepreneurs only rely on the market to finance their production. Specifically, entrepreneur-1 and entrepreneur-2 sell x at time 0 to raise cash. As we argued before, the entrepreneur with the less profitable production technology may end up buying x. To make matters simple, we can think of both entrepreneur-1 and entrepreneur-2 as having similar productions so that both are selling x in equilibrium. Finally, we add a superscript $'$ to all notations regarding the market economy to be distinguished from the previous banking economy.

In the market equilibrium at time 0, we denote by $\beta_1'(0)$ and $\beta_2'(0)$ the amount of x sold by entrepreneur-1 and entrepreneur-2, respectively. The equilibrium price is then $p_0' = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - (\beta_1'(0) + \beta_2'(0))\frac{2\sigma^2}{\gamma}$. This is the price for entrepreneur-0 to hold $\frac{1}{3} + \beta_1'(0) + \beta_2'(0)$ units of x according to his demand function $a_0'(0)(p_0') = \frac{\gamma(1-p_0')}{2\sigma^2}$.

A similar argument gives the price paths of x for the three periods. The result is summarized as:

$$p_0' = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - (\beta_1'(0) + \beta_2'(0))\frac{2\sigma^2}{\gamma} \tag{2.47}$$

$$p_1' = p_0', p_2' = p_1', \quad \text{if } (y_1, y_2) = (0, 0) \tag{2.48}$$

$$p_1' = p_0', p_2' = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - \beta_1'(0)\frac{\sigma^2}{\gamma}, \quad \text{if } (y_1, y_2) = (0, r) \tag{2.49}$$

$$p_1' = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - \beta_2'(0)\frac{\sigma^2}{\gamma}, p_2' = p_1', \quad \text{if } (y_1, y_2) = (r, 0) \tag{2.50}$$

$$p_1' = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma} - \beta_2'(0)\frac{\sigma^2}{\gamma}, p_2' = 1 - \frac{2}{3}\frac{\sigma^2}{\gamma}, \quad \text{if } (y_1, y_2) = (r, r). \tag{2.51}$$

2.4.3 The Comparison

Figure 2.5-2.8 illustrates the comparisons.

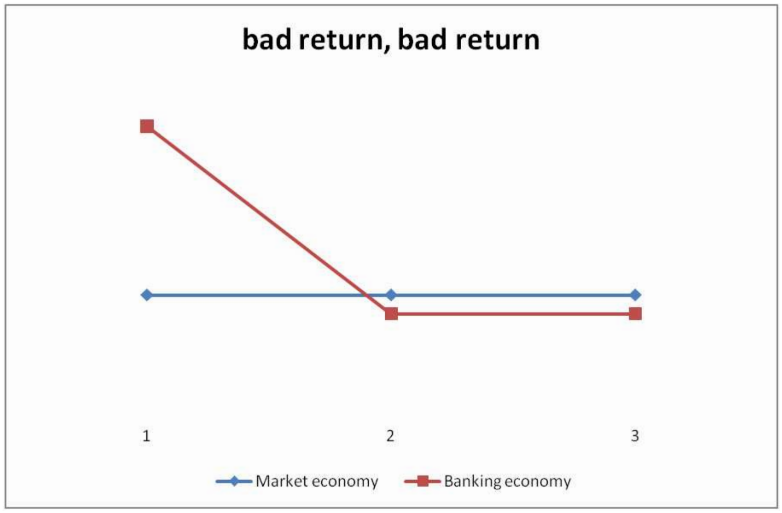

Figure 2.5: The price path of x in the economy with two bad returns $(y_1 = y_2 = 0)$

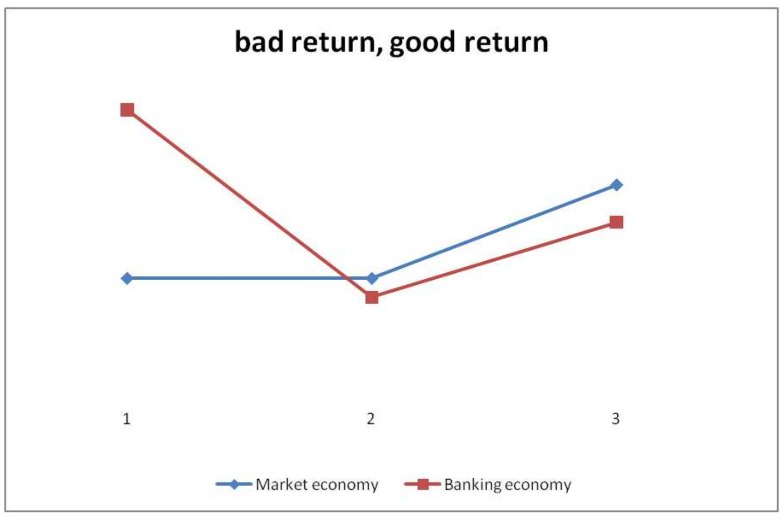

Figure 2.6: The price path of x in the economy with bad return preceding good return $(y_1 = 0, y_2 = r)$

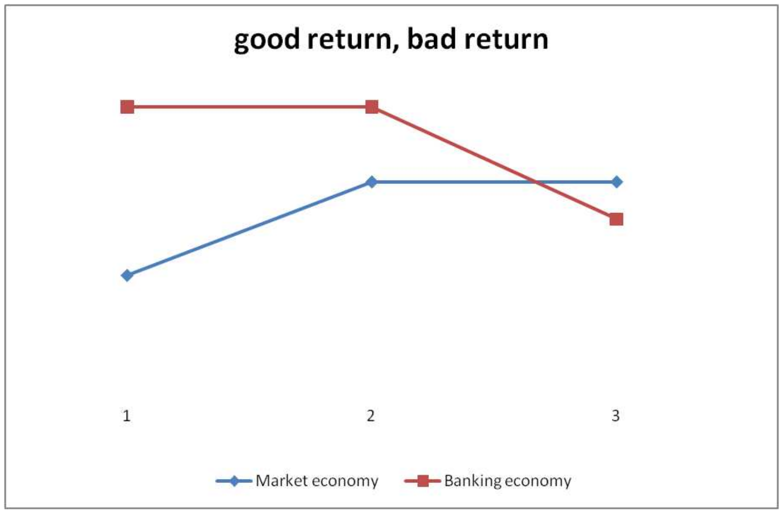

Figure 2.7: The price path of x in the economy with good return preceding bad return $(y_1 = r, y_2 = 0)$

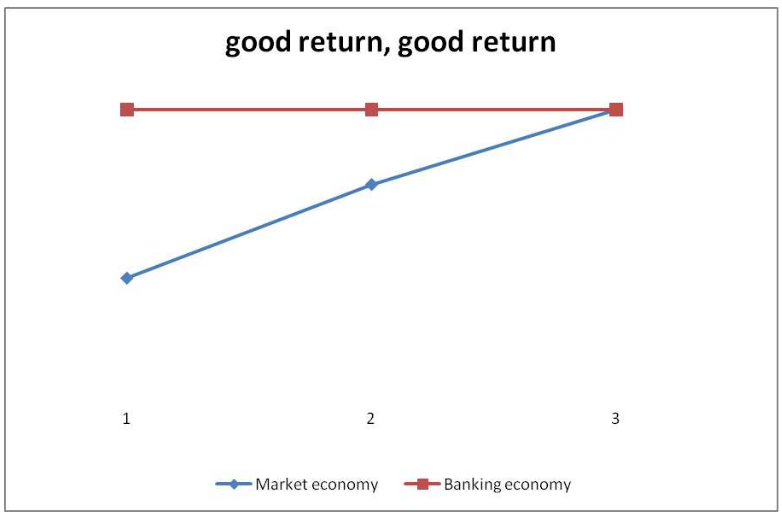

Figure 2.8: The price path of x in the economy with both good returns $(y_1 = r, y_2 = r)$

The timing of the lowest price in the market economy is in the very beginning when entrepreneurs are selling x to raise cash. The price then reflects the entrepreneurs' opportunity cost in addition to the risk of x. In the banking economy, the price at the beginning is artificially high, reflecting the risk of x only.

Both of these observations can be seen from the time 1 price equations: in the market economy $p_0 = \frac{3-2\frac{\sigma^2}{\gamma}}{r^3+q_1\theta r^2+q_2\theta r}$ and in the banking economy $p_0 = 1 - \frac{2\sigma^2}{3\gamma}$. The inflated price can be deflated by a default of one entrepreneur at time 1 when all banks realize that their models are invalid. Without being able to secure more collateral, all banks liquidate the collateral in the market–dumping to a single buyer, entrepreneur-0–driving the price to the bottom. Considering that entrepreneur-2 still has an on going production that allows him to possibly repay the loan upon maturity, it's advisable for bank-2 to wait. But the waiting only makes sense under a book cost accounting rule, which permits banks to record losses later. Mark-to-market accounting rule doesn't favor waiting.

2.4.4 Summary

With the multi-period model, we are able to examine the entrepreneurs' default and the banks' liquidation in detail. First, by using the same asset x as collateral, the entrepreneurs' balance sheets are tied together. Meanwhile, each bank's valuation on collateral is subject to the aggregate buying power of entrepreneurs. For such a closely linked web, even if there's one entrepreneur that defaults, all entrepreneurs may end up defaulting. This is because banks, observing a single default, need to reevaluate the collateral, sending margin calls to their borrowers. When some of the borrowers cannot satisfy the margin call, the collateral value goes down further, which forces banks to request more collateral. The vicious cycle could ultimately bankrupt all entrepreneurs. To illustrate the above effects, we assumed two things in the model. First is that there's no haircut for the collateral so that borrowers are vulnerable to a small change of collateral value. Second is that there's no liquidity for the borrowers' production, in that it cannot be sold to satisfy margin call before maturity.

2.5 Conclusion

We have built a model by adding banks to a general equilibrium setting and have shown that banks attract all entrepreneurs in need of financing. Extending the framework to a multi-period model, we explore the characteristics of the collateral's price. Because banks are able to lure away from the market all entrepreneurs that are searching for funds for production, the market price stays high in the beginning under no selling pressure. But this high price is an illusion, for banks cannot remove the low asset price, rather, they merely postpone it. The asset price will eventually be in line with the aggregate wealth in the economy.

The model used in the paper is the first step to research how collateral borrowing affects entrepreneurs and hence the economy in general. A more general model should include the hair cut or lending rates that are higher than the market free rate. After imposing the two, bank loan financing will look less attractive, and there might exist an equilibrium where entrepreneurs are indifferent between borrowing from banks and selling the asset in the market. Another important direction is to allow banks to base their loans not just on a single market price but also on the trading volume. After all, when banks model the value of the collateral, they're looking for the demand curve. A single price is insufficient to determine a curve.

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