HILBERT FUNCTIONS AND FREE RESOLUTIONS

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Hilbert functions and free resolutions are central concepts in the field of Commutative Algebra. In chapter 3 we prove some cases of the well-known Eisenbud-Green-Harris Conjecture. This conjecture characterizes the Hilbert functions of graded ideals containing a regular sequence in the polynomial ring. In chapter 4 we study the Hilbert functions of graded ideals in toric rings. We prove that Macaulay's Theorem holds for some projective monomial curves, and show that Macaulay's Theorem does not hold for all projective monomial curves. In the last chapter we construct explicitly the minimal free resolutions of linear edge ideals.

BIOGRAPHICAL SKETCH

Ri-Xiang Chen was born on May 9, 1981, in Jiangyan City, Jiangsu Province, P.R.China. When he was about 10, Chen found himself good at doing word problems. And gradually, math became his favorite subject.

Chen's interest in math increased during his 3-year study at Shengao Middle School, where he was lucky to have several good math teachers. Then in 1995 Chen went to Jiangyan high school, where he was fascinated by the process of solving math problems. During his last year in high school, he decided to pursue a career in mathematics.

In 1998 Chen began his math journey at University of Science and Technology of China, where he spent 5 years on studying classical materials in mathematics. After getting a bachelor's degree in 2003, Chen came to the US and became a graduate student at University of California, Santa Barbara, studying differential geometry. After realizing that he liked algebra better, Chen transferred to Cornell University, where he has studied commutative algebra ever since.

Dedicated to my mother, Guilan Zhu.

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CHAPTER 1

INTRODUCTION

In the late nineteenth century, David Hilbert [Hi] introduced the notions of Hilbert functions and free resolutions. From then on, Hilbert functions and free resolutions became central concepts in the study of commutative rings and their modules.

Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k with $\deg(x_i) = 1$ for $1 \leq i \leq n$. We say that a polynomial f in S is *homogeneous* if every term of f has the same degree. A homogeneous polynomial of degree d is also called a *d-form*. An ideal I in S is called a *graded* ideal if I can be generated by a set of homogeneous polynomials. With the graded structure, we can study the ideal I degree by degree. More precisely, for any integer $d \geq 0$ the set of all homogeneous polynomials of degree d in I forms a finite dimensional vector space over k, which is denoted by I_d.

For a graded ideal I in S, the *Hilbert function* of I is the sequence $\{\dim_k I_d\}_{d\geq 0}$, which measures the degree-by-degree dimensions of I. For example, the Hilbert function of S is the sequence $\{\binom{n-1+d}{d}\}_{d\geq 0}$.

Which sequences of nonnegative integers can be the Hilbert functions of graded ideals in S? This question is answered by the celebrated Macaulay's Theorem [Ma], which says that given any graded ideal I in S there exists a lex ideal L in S with the same Hilbert function. Lex ideals (see Definition 2.1.6) have nice structures and their Hilbert functions are easy to describe.

My research about Hilbert functions is to study the generalizations of Macaulay's Theorem in two different directions. In one direction, I study the

Hilbert functions of some special classes of graded ideals in the polynomial ring S ; in the other direction, I study the Hilbert functions of graded ideals in graded quotient rings S/J where J is a graded ideal in S; see Chapter 3 and Chapter 4 for details.

Let I be a graded ideal in S minimally generated by homegeneous polynomials $f_1, \ldots, f_r$. The Hilbert function of I is closely related to the relations that $f_1, \ldots, f_r$ have. That is, we are interested in homogeneous polynomials $g_1, \ldots, g_r \in S$ such that

$$g_1 f_1 + \cdots + g_r f_r = 0.$$

The solutions to the above equation are called *syzygies*. Similarly, we can look at the relations on the syzygies, the relations on the relations on the syzygies, etc. By Hilbert Syzygy Theorem (Theorem 2.2.2), this process stops in at most n steps, and eventually we will get an exact sequence in the following form:

$$0 \to \bigoplus_j S(-j)^{\beta_{l,j}} \xrightarrow{d_l} \bigoplus_j S(-j)^{\beta_{l-1,j}} \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_2} \bigoplus_j S(-j)^{\beta_{1,j}} \xrightarrow{d_1} \bigoplus_j S(-j)^{\beta_{0,j}} \xrightarrow{d_0} I \to 0,$$

where $l \leq n$, $S(-j)$ is the ring S but with a shift in grading (i.e. $S(-j)_d = S_{-j+d}$, for example, in $S(-1)$, x_1 has degree 2), and the differential maps $d_0, \ldots, d_l$ are graded of degree 0 and are given by matrices whose entries are homogeneous polynomials in S. This exact sequence is called a *free resolution* of the graded ideal I over the polynomial ring S. A free resolution is called *minimal* if the graded maps $d_0, \ldots, d_l$ are given by matrices whose entries are homogeneous polynomials in the maximal ideal $(x_1, \ldots, x_n)$. In the minimal case, the numbers $\beta_{i,j}$ are called the *graded Betti numbers* of I.

There is a formula (Theorem 2.2.3) for calculating the Hilbert function of I in terms of the graded Betti numbers of I. So minimal free resolutions have more information than Hilbert functions and are often harder to be obtained.

My research on minimal free resolutions is mainly about monomial resolutions. Namely, I study the minimal free resolutions of monomial ideals: such ideals are generated by monomials. In Chapter 5 we will construct the minimal free resolutions of a class of monomial ideals.

CHAPTER 2

BACKGROUND

2.1 Hilbert functions and lex ideals

In Chapter 1, we have defined the Hilbert functions of graded ideals in the polynomial ring S. In general, we can define the Hilbert functions of finitely generated graded S-modules.

Definition 2.1.1. A finitely generated S-module M is *graded* if

$$M = \bigoplus_{d\in\mathbb{Z}} M_d \quad \text{and} \quad S_i M_d \subseteq M_{i+d} \text{ for all } i \text{ and } d,$$

where $M_d = \{m \in M \mid \deg m = d\}$ is the k-vector space of degree-d elements of M.

If I is a graded ideal in S, then I and S/I are finitely generated graded S-modules. Also, if J is a graded ideal in S/I, then J is a finitely generated graded S-module. Actually, these are the only finitely generated graded S-modules we will study in this thesis, and we will always assume $M_d = 0$ for $d < 0$. Since S is a finitely generated k-algebra and M is a finitely generated S-module, each M_d is a finite dimensional vector space over k.

Definition 2.1.2. Let M be a finitely generated graded S-module. The *Hilbert function* of M is the sequence of non-negative integers $\{\dim_k M_d\}_{d\geq 0}$. The generating function of this sequence is called the *Hilbert series* of M, which is denoted by

$$\mathrm{Hilb}_M(t) := \sum_{d\geq 0} (\dim_k M_d) t^d.$$

Example 2.1.3. Let $j \geq 0$, then the Hilbert series of $S(-j)$ is

$$\begin{aligned}
\mathrm{Hilb}_{S(-j)}(t) &= \sum_{d \geq 0} (\dim_k S(-j)_d) t^d \\
&= \sum_{d \geq j} (\dim_k S_{d-j}) t^d \\
&= t^j \sum_{d \geq 0} (\dim_k S_d) t^d \\
&= t^j \sum_{d \geq 0} \binom{n-1+d}{d} t^d \\
&= \frac{t^j}{(1-t)^n}.
\end{aligned}$$

The study of Hilbert functions is closely related to lex ideals because of the celebrated Macaulay's Theorem.

Definition 2.1.4. The *lexicographic order* on S is a total order $>_{lex}$ on the monomials of S such that $u = x_1^{a_1} \cdots x_n^{a_n} >_{lex} v = x_1^{b_1} \cdots x_n^{b_n}$ if and only if $\deg(u) > \deg(v)$ or $\deg(u) = \deg(v)$ and $a_i > b_i$, where $i = \min\{ j \mid a_j \neq b_j\}$.

Remark 2.1.5. Besides lexicographic order, there is another important monomial order, called the *reverse lexicographic order* $>_{rlex}$, which is defined on the monomials of S such that $u = x_1^{a_1} \cdots x_n^{a_n} >_{rlex} v = x_1^{b_1} \cdots x_n^{b_n}$ if and only if $\deg(u) > \deg(v)$ or $\deg(u) = \deg(v)$ and $a_i < b_i$, where $i = \max\{ j \mid a_j \neq b_j\}$.

Definition 2.1.6. Let L be an ideal in S minimally generated by monomials $m_1, \ldots, m_t$. We say that L is a *lex ideal* if the following property is satisfied: if m is a monomial that is greater lexicographically than m_i and $\deg(m) = \deg(m_i)$ for some $1 \leq i \leq t$, then $m \in L$.

Example 2.1.7. $(x_1^2, x_1x_2, x_1x_3, x_2^3)$ is a lex ideal in $k[x_1, x_2, x_3]$ with the Hilbert function $(0, 0, 3, 7, 12, \cdots)$.

Theorem 2.1.8. *(Macaulay)[Ma] Let I be a graded ideal in S. Then there exists a lex ideal L in S with the same Hilbert function.*

Let J be a graded ideal in S. Can we generalize Macaulay's Theorem to the graded quotient ring S/J? To do this, we first need to generalize the definition of lex ideals in this quotient ring. This is possible when J is a monomial ideal.

Definition 2.1.9. Let M be a monomial ideal in the polynomial ring S. Let I be an ideal in S/M generated by some monomials. Then I is called a *lex ideal* in S/M if there is a lex ieals L in S such that

$$I = \frac{L+M}{M}.$$

By Theorem 2.1.8 we see that if M is a lex ideal in S, then for any graded ideal in S/M, there exists a lex ideal in S/M with the same Hilbert function. Therefore, we say that Macaulay's Theorem holds over S/M when M is a lex ideal. However, if M is not a lex ideal, Macaulay's Theorem may not hold over S/M.

Example 2.1.10. Let $S = k[x_1, x_2, x_3, x_4]$ and $M = (x_1x_2, x_3x_4)$. Let I be the ideal in S/M generated by x_2x_3. Then $\dim_k I_2 = 1$ and $\dim_k I_3 = 2$. Assume that there is a lex ideal L in S/M with the same Hilbert function as I, then x_1^2 must be a generator of L, but then $\dim_k L_3 \geq 3$. Hence, L can not have the same Hilbert function as I, which is a contradiction. So Macaulay's Theorem does not hold over S/M.

The first nontrivial generalization of Macaulay's Theorem is the following Clements-Lindström's Theorem.

Theorem 2.1.11. *(Clements-Lindström)[CL] Let* $R = k[x_1, \ldots, x_n]/(x_1^{a_1}, \cdots, x_n^{a_n})$ *with* $2 \le a_1 \le \cdots \le a_n \le \infty$ *(here we assume* $x_i^\infty = 0$*). Then Macaulay's Theorem holds over R, that is, for any graded ideal in R there is a lex ideal in R with the same Hilbert function; or equivalently, for any graded ideal I in S containing* $x_1^{a_1}, \cdots, x_n^{a_n}$*, there is a lex ideal L in S such that* $L + (x_1^{a_1}, \cdots, x_n^{a_n})$ *has the same Hilbert function as I.*

Note that in the case $a_1 = \cdots = a_n = 2$, the result was obtained earlier by Katona [Ka] and Kruskal [Kr].

If J is not a monomial ideal then in general, we can not define lex ideals in S/J. However, if J is a toric ideal, there is a notion of lex ideals in the toric ring S/J introduced by Gasharov, Horwitz and Peeva [GHP].

Definition 2.1.12. Let $\mathcal{A} = \left\{ \begin{pmatrix} a_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} a_n \\ 1 \end{pmatrix} \right\}$ be a subset of $\mathbb{N}^2 \backslash \{\vec{0}\}$. We set $A = \begin{pmatrix} a_1 & \cdots & a_n \\ 1 & \cdots & 1 \end{pmatrix}$ to be the matrix associated to $\mathcal{A}$, and assume $\text{rank} A = 2$. The *toric ideal* associated to $\mathcal{A}$ is the kernel $I_{\mathcal{A}}$ of the homomorphism:

$$\begin{aligned} \varphi : \quad k[x_1, \ldots, x_n] &\longrightarrow k[u, v] \\ x_i &\longmapsto u^{a_i} v. \end{aligned}$$

The ideal $I_{\mathcal{A}}$ is graded and prime. Set $R = S/I_{\mathcal{A}} \cong k[u^{a_1}v, \ldots, u^{a_n}v]$. Then R is a graded ring with $\deg(x_i) = 1$ for $1 \le i \le n$. We call $R = S/I_{\mathcal{A}}$ the *toric ring* associated to $\mathcal{A}$.

Theorem 2.1.13. *The toric ideal* $I_{\mathcal{A}}$ *is generated by the set of binomials*

$$\{x_1^{p_1} \cdots x_n^{p_n} - x_1^{q_1} \cdots x_n^{q_n} \mid (p_1 - q_1, \ldots, p_n - q_n) \in Ker(A)\}.$$

Definition 2.1.14. An element m in the toric ring $R = S/I_{\mathcal{A}}$ is a *monomial* if there exists a monomial preimage $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ of m in S. For simplicity, by writing $m =$

$x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in R, we mean $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} + I_{\mathcal{A}}$ in R. An ideal in R is a *monomial ideal* if it can be generated by monomials in R. Let $m \in R$ be a monomial, the set of all monomial preimages of m in S is called the *fiber* of m. The lex-greatest monomial in a fiber is called the *top-representative* of the fiber.

Let $m, m' \in R_d$ be two monomials of degree d in R. Let p, p' be the top-representatives of the fibers of m and m' respectively. We say that $m >_{lex} m'$ in R_d if $p >_{lex} p'$ in S.

A *d-monomial space* W is a vector subspace of R_d spanned by some monomials of degree d. A d-monomial space W is *lex* if the following property holds: for monomials $m \in W$ and $q \in R_d$, if $q >_{lex} m$ then $q \in W$. A monomial ideal L in R is *lex* if for every $d \geq 0$, the d-monomial space L_d is lex.

Every projective monomial curve in $\mathbb{P}^{n-1}$ can be defined by $I_{\mathcal{A}}$ for some $\mathcal{A}$. For example, the rational normal curves are defined by the toric ideals associated to matrices of the form

$$A = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

We say that Macaulay's Theorem holds for a projective monomial curve defined by $I_{\mathcal{A}}$ if Macaulay's Theorem holds over the toric ring $R = S/I_{\mathcal{A}}$, that is, for any graded ideal J in R there exists a lex ideal L in R with the same Hilbert function. In Chapter 4 we will show that for some projective monomial curves Macaulay's Theorem holds and for some other projective monomial curves Macaulay's Theorem does not hold.

2.2 Free resolutions and Betti numbers

In Chapter 1, we have defined free resolutions of graded ideals in the polynomial ring S. In general, we can define free resolutions of finitely generated graded S-modules.

Definition 2.2.1. Let M be a finitely generated graded S-module. A *graded free resolution* of M over S is an exact complex

$$\mathbf{F}: 0 \to \bigoplus_j S(-j)^{\beta_{l,j}} \xrightarrow{d_l} \bigoplus_j S(-j)^{\beta_{l-1,j}} \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_2} \bigoplus_j S(-j)^{\beta_{1,j}} \xrightarrow{d_1} \bigoplus_j S(-j)^{\beta_{0,j}} \xrightarrow{d_0} M \to 0,$$

where the differentials d_i are graded maps of degree 0. The resolution $\mathbf{F}$ is called *minimal* if for $i \geq 1$ the maps d_i are given by matrices whose entries are homogeneous polynomials in the maximal ideal $(x_1, \ldots, x_n)$ of S. $\mathbf{F}$ is called a *linear* free resolution if for $i \geq 1$ the maps d_i are given by matrices whose entries are elements in the k-vector space $(x_1, \ldots, x_n)_1$. In the minimal case, the numbers $\beta_{i,j}$ are called the *graded Betti numbers* of M, denoted by $\beta_{i,j}(M)$.

Note that in the above definition the direct sum over j is always finite because M is a finitely generated S-module. It is well-known that any two minimal free resolutions of M are isomorphic; also, if $\mathbf{G}$ is a free resolution of M and $\mathbf{F}$ is a minimal free resolution of M, then $\mathbf{G}$ is isomorphic to the direct sum of $\mathbf{F}$ with a trivial complex.

For any finitely generated graded S-module, there exists a free resolution. Furthermore, we have a bound for the length l of the free resolution.

Theorem 2.2.2. *(Hilbert Syzygy Theorem) Every finitely generated graded S-module has a graded free resolution of length $\leq n$.*

As mentioned in Chapter 1, the Hilbert funtion of M can be calculated from a graded free resolution of M.

Theorem 2.2.3. *Let M be a finitely generated graded S-module with a graded free resolution* $\mathbf{F}$ *as in Definition 2.2.1. Then the Hilbert series of M is given by a rational function:*

$$Hilb_M(t) = \frac{p(t)}{(1-t)^n},$$

where $p(t) = \sum_{i=0}^{l}(-1)^i(\sum_{j\geq 0}\beta_{i,j}t^j) \in \mathbb{Z}[t]$.

Proof. The formula follows from Example 2.1.3 and the fact that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of finitely generated graded S-modules and the maps are graded of degree 0, then

$$\mathrm{Hilb}_{M_2}(t) = \mathrm{Hilb}_{M_1}(t) + \mathrm{Hilb}_{M_3}(t).$$

□

There are two important classes of minimal free resolutions. One is the Koszul complex of a regular sequence; another is the Eliahou-Kervaire resolution of a Borel ideal.

Construction 2.2.4. (Koszul Complex) Let I be a graded ideal in S minimally generated by homogeneous polynomials $f_1, \dots, f_r$ of positive degrees $a_1, \dots, a_r$. Let $K_0 = S$ and for $1 \leq p \leq r$,

$$K_p = \bigoplus_{1\leq i_1<\cdots<i_p\leq r} S(-a_{i_1} - \cdots - a_{i_p}).$$

Let $e_{i_1\dots i_p}$ be the basis element of $S(-a_{i_1} - \cdots - a_{i_p})$, then K_p is a free S-module of rand $\binom{r}{p}$ with basis $\{e_{i_1\dots i_p} \mid 1 \leq i_1 < \cdots < i_p \leq r\}$. Note that 1 is the basis element of

$K_0 = S$. We define the differential map $d_p : K_p \to K_{p-1}$ by setting $d_0(1) = 1 \in S/I$, $d_1(e_i) = f_i \in K_0$ for $1 \le i \le r$, and

$$d_p(e_{i_1 \dots i_p}) = \sum_{j=1}^{p} (-1)^{j-1} f_{i_j} e_{i_1 \dots \hat{i}_j \dots i_p}.$$

One checks easily that $d^2 = 0$. So we get a complex

$$0 \to K_r \xrightarrow{d_r} K_{r-1} \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0 \xrightarrow{d_0} S/(f_1, \dots, f_r) \to 0.$$

This complex is called the *Koszul Complex*, and denoted by $\mathbf{K}(f_1, \dots, f_r)$.

Koszul complexes are closely related to regular sequences.

Theorem 2.2.5. *The Koszul complex $\mathbf{K}(f_1, \dots, f_r)$ is exact if and only if $f_1, \dots, f_r$ is a regular sequence in S, that is, f_i is a non-zero-divisor of $S/(f_1, \dots, f_{i-1})$ for $1 \le i \le r$.*

Note that if $f_1, \dots, f_r$ is a regular sequence of positive degrees in S, then for $1 \le p \le r$ the maps d_p are obviously given by matrices with entries in the maximal ideal $(x_1, \dots, x_n)$ of S. So by the above theorem, $\mathbf{K}(f_1, \dots, f_r)$ is the minimal free resolution of $S/(f_1, \dots, f_r)$.

Example 2.2.6. Given positive integers $a_1, \dots, a_n$, $x_1^{a_1}, \dots, x_n^{a_n}$ is a regular sequence of S, then by Theorem 2.2.5 $\mathbf{K}(x_1^{a_1}, \dots, x_n^{a_n})$ is the minimal free resolution of $S/(x_1^{a_1}, \dots, x_n^{a_n})$. By Theorem 2.2.3 we see that

$$\mathrm{Hilb}_{S/(x_1^{a_1}, \dots, x_n^{a_n})}(t) = \frac{1 - t^{a_1} - \cdots - t^{a_n} + \cdots + (-1)^n t^{a_1 + \cdots + a_n}}{(1-t)^n} = \frac{\prod_{i=1}^n (1 - t^{a_i})}{(1-t)^n}.$$

Similarly, if $f_1, \dots, f_n$ is a regular sequence of homogeneous polynomials in S with degrees $a_1, \dots, a_n$, then

$$\mathrm{Hilb}_{S/(f_1, \dots, f_n)}(t) = \frac{\prod_{i=1}^n (1 - t^{a_i})}{(1-t)^n} = \mathrm{Hilb}_{S/(x_1^{a_1}, \dots, x_n^{a_n})}(t).$$

This equality is the starting point of the Eisenbud-Green-Harris Conjecture in Chapter 3.

Next, we will construct the minimal free resolutions of Borel ideals.

Definition 2.2.7. A monomial ideal M in S is a *Borel ideal* if the following condition is satisfied: if $i < j$ and $x_j m \in M$ for some monomial m, then $x_i m \in M$.

For any monomial $m \in S$, we define $\max(m) = \max\{ i \mid x_i \text{ divides } m\}$ and $\min(m) = \min\{ i \mid x_i \text{ divides } m\}$

Lemma 2.2.8. *Let M be a Borel ideal in S. If m is a monomial in M, then there exists a minimal monomial generator u of M such that u divides m and* $\max(u) \leq min(m/u)$*. We call u the* beginning *of m, denoted by $b(m)$.*

Construction 2.2.9. (Eliahou-Kervaire)[EK] Let M be a Borel ideal in S minimallly generated by monomials $m_1, \ldots, m_r$. We construct the Eliahou-Kervaire resolution $\mathbf{E}_M$ as follows.

For each sequence $1 \leq j_1 < \cdots < j_p < \max(m_i)$, let the symbol $(m_i; j_1, \ldots, j_p)$ denote the generator of the free S-module $S(-m_i x_{j_1} \cdots x_{j_p})$ in homological degree $p+1$ and multidegree $m_i x_{j_1} \cdots x_{j_p}$. Here in $S(-m_i x_{j_1} \cdots x_{j_p})$, the element 1 has multidegree $m_i x_{j_1} \cdots x_{j_p}$.

The Eliahou-Kervaire resolution $\mathbf{E}_M$ has basis

$$\mathcal{B} = \{1\} \cup \{(m_i; j_1, \ldots, j_p) \mid 1 \leq j_1 < \cdots < j_p < \max(m_i), 1 \leq i \leq r\},$$

where 1 is the basis in homological degree 0, and in homological degree 1, the basis elements are $(m_1; \emptyset), \ldots, (m_r; \emptyset)$.

We define the map d on the set $\mathcal{B}$ by setting $d(1) = 1$, $d(m_i; \emptyset) = m_i$ for $1 \leq i \leq r$, and for $p \geq 1$,

$$d(m_i; j_1, \ldots, j_p) = \sum_{q=1}^{p} (-1)^{q+1} x_{j_q} (m_i; j_1, \ldots, \hat{j_q}, \ldots, j_p)$$
$$- \sum_{q=1}^{p} (-1)^{q+1} \frac{m_i x_{j_q}}{b(m_i x_{j_q})} (b(m_i x_{j_q}); j_1, \ldots, \hat{j_q}, \ldots, j_p),$$

where the symbols not in $\mathcal{B}$ are regarded as zeros.

Theorem 2.2.10. *Let M be a Borel ideal in S, then the Eliahou-Kervaire resolution $\mathbf{E}_M$ is the minimal free resolution of M.*

Note that lex ideals are Borel ideals. So Construction 2.2.9 also gives the minimal free resolutions of lex ideals, and then it is easy to get the graded Betti numbers of lex ideals.

Example 2.2.11. Let $L = (x_1^2, x_1x_2, x_1x_3, x_2^3)$ be the lex ideal in $k[x_1, x_2, x_3]$ as in Example 2.1.7. By Construction 2.2.9, the minimal free resolution $\mathbf{E}_L$ of S/L has basis

$$1; (x_1^2; \emptyset), (x_1x_2; \emptyset), (x_1x_3; \emptyset), (x_2^3; \emptyset); (x_1x_2; 1), (x_1x_3; 1), (x_1x_3; 2), (x_2^3; 1); (x_1x_3; 1, 2).$$

And we have the map d such that

$$d(x_1x_2; 1) = x_1(x_1x_2; \emptyset) - x_2(x_1^2; \emptyset), \qquad d(x_1x_3; 1) = x_1(x_1x_3; \emptyset) - x_3(x_1^2; \emptyset),$$
$$d(x_1x_3; 2) = x_2(x_1x_3; \emptyset) - x_3(x_1x_2; \emptyset), \quad d(x_2^3; 1) = x_1(x_2^3; \emptyset) - x_2^2(x_1x_2; \emptyset),$$
$$d(x_1x_3; 1, 2) = x_1(x_1x_3; 2) - x_2(x_1x_3; 1) + x_3(x_1x_2; 1).$$

Therefore, the minimal free resolution of S/L is

$$0 \to S(-x_1^2x_2x_3) \xrightarrow{d_3} S(-x_1^2x_2) \oplus S(-x_1^2x_3) \oplus S(-x_1x_2x_3) \oplus S(-x_1x_2^3)$$
$$\xrightarrow{d_2} S(-x_1^2) \oplus S(-x_1x_2) \oplus S(-x_1x_3) \oplus S(-x_2^3) \xrightarrow{d_1} S \to S/L \to 0,$$

where

$$d_3 = \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \\ 0 \end{pmatrix}, \ d_2 = \begin{pmatrix} -x_2 & -x_3 & 0 & 0 \\ x_1 & 0 & -x_3 & -x_2^2 \\ 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & x_1 \end{pmatrix}, \ d_1 = \begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 & x_2^3 \end{pmatrix}.$$

So, $\beta_{1,2}(S/L) = 3, \beta_{1,3}(S/L) = 1;\ \beta_{2,3}(S/L) = 3, \beta_{2,4}(S/L) = 1;\ \beta_{3,4}(S/L) = 1.$

Macaulay's Theorem is a special case of the following theorem.

Theorem 2.2.12. *(Bigatti-Hulett-Pardue) Let I be a graded ideal in S. Let L be the lex ideal in S with the same Hilbert function as I. Then for all i, j,*

$$\beta_{i,j}(S/I) \leq \beta_{i,j}(S/L),$$

that is, every lex ideal has maximal graded Betti numbers among all graded ideals with the same Hilbert function.

The following mapping cone construction is helpful when constructing new resolutions from old ones.

Construction 2.2.13. (Mapping Cone) Let $0 \to M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3 \to 0$ be a short exact sequence of finitely generated graded S-modules. Let

$$\mathbf{F} : 0 \to F_n \xrightarrow{d_n^F} \cdots \xrightarrow{d_2^F} F_1 \xrightarrow{d_1^F} F_0 \xrightarrow{d_0^F} M_1 \to 0$$

be a graded free resolution of M_1. Let

$$\mathbf{G} : 0 \to G_n \xrightarrow{d_n^G} \cdots \xrightarrow{d_2^G} G_1 \xrightarrow{d_1^G} G_0 \xrightarrow{d_0^G} M_2 \to 0$$

be a graded free resolution of M_2. Let $\Phi : \mathbf{F} \to \mathbf{G}$ be be a morphism of complexes of degree zero which is over the map $\phi : M_1 \to M_2$.

Let $C_0 = G_0$ and $C_i = F_{i-1} \oplus G_i$ for $1 \le i \le n+1$. Let $d_0^C = \psi d_0^G$ and for $1 \le i \le n+1$,

$$d_i^C = \begin{pmatrix} -d_{i-1}^F & 0 \\ \Phi_{i-1} & d_i^G \end{pmatrix}.$$

It is easy to check that $d_{i-1}^C d_i^C = 0$. We call the new complex

$$MC(\Phi) : 0 \to C_{n+1} \xrightarrow{d_{n+1}^C} \cdots \xrightarrow{d_2^C} C_1 \xrightarrow{d_1^C} C_0 \xrightarrow{d_0^C} M_3 \to 0$$

the *mapping cone* of Φ.

Theorem 2.2.14. *In the above construction, the mapping cone $MC(\Phi)$ is a graded free resolution of M_3.*

Note that in Construction 2.2.13, even if both **F** and **G** are minimal free resolutions, $MC(\Phi)$ may not be minimal. In chapter 5, We will use the mapping cone construction to get the minimal free resolutions of a class of monomial ideals in the polynomial ring S.

CHAPTER 3

SOME CASES OF THE EISENBUD-GREEN-HARRIS CONJECTURE

3.1 Known results about the conjecture

Given any homogeneous ideal I in S, Macaulay (Theorem 2.1.8) proved that there exists a lex ideal L with the same Hilbert function. As a generalization of Macaulay's Theorem, Clements and Lindström (Theorem 2.1.11) proved that if $I \subset S$ is a homogeneous ideal containing $x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}$ for some integers $2 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ and $1 \leq r \leq n$, then there exists a lex ideal $L \subset S$ such that $L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ has the same Hilbert function as I. Here, $L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ is called *a lex-plus-powers ideal* in S. Motivated by Example 2.2.6, we have the following conjecture.

Conjecture 3.1.1. *(Eisenbud-Green-Harris)[EGH]If $I \subset S$ is a homogeneous ideal containing a regular sequence of forms $f_1, f_2, \ldots, f_r$ of degrees $a_1, a_2, \ldots, a_r$ where $2 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ and $1 \leq r \leq n$, then there exists a homogeneous ideal in S containing $x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}$ with the same Hilbert function.*

The above conjecture is called the EGH Conjecture. By the Clements-Lindström Theorem, the EGH Conjecture can be stated in the following equivalent form: If $I \subset S$ is a homogeneous ideal containing a regular sequence of forms $f_1, f_2, \ldots, f_r$ of degrees $a_1, a_2, \ldots, a_r$, then there exists a lex-plus-powers ideal $L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ in S with the same Hilbert function.

The following are some known cases of the EGH Conjecture.

Theorem 3.1.2. *(Mermin)[Me] If $I \subset S$ is a homogeneous ideal containing a regular* *sequence of monomials $m_1, m_2, \ldots, m_r$ of degrees $a_1, a_2, \ldots, a_r$, then there exists a lex-plus-powers ideal $L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ in S with the same Hilbert function.*

Note that the above theorem is trivial if $r = n$.

Theorem 3.1.3. *(Cooper)[Co1] Let k be an algebraically closed field of characteristic zero. The EGH Conjecture holds if $I \subset S = k[x_1, x_2, x_3]$ has minimal generators which are all in the same degree and two of the minimal generators form a regular sequence in $k[x_1, x_2]$.*

Cooper [Co2] also studied the conjecture for some cases with $r = n = 3$ in a geometric setting.

In [CM, Propositions 9 and 10], Caviglia and Maclagan proved that if the EGH conjecture holds for all regular sequences of length n, then it holds for all regular sequences of length $r \leq n$. So the rest of the paper will always assume $r = n$.

Definition 3.1.4. (Caviglia-Maclagan)[CM] Fix integers $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and let d be a non-negative integer. We say that *EGH(d) holds* if for any homogeneous ideal $I \subset S$ containing a regular sequence of forms of degrees $a_1, a_2, \ldots, a_n$, there exists an homogeneous ideal $J \subset S$ containing $x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}$ such that $\dim_k I_d = \dim_k J_d$ and $\dim_k I_{d+1} = \dim_k J_{d+1}$.

Note that given any non-negative integer d, there is a lex-plus-powers ideal $J = L + (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})$ such that $\dim_k I_d = \dim_k J_d$. And the Clements-Lindström Theorem implies that EGH(d) holds if and only if $\dim_k I_{d+1} \geq \dim_k \{S_1 J_d \bigoplus (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})_{d+1}\}$. It follows that the EGH Conjecture holds if and only if EGH(d) holds for all non-negative integers d. In addition, we only need to check if EGH(d) holds for $d < \sum_{i=1}^{n}(a_i - 1)$ because $I_d = S_d$ for $d > \sum_{i=1}^{n}(a_i - 1)$.

Lemma 3.1.5. *(Caviglia-Maclagan)[CM] Fix integers* $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ *and set* $N = \sum_{i=1}^{n}(a_i - 1)$. *Then for any* $0 \leq d \leq N-1$, *EGH(d) holds if and only if EGH(*$N-1-d$*) holds.*

Theorem 3.1.6. *(Caviglia-Maclagan)[CM] Fix integers* $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$. *If* $a_i > \sum_{j=1}^{i-1}(a_j - 1)$ *for all* $2 \leq i \leq n$ *then the EGH Conjecture holds.*

An immediate consequence of the above theorem is that the EGH Conjecture holds for $n = 2$. Indeed, if $2 \leq a_1 \leq a_2$ then $a_2 > a_1 - 1$. The $n = 2$ case was also obtained by Richert [Ri].

Francisco [Fra] proved the following almost complete intersection case.

Theorem 3.1.7. *(Francisco)[Fra] Fix integers* $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ *and let* d *be an integer such that* $d \geq a_1$. *Let* $I \subset S$ *be a homogeneous ideal minimally generated by forms* $f_1, \ldots, f_n, g$ *where* $f_1, \ldots, f_n$ *is a regular sequence,* $\deg f_i = a_i$ *and* $\deg g = d$. *Let* $J = (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}, m)$, *where* m *is the greatest monomial in lex order in degree* d *not in* $(x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})$. *Then* $\dim_k I_{d+1} \geq \dim_k J_{d+1}$.

In section 3.2 and section 3.3, we will focus on the case $a_1 = a_2 = \cdots = a_n = 2$. The EGH Conjecture was originally stated in this case [EGH]. Richert [Ri] says that he verified the EGH Conjecture for $a_1 = a_2 = \cdots = a_n = 2$ and $n \leq 5$, but this result was not published. Herzog and Popescu [HP] proved that if k is a field of characteristic zero and I is minimally generated by generic quadratic forms, then the EGH Conjecture holds.

3.2 Some new cases of the conjecture

The following proposition implies that EGH(1) holds if $a_1 = \cdots = a_n = 2$.

Proposition 3.2.1. *Let $I = (f_1, \dots, f_n, g_1, \dots, g_m)$ be an ideal in S, where $f_1, \dots, f_n$ is a regular sequence of 2-forms and $g_1, \dots, g_m$ are linearly independent 1-forms over k with $1 \le m \le n$. Set $J = (x_1^2, x_2^2, \dots, x_n^2, x_1, \dots, x_m) \subset S$. Then*

$$\dim_k I_2 \ge \dim_k J_2.$$

Proof. Since $J_2 = (x_1, \dots, x_m)_2 \bigoplus \mathrm{span}\{x_{m+1}^2, \dots, x_n^2\}$, it follows that

$$\dim_k J_2 = \dim_k (x_1, \dots, x_m)_2 + (n - m).$$

Without the loss of generality we can assume that $g_1 = x_1, \dots, g_m = x_m$ and then $I = (x_1, \dots, x_m, f_1, \dots, f_n)$. Hence,

$$\dim_k I_2 = \dim_k (x_1, \dots, x_m)_2 + \dim_k (I/(x_1, \dots, x_m))_2.$$

Set $t = \dim_k (I/(x_1, \dots, x_m))_2$. Then there exists $1 \le i_1 < \cdots < i_t \le n$ such that $\bar{f}_{i_1}, \dots, \bar{f}_{i_t}$ form a basis of the k-vector space $(I/(x_1, \dots, x_m))_2$. Thus we have $I = (x_1, \dots, x_m, f_{i_1}, \dots, f_{i_t})$ which implies that $\mathrm{ht}(I) \le m + t$. Since $f_1, \dots, f_n$ is a regular sequence it follows that $\mathrm{ht}(f_1, \dots, f_n) = n$. But $(f_1, \dots, f_n) \subset I \subset (x_1, \dots, x_n)$ and $\mathrm{ht}(x_1, \dots, x_n) = n$, thus $\mathrm{ht}(I) = n$ which implies $n \le m + t$ and then $t \ge n - m$. So $\dim_k I_2 \ge \dim_k J_2$ and the theorem is proved. □

Theorem 3.2.2. *If $a_1 = a_2 = \cdots = a_n = 2$ and $2 \le n \le 4$ then the EGH Conjecture holds.*

Proof. Let $N = \sum_{i=1}^{n} (a_i - 1)$. Note that EGH(0) always holds trivially and EGH(1) holds by Proposition 3.2.1, so we only need to show that EGH(2),$\dots$, EGH($N - 1$) hold.

If $n = 2$ then $N - 1 = 1$ and there is nothing to prove, so that the EGH Conjecture is true.

If $n = 3$ then $N - 1 = 2$. By Lemma 3.1.5, EGH(2) holds if and only if EGH(0) holds. So EGH(2) holds and the EGH Conjecture is true.

If $n = 4$ then $N - 1 = 3$. By Lemma 3.1.5, EGH(3) holds if and only if EGH(0) holds; EGH(2) holds if and only if EGH(1)holds. Therefore, EGH(2) and EGH(3) hold, and the EGH Conjecture is true. □

Note that if we want to show the cases $n = 5$ and $n = 6$ then EGH(2) needs to be proved directly which is not as simple as Proposition 3.2.1. Richert [Ri] claimed that he had a proof for $n \leq 5$ but not for $n = 6$ because his proof is different from mine.

The EGH Conjecture also holds in the following two simple cases where regular sequences have nice structures.

Proposition 3.2.3. *Let $f_1, \dots, f_n$ be a regular sequence of 2-forms in S. Then the EGH Conjecture holds in the following two cases:*

(1) $f_1 = l_1^2, \dots, f_n = l_n^2$, where $l_i = \sum_{j=1}^{n} a_{ij}x_j$ for $1 \leq i \leq n$, $a_{ij} \in k$ and $\det(a_{ij}) \neq 0$.

(2) For $1 \leq i \leq n$, $f_i = \sum_{m \in S_2} a_{i,m}m$, where the sum is over all monomials m in S_2, $a_{i,m} \in k$ and $a_{i,m} = 0$ for $m <_{lex} x_i^2$. Here we assume $x_1 > x_2 > \cdots > x_n$ and use the lex order.

Proof. (1) Note that the k-algebra map $F : S \longrightarrow S$ defined by $F(x_i) = l_i$ for $1 \leq i \leq n$ is an graded isomorphism. So the Hilbert function is preserved under F^{-1}. It follows that the EGH Conjecture holds.

(2) First we claim that $a_{i,x_i^2} \neq 0$ for all $1 \leq i \leq n$. Indeed, if not, then let j be the smallest integer such that $a_{j,x_j^2} = 0$. If $j = 1$ then $f_1 = 0$ which is a contradiction.

Hence $j > 1$. Since $a_{i,m} = 0$ for $m <_{lex} x_i^2$, it follows that $(f_1, \ldots, f_j) \subseteq (x_1, \ldots, x_{j-1})$, so that

$$(f_1, \ldots, f_n) \subseteq (x_1, \ldots, x_{j-1}, f_{j+1}, \ldots, f_n).$$

Since $f_1, \ldots, f_n$ is a regular sequence, we have that $\mathrm{ht}(f_1, \ldots, f_n) = n$ which implies $\mathrm{ht}(x_1, \ldots, x_{j-1}, f_{j+1}, \ldots, f_n) = n$, but $(x_1, \ldots, x_{j-1}, f_{j+1}, \ldots, f_n)$ is generated by $n-1$ elements and its height can not be n. So we get a contradiction and the claim is proved.

Now we consider the initial ideal $\mathrm{in}_{<_{rlex}}(f_1, \ldots, f_n)$ with respect to the reverse lex order such that $x_n > \cdots > x_1$. With this monomial order, by the above claim it is easy to see that $\mathrm{in}_{<_{rlex}} f_i = x_i^2$. Thus, $\mathrm{in}_{<_{rlex}}(f_1, \ldots, f_n) = (x_1^2, \ldots, x_n^2)$. Given any homogeneous ideal I containing $f_1, \ldots, f_n$, since $\mathrm{in}_{<_{rlex}}(I)$ contains $\mathrm{in}_{<_{rlex}}(f_1, \ldots, f_n) = (x_1^2, \ldots, x_n^2)$ and $\mathrm{in}_{<_{rlex}}(I)$ has the same Hilbert function as I, it follows that I has the same Hilbert function as a monomial ideal containing $x_1^2, \ldots, x_n^2$. So the EGH Conjecture holds.

□

Remark 3.2.4. The above proposition is actually an easy consequence of the fact that the Hilbert function is preserved under $GL(n, k)$ actions on the variables or by taking initial ideas. In part (2) of the above proposition, if we replace "lex" by "reverse lex", or replace "$m <_{lex} x_i^2$" by "$m >_{lex} x_i^2$", then the result still holds.

In general, $f_1, \ldots, f_n$ do not satisfy the assumptions in the above proposition.

By part (2) of the above proposition, the EGH Conjecture in the case of $a_1 = \cdots = a_n = 2$ can be stated in the following equivalent form: If $I \subset S$ is a homogeneous ideal containing a regular sequence of n 2-forms, then there exists a homogeneous ideal in S containing $f_1, \ldots, f_n$ with the same Hilbert function, where $f_1, \ldots, f_n$ are some 2-forms satisfying part (2) of the above proposition.

3.3 Almost complete intersections

This section proves Theorem 3.1.7 for the case $a_1 = \cdots = a_n = 2$. The key ingredient of any proof of the EGH Conjecture should be about the use of the assumption that $f_1, f_2, \ldots, f_n$ is a regular sequence in S. In [Fra], Francisco made use of the fact that if $f_1, f_2, \ldots, f_n$ is a regular sequence in S then the minimal free resolution of $S/(f_1, \ldots, f_n)$ over S is given by the Koszul complex. In this section we will use the regular sequence assumption in different ways. Before proving Theorem 3.3.4, we look at some lemmas about regular sequences. The following lemma is a special case of Proposition 7 in [CM], which was originally proved in [DGO].

Lemma 3.3.1. *(Davis-Geramita-Orecchia)[DGO] Let $f_1, \ldots, f_n$ be a regular sequence of 2-forms in S. Let I be a homogeneous ideal containing $f_1, \ldots, f_n$. Then for all $0 \leq d \leq n$, we have*

$$\dim_k(S/(f_1, \ldots, f_n))_d = \dim_k(S/I)_d + \dim_k(S/((f_1, \ldots, f_n) : I))_{n-d},$$

or equivalently,

$$\dim_k(I/(f_1, \ldots, f_n))_d = \dim_k(S/((f_1, \ldots, f_n) : I))_{n-d}.$$

Lemma 3.3.2. *Let I be an ideal in S minimally generated by some 2-forms. If the height of I is $r \geq 1$, that is, $ht(I) = r$, then I contains a regular sequence $f_1, \ldots, f_r$ of 2-forms.*

Proof. Let s be the maximal integer such that I contains a regular sequence $f_1, \ldots,$ f_s of 2-forms. Then it is easy to see that $s \geq 1$ and we have

$$s = \mathrm{ht}(f_1, \ldots, f_s) \leq \mathrm{ht}(I) = r.$$

Hence, it suffices to show that $s = r$.

To prove by contradiction, we assume $s < r$. Let $f_1, \ldots, f_s$ be a regular sequence of 2-forms contained in I, then $\mathrm{ht}(f_1, \ldots, f_s) = s < r$. Let $P_1, \ldots, P_l$ be the prime divisors of the ideal $(f_1, \ldots, f_s)$. Since S is Cohen-Macaulay, we have $\mathrm{ht}(P_i) = s$ for $1 \leq i \leq l$. If $I \subseteq P_1 \cup \cdots \cup P_l$, then there exists i such that $I \subseteq P_i$, which implies $\mathrm{ht}(I) \leq \mathrm{ht}(P_i) = s < r$; but $\mathrm{ht}(I) = r$, thus I is not contained in $P_1 \cup \cdots \cup P_l$. Since I is generated by 2-forms, it follows that there exists a 2-form f_{s+1} in I such that $f_{s+1} \notin P_1 \cup \cdots \cup P_l$. Thus, f_{s+1} is a non-zero-divisor of $S/(f_1, \ldots, f_s)$. Therefore, I contains a regular sequence $f_1, \ldots, f_s, f_{s+1}$ of 2-forms, which contradicts the definition of s. So $s = r$ and the lemma is proved. □

Lemma 3.3.3. *If $f_1, \ldots, f_n$ is a regular sequence of 2-forms in S and $g_1 f_1 + g_2 f_2 + \cdots + g_n f_n = 0$ for some q-forms $g_1, g_2, \ldots, g_n$, then $g_1, g_2, \ldots, g_n \in (f_1, \ldots, f_n)_q$. More precisely, we have $q \geq 2$ and there exists a skew-symmetric $n \times n$ matrix A of $(q-2)$-forms such that*

$$\begin{pmatrix} g_1 & g_2 & \ldots & g_n \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & \ldots & f_n \end{pmatrix} A.$$

Proof. Let $K(f_1, \ldots, f_n)$ be the Koszul complex with $e_1, \ldots, e_n$ the basis in homological degree 1. Since $f_1, \ldots, f_n$ is a regular sequence, we have $H_1(K(f_1, \ldots, f_n)) = 0$. Thus, if $g_1 f_1 + \cdots + g_n f_n = 0$ then there exists $(q-2)$-forms h_{ij} for $1 \leq i < j \leq n$ such that

$$g_1 e_1 + \cdots + g_n e_n = \sum_{1 \leq i < j \leq n} h_{ij} (f_j e_i - f_i e_j).$$

Comparing the coefficients of $e_1, \ldots, e_n$, we get

$$\begin{pmatrix} g_1 & g_2 & \ldots & g_n \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & \ldots & f_n \end{pmatrix} A,$$

where A is a skew-symmetric matrix with the $(i, j)^{\text{th}}$ entry given by $-h_{ij}$ for $i < j$. □

Theorem 3.3.4. *Let $I \subset S$ be a homogeneous ideal minimally generated by a regular sequence of 2-forms $f_1, \dots, f_n$ and a d-form g with $d \geq 2$. Let $J = (x_1^2, x_2^2, \dots, x_n^2, m)$, where m is the greatest monomial in lex order in degree d not in $(x_1^2, x_2^2, \dots, x_n^2)$. Then* $\dim_k I_{d+1} \geq \dim_k J_{d+1}$.

We will prove this theorem by two different methods. The first method uses Lemma 3.3.1 and Lemma 3.3.2.

Proof. Note that $(f_1, \dots, f_n)_{n+1} = (x_1^2, \dots, x_n^2)_{n+1} = S_{n+1}$, hence $d \leq n$. Since the $d = n$ case is also trivial, we will assume that $2 \leq d \leq n-1$. It is easy to see that $m = x_1 \cdots x_d$ and then $\dim_k J_{d+1} = \dim_k(x_1^2, \dots, x_n^2)_{d+1} + n - d$. On the other hand,

$$\dim_k I_{d+1} = \dim_k(f_1, \dots, f_n)_{d+1} + n - \dim_k \left((f_1, \dots, f_n)_{d+1} \cap S_1 \text{span}\{g\}\right).$$

Let $r = \dim_k \left((f_1, \dots, f_n)_{d+1} \cap S_1 \text{span}\{g\}\right) \leq n$. Since $\dim_k(x_1^2, \dots, x_n^2)_{d+1} = \dim_k(f_1, \dots, f_n)_{d+1}$ we need only to show $r \leq d$.

To prove by contradiction, we assume that $r > d$. Then without the loss of generality, we can assume that $x_1 g, \dots, x_r g \in (f_1, \dots, f_n)_{d+1}$. Then we have $x_1, \dots, x_r \in ((f_1, \dots, f_n) : I)$. Note that

$$\frac{S}{(x_1, \dots, x_r, f_1, \dots, f_n)} \cong \frac{k[x_{r+1}, \dots, x_n]}{(\bar{f}_1, \dots, \bar{f}_n)},$$

where $\bar{f}_1, \dots, \bar{f}_n$ are the images of $f_1, \dots, f_n$ in the quotient ring $S/(x_1, \dots, x_r) \cong k[x_{r+1}, \dots, x_n]$. Since $k[x_{r+1}, \dots, x_n]/(\bar{f}_1, \dots, \bar{f}_n)$ has dimension zero, we have $\text{ht}(\bar{f}_1, \dots, \bar{f}_n) = n - r$. Hence, by Lemma 3.3.2, $(\bar{f}_1, \dots, \bar{f}_n)$ contains a regular sequence $g_1, \dots, g_{n-r}$ of 2-forms in the polynomial ring $k[x_{r+1}, \dots, x_n]$. Thus, for all $i \geq 0$,

$$\dim_k(k[x_{r+1}, \dots, x_n]/(\bar{f}_1, \dots, \bar{f}_n))_i \leq \binom{n-r}{i}.$$

Therefore, by Lemma 3.3.1, we have

$$\begin{aligned}
1 &= \dim_k(I/(f_1,\dots,f_n))_d \\
&= \dim_k(S/((f_1,\dots,f_n):I))_{n-d} \\
&\le \dim_k(S/(x_1,\dots,x_r,f_1,\dots,f_n))_{n-d} \\
&= \dim_k(k[x_{r+1},\dots,x_n]/(\bar{f}_1,\dots,\bar{f}_n))_{n-d} \\
&\le \binom{n-r}{n-d} \\
&= 0, \text{ since } r > d.
\end{aligned}$$

So we get a contradiction and $r \le d$. □

The following proof of Theorem 3.3.4 uses Lemma 3.3.3.

Proof. As in the previous proof, we can assume $2 \le d \le n-1$.

First we consider the case $d = 2$ and $n \ge 3$. Now $J = (x_1^2, x_2^2, \dots, x_n^2, x_1x_2)$ and $\dim_k J_3 = \dim_k(x_1^2,\dots,x_n^2)_3 + n - 2$. On the other hand,

$$\dim_k I_3 = \dim_k(f_1,\dots,f_n)_3 + n - \dim_k\left((f_1,\dots,f_n)_3 \cap S_1\text{span}\{\text{g}\}\right).$$

Since $\dim_k(x_1^2,\dots,x_n^2)_3 = \dim_k(f_1,\dots,f_n)_3$ we need only to show that

$$\dim_k((f_1,\dots,f_n)_3 \cap S_1\text{span}\{\text{g}\}) \le 2.$$

We prove by contradiction, so assume $\dim_k((f_1,\dots,f_n)_3 \cap S_1\text{span}\{\text{g}\}) \ge 3$. Then without the loss of generality we can assume that

$$\begin{aligned}
x_1 g &= \vec{f}\cdot\vec{p}_1, \\
x_2 g &= \vec{f}\cdot\vec{p}_2, \\
x_3 g &= \vec{f}\cdot\vec{p}_3,
\end{aligned}$$

where $\vec{f}$ is the row vector $\begin{pmatrix} f_1, f_2, \ldots, f_n \end{pmatrix}$ and $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are some column vectors of 1-forms. Hence we have

$$g\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \vec{f} \cdot \begin{pmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 \end{pmatrix}.$$

Since

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}\begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix} = 0,$$

it follows that

$$\begin{aligned} &\vec{f} \cdot \begin{pmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 \end{pmatrix}\begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix} \\ &= \vec{f} \cdot \begin{pmatrix} x_2\vec{p}_1 - x_1\vec{p}_2 & x_3\vec{p}_1 - x_1\vec{p}_3 & x_3\vec{p}_2 - x_2\vec{p}_3 \end{pmatrix} = 0. \end{aligned}$$

By Lemma 3.3.3 there are skew-symmetric $n \times n$ matices A_{12}, A_{13}, A_{23} of scalars such that

$$\begin{pmatrix} x_2\vec{p}_1 - x_1\vec{p}_2 & x_3\vec{p}_1 - x_1\vec{p}_3 & x_3\vec{p}_2 - x_2\vec{p}_3 \end{pmatrix} = \begin{pmatrix} A_{12}\vec{f}^{\,T} & A_{13}\vec{f}^{\,T} & A_{23}\vec{f}^{\,T} \end{pmatrix}.$$

Since

$$\begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix}\begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix} = 0,$$

it follows that

$$\begin{pmatrix} A_{12}\vec{f}^{\,T} & A_{13}\vec{f}^{\,T} & A_{23}\vec{f}^{\,T} \end{pmatrix}\begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix} = 0,$$

so that $(x_3A_{12} - x_2A_{13} + x_1A_{23})\vec{f}^{\,T} = 0$. Since $x_3A_{12} - x_2A_{13} + x_1A_{23}$ is an $n \times n$ matrix of 1-forms, it follows from Lemma 3.3.3 that $x_3A_{12} - x_2A_{13} + x_1A_{23} = 0$ and then

$A_{12} = A_{13} = A_{23} = 0$. Thus, $x_2\vec{p}_1 - x_1\vec{p}_2 = 0$ which implies that $\vec{p}_1$ can be divided by x_1. So $g = \vec{f} \cdot (\vec{p}_1/x_1)$ and then $g \in (f_1, \dots, f_n)_2$ which contradicts the assumption that I is minimally generated by $f_1, \dots, f_n, g$. So we have proved the case $d = 2$.

Then we consider the case $d = 3$ and $n \geq 4$. Now $J = (x_1^2, \dots, x_n^2, x_1x_2x_3)$ and $\dim_k J_4 = \dim_k(x_1^2, \dots, x_n^2)_4 + n - 3$. On the other hand,

$$\dim_k I_4 = \dim_k(f_1, \dots, f_n)_4 + n - \dim_k((f_1, \dots, f_n)_4 \cap S_1\text{span}\{\text{g}\}).$$

Since $\dim_k(x_1^2, \dots, x_n^2)_4 = \dim_k(f_1, \dots, f_n)_4$ we need only to show that

$$\dim_k((f_1, \dots, f_n)_4 \cap S_1\text{span}\{\text{g}\}) \leq 3.$$

We prove by contradiction, so assume $\dim_k((f_1, \dots, f_n)_4 \cap S_1\text{span}\{\text{g}\}) \geq 4$. Then without the loss of generality we can assume that

$$x_1g = \vec{f} \cdot \vec{p}_1,$$
$$x_2g = \vec{f} \cdot \vec{p}_2,$$
$$x_3g = \vec{f} \cdot \vec{p}_3,$$
$$x_4g = \vec{f} \cdot \vec{p}_4,$$

where $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$ are some column vectors of 2-forms. Hence we have

$$g\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} = \vec{f} \cdot \begin{pmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \vec{p}_4 \end{pmatrix}.$$

Since

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix} = 0,$$

it follows that

$$\vec{f}\cdot\begin{pmatrix}\vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \vec{p}_4\end{pmatrix}\begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix}$$

$$= \vec{f}\cdot\begin{pmatrix} x_2\vec{p}_1 - x_1\vec{p}_2 & \cdots & x_4\vec{p}_3 - x_3\vec{p}_4\end{pmatrix} = 0.$$

By Lemma 3.3.3 there are skew-symmetric $n \times n$ matices $A_{12}, A_{13}, \ldots, A_{34}$ of 1-forms such that

$$\begin{pmatrix} x_2\vec{p}_1 - x_1\vec{p}_2 & \cdots & x_4\vec{p}_3 - x_3\vec{p}_4\end{pmatrix} = \begin{pmatrix} A_{12}\vec{f}^{\,T} & \cdots & A_{34}\vec{f}^{\,T}\end{pmatrix}.$$

Since

$$\begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix}\begin{pmatrix} x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & x_4 & 0 \\ 0 & -x_2 & -x_3 & 0 \\ x_1 & 0 & 0 & x_4 \\ 0 & x_1 & 0 & -x_3 \\ 0 & 0 & x_1 & x_2 \end{pmatrix} = 0,$$

it follows that

$$\begin{pmatrix} A_{12}\vec{f}^{\,T} & \cdots & A_{34}\vec{f}^{\,T}\end{pmatrix}\begin{pmatrix} x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & x_4 & 0 \\ 0 & -x_2 & -x_3 & 0 \\ x_1 & 0 & 0 & x_4 \\ 0 & x_1 & 0 & -x_3 \\ 0 & 0 & x_1 & x_2 \end{pmatrix} = 0,$$

that is,

$$\begin{pmatrix} (x_3A_{12} - x_2A_{13} + x_1A_{23})\vec{f}^{\,T} & \cdots & (x_4A_{23} - x_3A_{24} + x_2A_{34})\vec{f}^{\,T}\end{pmatrix} = 0.$$

By Lemma 3.3.3 there are skew-symmetric $n \times n$ matices $B_{123,1}, \ldots, B_{123,n}, \ldots, B_{234,n}$ of scalars such that

$$x_3A_{12} - x_2A_{13} + x_1A_{23} = \begin{pmatrix} \vec{f}B_{123,1} \\ \vdots \\ \vec{f}B_{123,n} \end{pmatrix},$$

$$x_4A_{12} - x_2A_{14} + x_1A_{24} = \begin{pmatrix} \vec{f}B_{124,1} \\ \vdots \\ \vec{f}B_{124,n} \end{pmatrix},$$

$$x_4A_{13} - x_3A_{14} + x_1A_{34} = \begin{pmatrix} \vec{f}B_{134,1} \\ \vdots \\ \vec{f}B_{134,n} \end{pmatrix},$$

$$x_4A_{23} - x_3A_{24} + x_2A_{34} = \begin{pmatrix} \vec{f}B_{234,1} \\ \vdots \\ \vec{f}B_{234,n} \end{pmatrix}.$$

Since

$$\begin{pmatrix} x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & x_4 & 0 \\ 0 & -x_2 & -x_3 & 0 \\ x_1 & 0 & 0 & x_4 \\ 0 & x_1 & 0 & -x_3 \\ 0 & 0 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_4 \\ -x_3 \\ x_2 \\ -x_1 \end{pmatrix} = 0,$$

it follows that for any $1 \leq i \leq n$,

$$\vec{f}(x_4B_{123,i} - x_3B_{124,i} + x_2B_{134,i} - x_1B_{234,i}) = 0.$$

Since $x_4B_{123,i} - x_3B_{124,i} + x_2B_{134,i} - x_1B_{234,i}$ is an $n \times n$ matrix of 1-forms, it follows from Lemma 3.3.3 that

$$x_4B_{123,i} - x_3B_{124,i} + x_2B_{134,i} - x_1B_{234,i} = 0,$$

and then $B_{123,1} = \cdots = B_{234,n} = 0$. Thus, $x_3A_{12} - x_2A_{13} + x_1A_{23} = 0$ which implies that $x_2A_{13} - x_1A_{23}$ can be divided by x_3. Let A'_{13} and A'_{23} be the skew-symmetric matrices of 1-forms obtained from A_{13} and A_{23} by keeping only the terms containing x_3, then we have

$$\begin{aligned} A_{12} &= \frac{1}{x_3}(x_2A_{13} - x_1A_{23}) \\ &= \frac{1}{x_3}(x_2A'_{13} - x_1A'_{23}) \\ &= x_2\frac{A'_{13}}{x_3} - x_1\frac{A'_{23}}{x_3}. \end{aligned} \tag{3.1}$$

Thus,

$$x_2\vec{p}_1 - x_1\vec{p}_2 = A_{12}\vec{f}^{\,T} = (x_2\frac{A'_{13}}{x_3} - x_1\frac{A'_{23}}{x_3})\vec{f}^{\,T},$$

and then,

$$x_1(\vec{p}_2 - \frac{A'_{23}}{x_3}\vec{f}^{\,T}) = x_2(\vec{p}_1 - \frac{A'_{13}}{x_3}\vec{f}^{\,T}),$$

so that $\vec{p}_1 - \frac{A'_{13}}{x_3}\vec{f}^{\,T}$ can be divided by x_1. Note that $\frac{A'_{13}}{x_3}$ is an $n\times n$ skew-symmetric matrix of scalars, which implies that $\vec{f}\frac{A'_{13}}{x_3}\vec{f}^{\,T} = 0$. So we have $x_1g = \vec{f}\cdot(\vec{p}_1 - \frac{A'_{13}}{x_3}\vec{f}^{\,T})$ and then $g = \vec{f}\cdot\frac{1}{x_1}(\vec{p}_1 - \frac{A'_{13}}{x_3}\vec{f}^{\,T}) \in (f_1, \ldots, f_n)_3$ which contradicts the assumption that I is minimally generated by $f_1, \ldots, f_n, g$. So we have proved the case $d = 3$.

Proceeding in the same way we can prove the theorem for all $2 \le d \le n-1$ and we are done. □

The second proof actually uses the minimal free resolution (Koszul complex) of $S/(x_1, x_2, \ldots, x_i)$. This is because we add only *one* polynomial g in degree d. If we add two or more polynomials in degree d, things get very complicated and the second proof does not work any more. The first proof also depends heavily on adding just *one* polynomial g. If we add two or more polynomials in degree d, then $((f_1, \ldots, f_n) : I)$ will not always contain many variables as in our first proof.

After proving theorem 3.3.4, it is natural to consider the following problem, which is a special case of the EGH Conjecture.

Problem 3.3.5. *Let* $f_1, \ldots, f_n$ *be a regular sequence of* 2*-forms in* S *with* $n \geq 3$*. Let* $g, h \in S$ *be* 2*-forms such that* $\dim_k(f_1, \ldots, f_n, g, h)_2 = n + 2$*. Is it true that* $\dim_k(f_1, \ldots, f_n, g, h)_3 \geq \dim_k(x_1^2, \ldots, x_n^2, x_1x_2, x_1x_3)_3 = n^2 + 2n - 5$?

From section 2, we know that it is true if $3 \leq n \leq 4$, or if $f_1, \ldots, f_n$ satisfy the assumption of Proposition 3.2.3. From [HP], we know that it is true if g and h are generic 2-forms and $\mathrm{Char}(k) = 0$.

By theorem 3.3.4 we see that $\dim_k((f_1, \ldots, f_n)_3 \cap S_1\mathrm{span}\{g\})$ can only be 0, 1 or 2. In the next proposition we study the case $\dim_k((f_1, \ldots, f_n)_3 \cap S_1\mathrm{span}\{g\}) = 2$ by using a combination of techniques used in the two proofs of Theorem 3.3.4.

Proposition 3.3.6. *Let* $f_1, \ldots, f_n$ *be a regular sequence of* 2*-forms in* S *with* $n \geq 3$*. Let* g, h *be* 2*-forms such that* $\dim_k(f_1, \ldots, f_n, g, h)_2 = n+2$*. If* $\dim_k((f_1, \ldots, f_n)_3 \cap S_1 span\{g\}) = 2$*, then*

$$\dim_k(f_1, \ldots, f_n, g, h)_3 \geq n^2 + 2n - 5.$$

Proof. Since $\dim_k((f_1, \ldots, f_n)_3 \cap S_1\mathrm{span}\{g\}) = 2$, there exists linearly independent 1-forms l_1 and l_2 such that

$$l_1 g = \vec{f} \cdot \vec{p}_1,$$
$$l_2 g = \vec{f} \cdot \vec{p}_2,$$

where $\vec{f}$ is the row vector $\left(f_1, f_2, \ldots, f_n\right)$ and $\vec{p}_1, \vec{p}_2$ are some column vectors of 1-forms.

To prove by contradiction, we assume that $\dim_k(f_1, \ldots, f_n, g, h)_3 < n^2 + 2n - 5$.

Since

$$\begin{aligned}
&\dim_k(f_1, \ldots, f_n, g, h)_3 \\
&= \dim_k(f_1, \ldots, f_n, g)_3 + n - \dim_k((f_1, \ldots, f_n, g)_3 \cap S_1\text{span}\{\text{h}\}) \\
&= (\dim_k(f_1, \ldots, f_n)_3 + n - 2) + n - \dim_k((f_1, \ldots, f_n, g)_3 \cap S_1\text{span}\{\text{h}\}) \\
&= n^2 + 2n - 2 - \dim_k((f_1, \ldots, f_n, g)_3 \cap S_1\text{span}\{\text{h}\}),
\end{aligned}$$

it follows that $\dim_k((f_1, \ldots, f_n, g)_3 \cap S_1\text{span}\{\text{h}\}) \geq 4$. Without the loss of generality, we can assume that

$$\begin{aligned}
x_1h &= l_3g + \vec{f} \cdot \vec{p}_3, \\
x_2h &= l_4g + \vec{f} \cdot \vec{p}_4, \\
x_3h &= l_5g + \vec{f} \cdot \vec{p}_5, \\
x_4h &= l_6g + \vec{f} \cdot \vec{p}_6,
\end{aligned}$$

where l_3, l_4, l_5, l_6 are some 1-forms and $\vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_6$ are some column vectors of 1-forms. Multiplying the above 4 equations by l_1, because $l_1g = \vec{f} \cdot \vec{p}_1$, we get that

$$x_1(l_1h), x_2(l_1h), x_3(l_1h), x_4(l_1h) \in (f_1, \ldots, f_n)_4.$$

By the second proof of Theorem 3.3.4, we conclude that $l_1h \in (f_1, \ldots, f_n)_3$. Similarly, we have $l_2h \in (f_1, \ldots, f_n)_3$. Thus,

$$l_1, l_2 \in ((f_1, \ldots, f_n) : (f_1, \ldots, f_n, g, h)).$$

Without the loss of generality we can assume that $l_1 = x_1$ and $l_2 = x_2$. Therefore, similar to the first proof of Theorem 3.3.4, we have

$$\begin{aligned}
2 &= \dim_k((f_1, \dots, f_n, g, h)/(f_1, \dots, f_n))_2 \\
&= \dim_k(S/((f_1, \dots, f_n) : (f_1, \dots, f_n, g, h)))_{n-2} \\
&\leq \dim_k(S/(x_1, x_2, f_1, \dots, f_n))_{n-2} \\
&= \dim_k(k[x_3, \dots, x_n]/(\bar{f}_1, \dots, \bar{f}_n))_{n-2} \\
&\leq \binom{n-2}{n-2} \\
&= 1,
\end{aligned}$$

which is a contradiction. So $\dim_k(f_1, \dots, f_n, g, h)_3 \geq n^2+2n-5$ and we are done. □

Remark 3.3.7. The key point of the above proof is that there exist two 1-forms l_1 and l_2 such that $l_1, l_2 \in ((f_1, \dots, f_n) : (f_1, \dots, f_n, g, h))$, which is not the case if

$$\dim_k((f_1, \dots, f_n)_3 \cap S_1\text{span}\{\text{g}\}) \neq 2 \text{ and } \dim_k((f_1, \dots, f_n)_3 \cap S_1\text{span}\{\text{h}\}) \neq 2.$$

It would be interesting to study the other two cases of Problem 3.3.5.

We end this section by looking at two criteria and one example about regular sequences. Here we do not assume that $f_1, f_2, \dots, f_n$ are of degrees 2. One simple criterion for $f_1, f_2, \dots, f_n$ being a regular sequence in S is the following:

$$f_1, f_2, \dots, f_n \text{ is a regular sequence } \Longleftrightarrow \text{Rad}(f_1, \dots, f_n) = (x_1, \dots, x_n).$$

The other criterion follows easily from [Mt, Corollary on Page 161], which says: $f_1, \dots, f_n$ is a regular sequence in S if and only if the following condition holds:

$$\text{if } g_1 f_1 + \cdots + g_n f_n = 0 \text{ for some } g_1, \dots, g_n \in S, \text{ then } g_1, \dots, g_n \in (f_1, \dots, f_n).$$

In general, given homogeneous polynomials $f_1, \dots, f_n$ of degrees 2 in S, it is hard to check by hand whether $f_1, \dots, f_n$ form a regular sequence, although generically $f_1, \ldots, f_n$ form a regular sequence. The following example gives a characterization of a special class of regular sequences.

Example 3.3.8. Let $f_1 = x_1 l_1, \ldots, f_n = x_n l_n$ be a sequence of homogeneous polynomials in S, where $l_i = \sum_{j=1}^{n} a_{ij} x_j$ with $a_{ij} \in k$ and $i = 1, \ldots, n$. Let A be the $n \times n$ matrix (a_{ij}). For any $1 \leq r \leq n$ and $1 \leq i_1 < \cdots < i_r \leq n$, let $A[i_1, \ldots, i_r]$ be the submatrix of A formed by rows $i_1, \ldots, i_r$ and columns $i_1, \ldots, i_r$. By looking at the primary decomposition of the ideal $(f_1, \ldots, f_n)$, we see that $f_1, \ldots, f_n$ is a regular sequence if and only if $\det(A[i_1, \ldots, i_r]) \neq 0$ for all $1 \leq r \leq n$ and $1 \leq i_1 < \cdots < i_r \leq n$. It would be interesting to know if the EGH Conjecture holds in this special case.

CHAPTER 4

MACAULAY'S THEOREM FOR SOME PROJECTIVE MONOMIAL CURVES

4.1 Introduction

In 1927, Macaulay proved that for every graded ideal in $S = k[x_1, \ldots, x_n]$ there exists a lex ideal with the same Hilbert function (see Theorem 2.1.8). Then it is interesting to know if similar results hold for graded quotient rings of the polynomial ring S. From 2.1.11 we see that Macaulay's Theorem holds over the quotient ring $k[x_1, \ldots, x_n]/(x_1^{a_1}, \cdots, x_n^{a_n})$, where $2 \leq a_1 \leq \cdots \leq a_n \leq \infty$. Recently, Mermin and Peeva [MP] raised the problem to find other graded quotient rings over which Macaulay's Theorem holds.

Toric varieties, cf. [Fu], have been extensively studied in Algebraic Geometry. They are very interesting because they can be studied with methods and ideas from Algebraic Geometry, Combinatorics, Commutative Algebra and Computational Algebra. In [GHP], Gasharov, Horwitz and Peeva introduced the notion of a lex ideal in the toric ring (see Definition 2.1.12 and Definition 2.1.14) and raised the question [GHP, 4.1] to find projective toric rings over which Macaulay's Theorem holds. They proved in [GHP, Theorem 5.1] that Macaulay's Theorem holds for the rational normal curves.

The goal of this chapter is to study whether Macaulay's Theorem holds for other projective monomial curves.

In Theorem 4.3.1 we prove that Macaulay's Theorem holds for projective monomial curves defined by the toric ideals associated to matrices of the form

$$A = \begin{pmatrix} 0 & 1 & \cdots & n-2 & n-1+h \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \text{ where } n \geq 3, h \in \mathbb{Z}^+.$$

In Theorem 4.4.1 we consider matrices of the form

$$A = \begin{pmatrix} 0 & 1+h & 2+h & \cdots & n-1+h \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}, \text{ where } n \geq 3, h \in \mathbb{Z}^+,$$

and prove that if $h = 1$ or $n = 3$, Macaulay's Theorem holds; otherwise, Macaulay's Thereom does not hold.

Finally, in Theorem 4.4.5 we prove that Macaulay's Theorem does not hold if

$$A = \begin{pmatrix} 0 & 1 & \cdots & m-1 & m+h & \cdots & n-1+h \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $n \geq 4$, $2 \leq m \leq n-2$ and $h \in \mathbb{Z}^+$.

In section 2.1,We have defined toric rings and lex ideals in toric rings. Before invesgating Macaulay's Theorem over toric rings, we list some known results and make some small but useful observations.

By [GHP, Theorem 2.5], we know that for any homogeneous ideal J in R, there exists a monomial ideal M in R such that M has the same Hilbert function as J. So, to show that Macaulay's Theorem holds over R, we only need to prove that given any monomial ideal M in R, there exists a lex ideal L in R with the same Hilbert function. Furthermore, we will use [GHP, Lemma 4.2], which states:

Lemma 4.1.1 (Gasharov-Horwitz-Peeva)**.** *Macaulay's Theorem holds over R if and only if for every $d \geq 0$ and for every d-monomial space W, we have the inequality:*

$$\dim_k R_1 L_W \leq \dim_k R_1 W,$$

where L_W is the lex d-monomial space in R_d such that $\dim_k L_W = \dim_k W$.

Remark 4.1.2. Let W be a d-monomial space spanned by monomials $w_1, \dots, w_s \in R_d$, then we have that

$$\dim_k W = |\{w_1, \dots, w_s\}| \text{ and } \dim_k R_1 W = |\{x_i w_j \in R_{d+1} \mid 1 \le i \le n, 1 \le j \le s\}|.$$

If W' is another d-monomial space spanned by monomials $w'_1, \dots, w'_t \in R_d$, then we have

$$\dim_k W \cap W' = |\{w_1, \dots, w_s\} \cap \{w'_1, \dots, w'_t\}|.$$

Remark 4.1.3. Let m be a monomial in R. Pick a representative $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ from the fiber of m. Then $\varphi(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = u^{\alpha_1 a_1 + \cdots + \alpha_n a_n} v^{\alpha_1 + \cdots + \alpha_n}$, where φ is defined in Definition 2.1.12. This is independent of the choice of the representative. Define

$$u(m) = u(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) := \alpha_1 a_1 + \cdots + \alpha_n a_n.$$

Note that $\deg m = \alpha_1 + \cdots + \alpha_n$, then for monomials $m, m' \in R$,

$$m = m' \iff u(m) = u(m') \text{ and } \deg m = \deg m'.$$

Hence, for any $d \ge 1$, we have a natural order $>_u$ on the monomials in R_d: for monomials $m, m' \in R_d$, we say that $m >_u m'$ if $u(m) < u(m')$. Note that the lex order $>_{lex}$ may not concide with the natural order $>_u$. This is illustrated in the following example.

Example 4.1.4. Let $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, then in R_2, $x_1x_3 >_{lex} x_2^2$, but $x_2^2 >_u x_1x_3$.

We use lex order $>_{lex}$ instead of $>_u$ to define lex ideals in R because we want to have the following crucial property: *If L_d is a lex d-monomial space in R_d, then $R_1 L_d$ is a lex $(d+1)$-monomial space in R_{d+1}.* By [GHP, Theorem 3.4], we know that this property holds for the lex order $>_{lex}$. However, by the above example, it is easy to see that this property does not hold for the natural order $>_u$. Indeed, let $L_1 = \text{span}\{x_1\} \subseteq R_1$, then L_1 is lex with respect to the natural order $>_u$ and $R_1L_1 = \text{span}\{x_1^2, x_1x_2, x_1x_3\} \subseteq R_2$; but in R_2, since $x_1^2 >_u x_1x_2 >_u x_2^2 >_u x_1x_3$, one sees that R_1L_1 is not lex with respect to the natural order $>_u$.

Remark 4.1.5. In the polynomial ring S we have the following property: if L_d is a lex d-monomial space in S_d and m is the first monomial in $S_d \backslash L_d$, then

$$\dim_k S_1(L_d + km) > \dim_k S_1 L_d, \tag{*}$$

and in particular, $x_n m \notin S_1 L_d$. However, this may not be true in R, and we have the following example.

Example 4.1.6. Let $A = \begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $L_2 = \text{span}\{x_1^2, x_1x_2, x_1x_3, x_1x_4\}$ and $m = x_2^2$, then L_2 is lex in R_2 and m is the first monomial after x_1x_4. Since

$$u(x_1x_2^2) = u(x_2x_1x_2), \qquad u(x_2x_2^2) = u(x_1x_1x_3),$$

$$u(x_3x_2^2) = u(x_2x_1x_4), \qquad u(x_4x_2^2) = u(x_3x_1x_3),$$

it follows that $R_1(L_2 + km) = R_1L_2$ and $x_4m \in R_1L_2$. Thus, $\dim_k R_1(L_2 + km) = \dim_k R_1L_2$ and (*) fails.

4.2 Lemmas for general projective monomial curves

In this section, we prove three lemmas which hold for projective monomial curves. These lemmas will be used later in section 4.3 and section 4.4.

First we make the following observation. Let $I_{\mathcal{A}}$ be the toric ideal associated to $\mathcal{A} = \left\{ \begin{pmatrix} a_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ 1 \end{pmatrix} \right\}$; then without the loss of generality, we can assume that $a_i \neq a_j$ for $i \neq j$. By changing the order of the variables in S, we can assume $a_1 < \cdots < a_n$. Let $B = \begin{pmatrix} 1 & -a_1 \\ 0 & 1 \end{pmatrix}$ and $p = \gcd(a_2 - a_1, \cdots, a_n - a_1)$, then we have

$$\frac{1}{p} BA = \begin{pmatrix} 0 & (a_2 - a_1)/p & \cdots & (a_n - a_1)/p \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Since A and $\frac{1}{p}BA$ have the same kernel, by Theorem 2.1.13 they define the same toric ideal, so that we can always assume that $0 = a_1 < a_2 < \cdots < a_n$ and $\gcd(a_2, \cdots, a_n) = 1$.

Given a d-monomial space W, in order to calculate $\dim_k R_1 W$ efficiently, we have the following lemma.

Lemma 4.2.1. *Let W be a d-monomial space spanned by monomials $w_1, \dots, w_s \in R_d$ with $u(w_1) < \cdots < u(w_s)$. Then*

$$\dim_k R_1 W = sn - \sum_{1 \leq i < j \leq s} \lambda(w_i, w_j),$$

where

$$\lambda(w_i, w_j) = |\{(p, q) \mid 1 \leq p < q \leq n, u(x_q) - u(x_p) = u(w_j) - u(w_i), \text{and there exist}$$
$$\text{no } p < r < q, i < k < j \text{ such that } u(x_r) - u(x_p) = u(w_j) - u(w_k)\}|.$$

Proof. By induction on s. If $s = 1$, then the assertion is clear. If $s > 1$, then setting $W' = \operatorname{span}\{w_1, \cdots, w_{s-1}\}$, we get

$$\begin{aligned} \dim_k R_1 W &= \dim_k R_1(W' + kw_s) \\ &= \dim_k (R_1 W' + R_1(kw_s)) \\ &= \dim_k R_1 W' + \dim_k R_1(kw_s) - \dim_k R_1 W' \cap R_1(kw_s). \end{aligned}$$

By the induction hypothesis, we have that

$$\dim_k R_1 W' = (s-1)n - \sum_{1 \le i < j \le s-1} \lambda(w_i, w_j), \quad \text{and} \quad \dim_k R_1(kw_s) = n.$$

Note that

$$\begin{aligned}
&\dim_k R_1 W' \cap R_1(kw_s) \\
&= |\{1 \le p \le n \mid x_p w_s = x_q w_i \text{ in } R_{d+1}, \text{ for some } 1 \le i \le s-1, q > p\}| \\
&= \sum_{1 \le i \le s-1} |\{1 \le p \le n \mid x_p w_s = x_q w_i \text{ in } R_{d+1}, \text{ for some } q > p, \text{ and there exists} \\
&\qquad\qquad \text{no } i < k < s \text{ such that } x_p w_s = x_r w_k \text{ for some } r > p\}| \\
&= \sum_{1 \le i \le s-1} \lambda(w_i, w_s).
\end{aligned}$$

So we have

$$\begin{aligned}
\dim_k R_1 W &= (s-1)n - \sum_{1 \le i < j \le s-1} \lambda(w_i, w_j) + n - \sum_{1 \le i \le s-1} \lambda(w_i, w_s) \\
&= sn - \sum_{1 \le i < j \le s} \lambda(w_i, w_j).
\end{aligned}$$

□

The following two lemmas will be helpful when we prove Theorem 4.4.1.

Lemma 4.2.2. *Let* $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ *and* $A' = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ *be such that* $0 = a_1 < a_2 < \cdots < a_n$*,* $0 = b_1 < b_2 < \cdots < b_n$ *and* $a_i + b_{n+1-i} = a_n$ *for* $i = 1, \ldots, n$*. Set* $S = k[x_1, \ldots, x_n]$ *and* $S' = k[y_1, \ldots, y_n]$*. Then we have an isomorphism* $\hat{f} : S \longrightarrow S'$ *with* $\hat{f}(x_i) = y_{n+1-i}$*. Let* $R = S/I_{\mathcal{A}}$ *be the toric ring associated to* A *and* $R' = S'/I_{\mathcal{A}'}$ *the toric ring associted to* A'*; then* $\hat{f}$ *induces an isomorphism* $f : R \longrightarrow R'$ *such that* $f(x_i + I_{\mathcal{A}}) = y_{n+1-i} + I_{\mathcal{A}'}$*.*

Proof. Given a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in S, we have

$$\begin{aligned} u(m) + u(\hat{f}(m)) &= u(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) + u(y_n^{\alpha_1} \cdots y_1^{\alpha_n}) \\ &= \alpha_1 a_1 + \cdots + \alpha_n a_n + \alpha_1 b_n + \cdots + \alpha_n b_1 \\ &= \alpha_1(a_1 + b_n) + \cdots + \alpha_n(a_n + b_1) \\ &= (\alpha_1 + \cdots + \alpha_n)a_n \\ &= \deg(m)a_n. \end{aligned}$$

If $m - m' \in I_{\mathcal{A}}$ for some monomials $m, m' \in S$, then by Remark 4.1.3 we have that $u(m) = u(m')$ and $\deg(m) = \deg(m')$. Hence $u(\hat{f}(m)) = u(\hat{f}(m'))$ and $\deg(\hat{f}(m)) = \deg(\hat{f}(m'))$, so that $\hat{f}(m) - \hat{f}(m') = \hat{f}(m - m') \in I_{\mathcal{A}'}$. Similarly, if $m - m' \in I_{\mathcal{A}'}$, then $\hat{f}^{-1}(m - m') \in I_{\mathcal{A}}$. Thus, $\hat{f}(I_{\mathcal{A}}) = I_{\mathcal{A}'}$ and therefore, $\hat{f}$ induces an isomorphism f from R to R' such that $f(x_i + I_{\mathcal{A}}) = y_{n+1-i} + I_{\mathcal{A}'}$. □

Lemma 4.2.3. *Under the assumption of Lemma 4.2.2,we have the following two properties.*

(1) *If $W \subseteq R_d$ is a d-monomial space spanned by monomials $m_1, \ldots, m_r \in R_d$ with $u(w_1) < \cdots < u(w_r)$, then $f(W) \subseteq R'_d$ is a d-monomial space spanned by monomials $f(w_1), \ldots, f(w_r) \in R'_d$ with $u(f(w_1)) > \cdots > u(f(w_r))$, and* $\dim_k R_1 W = \dim_k R'_1 f(W)$.

(2) *Note that we have defined a lex order $>_{lex}$ in R_d. Now setting $y_n > \cdots > y_1$, we have a lex order $>_{lex'}$ in S' which induces a lex order $>_{lex'}$ in R'_d. Let m be a monomial in R_d with top representative $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, then $f(m)$ is a monomial in R'_d with top representative $\hat{f}(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = y_n^{\alpha_1} \cdots y_1^{\alpha_n}$. Furthermore, if monomials $m, m' \in R_d$ are such that $m >_{lex} m'$, then $f(m) >_{lex'} f(m')$ in R'_d; if L_d is a lex d-monomial space in R_d, then $f(L_d)$ is a lex d-monomial space in R'_d; if Macaulay's Theorem holds over R, then Macaulay's Theorem holds over R'.*

Proof. (1)It is clear that $f(W)$ is a d-monomial space in R'_d. By the proof of Lemma 4.2.2, we see that $u(w_i) + u(f(w_i)) = da_n$, which implies that $u(f(w_i)) > u(f(w_j))$ for $i < j$. Note that $a_p - a_q = b_q - b_p$ for any $p \neq q$ and $u(w_i) - u(w_j) = u(f(w_j)) - u(f(w_i))$ for any $i \neq j$, so that the last part of the assertion follows directly from Lemma 4.2.1.

(2)By contradiction, we assume that $y_n^{\beta_1} \cdots y_1^{\beta_n}$ is in the fiber of $f(m)$ and $y_n^{\beta_1} \cdots y_1^{\beta_n} >_{lex'} y_n^{\alpha_1} \cdots y_1^{\alpha_n}$ in S', then $\hat{f}^{-1}(y_n^{\beta_1} \cdots y_1^{\beta_n}) = x_1^{\beta_1} \cdots x_n^{\beta_n}$ is also in the fiber of m and $x_1^{\beta_1} \cdots x_n^{\beta_n} >_{lex} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in S, which is a contradiction. So we have proved the first part of the assertion, and the rest of the assertion follows easily. □

Remark 4.2.4. If we set $y_1 > \cdots > y_n$ in Lemma 4.2.3 (2), then the assertion may not hold. Indeed, considering Example 4.1.6, we have that $A = A'$; let $m = x_1x_3^2$ in R, then $x_1x_3^2$ is the top-representative of the fiber of m, but $\hat{f}(x_1x_3^2) = y_4y_2^2$ is not the top-represtative of the fiber of $f(m)$. Also, by Theorems 4.3.1 and 4.4.1, we will see that even if Macaulay's Theorem holds over R, it may not hold over R'.

4.3 A class of projective monomial curves

Throughout this section,

$$A = \begin{pmatrix} 0 & 1 & \cdots & n-2 & n-1+h \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \text{ where } n \geq 3, h \in \mathbb{Z}^+,$$

and R is the toric ring associated to A. We prove:

Theorem 4.3.1. *Macaulay's Theorem holds over R.*

For the proof of Theorem 4.3.1, we need the following lemmas 4.3.2, 4.3.3, 4.3.5, 4.3.7, 4.3.8, 4.3.9, 4.3.10, 4.3.11.

Lemma 4.3.2. *Let m be a monomial in R. Suppose that*

$$u(m) = \alpha(n-1+h) + \beta(n-2) + \gamma,$$

where α,β and γ are non-negative integers such that $\beta(n-2)+\gamma < n-1+h$ and $\gamma < n-2$. If $\gamma \neq 0$, then $x_1^{\deg(m)-\alpha-\beta-1}x_{r+1}x_{n-1}^{\beta}x_n^{\alpha}$ is the top-representative of the fiber of m. If $\gamma = 0$, then $x_1^{\deg(m)-\alpha-\beta}x_{n-1}^{\beta}x_n^{\alpha}$ is the top-representative of the fiber of m.

Proof. Pick a monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ from the fiber of m, and run the following algorithm.

Input: $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$

Step 1: If $\sum_{i=1}^{n-1}\alpha_i(i-1) < n-1+h$, go to Step 2. Otherwise, choose $\beta_2,\dots,\beta_{n-1} \in \mathbb{Z}$ such that $0 \le \beta_2 \le \alpha_2, \dots, 0 \le \beta_{n-1} \le \alpha_{n-1}$, $\sum_{i=2}^{n-1}\beta_i(i-1) \ge n-1+h$ and $\sum_{i=2}^{n-1}\beta_i(i-1)$ is minimial with respect to this property. Run the division algorithm, we get $\sum_{i=2}^{n-1}\beta_i(i-1) = \beta_n(n-1+h)+\delta$, for some $\beta_n \ge 1$ and $0 \le \delta < n-1+h$. Let $j = \min\{i \mid \beta_i \neq 0\}$. Then $\delta < j-1$, otherwise, it contradicts to the minimality of $\sum_{i=1}^{n-1}\beta_i(i-1)$. Setting

$$\begin{aligned}
\alpha_j &:= \alpha_j - \beta_j,\\
&\cdots\cdots,\\
\alpha_{n-1} &:= \alpha_{n-1} - \beta_{n-1},\\
\alpha_n &:= \alpha_n + \beta_n,\\
\alpha_{\delta+1} &:= \alpha_{\delta+1} + 1,\\
\alpha_1 &:= \alpha_1 + (\beta_j + \cdots + \beta_{n-1}) - \beta_n - 1,
\end{aligned}$$

we get a new monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ which is still in the fiber of m and is strictly bigger with respect to $>_{lex}$ in S. Go back to step 1.

Step 2: If $\sum_{i=1}^{n-2} \alpha_i(i-1) < n-2$, stop. Otherwise, choose $\beta_2, \ldots, \beta_{n-2} \in \mathbb{Z}$ such that $0 \leq \beta_2 \leq \alpha_2, \cdots, 0 \leq \beta_{n-2} \leq \alpha_{n-2}$, $\sum_{i=2}^{n-2} \beta_i(i-1) \geq n-2$ and $\sum_{i=2}^{n-2} \beta_i(i-1)$ is minimial with respect to this property. Run the division algorithm, we get $\sum_{i=2}^{n-2} \beta_i(i-1) = \beta_{n-1}(n-2) + \delta$, for some $\beta_{n-1} \geq 1$ and $0 \leq \delta < n-2$. Let $j = \min\{i \mid \beta_i \neq 0\}$. Then $\delta < j-1$, otherwise, it contradicts to the minimality of $\sum_{i=2}^{n-2} \beta_i(i-1)$. Setting

$$\begin{aligned}
\alpha_j &:= \alpha_j - \beta_j, \\
&\cdots\cdots, \\
\alpha_{n-2} &:= \alpha_{n-2} - \beta_{n-2}, \\
\alpha_{n-1} &:= \alpha_{n-1} + \beta_{n-1}, \\
\alpha_{\delta+1} &:= \alpha_{\delta+1} + 1, \\
\alpha_1 &:= \alpha_1 + (\beta_j + \cdots + \beta_{n-2}) - \beta_{n-1} - 1,
\end{aligned}$$

we get a new monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ which is still in the fiber of m and is strictly bigger with respect to $>_{lex}$ in S. Go back to step 2.

The algorithm stops after finitely many steps and the output of the algorithm is the monomial described in the lemma. If the top-representative of the fiber of m is different from the monomial given in the lemma, then we can run the algorithm on the top-representative to get a bigger monomial in the fiber, which is a contradiction. So the monomial given in the lemma is the top-representative of the fiber of m. □

Lemma 4.3.3. *R has the following two properties.*

(1) Let m be a monomial in R_d; if $w \in S$ is the top-representative of the fiber of m, then $x_n w \in S$ is the top-representative of the fiber of $x_n m \in R_{d+1}$.

(2) *If L_d is a lex d-monomial space in R_d and m is the first monomial in $R_d \backslash L_d$, then* $\dim_k R_1(L_d + km) > \dim_k R_1 L_d$ *and* $x_n m \notin R_1 L_d$.

Proof. (1) Let $\widehat{m} \in S$ be the top-representative of the fiber of $x_n m$. Since $u(x_n m) \geq n-1+h$, by Lemma 4.3.2 we have $x_n | \widehat{m}$. Suppose that $\widehat{m} = x_n w'$ for some monomial $w' \in S$, then it is easy to see that w' is the top-representative of the fiber of m, so that $w' = w$ and $\widehat{m} = x_n w$. So $x_n w$ is the top-representative of the fiber of $x_n m$.

(2) It suffices to prove that $x_n m \notin R_1 L_d$. By contradiction, we assume $x_n m \in R_1 L_d$, then there exist x_i, $1 \leq i < n$ and $m' \in L_d$ such that $x_n m = x_i m'$ in R_{d+1}. Let w, w' be the top-representatives of the fibers of m and m', respectively; then by (1), $x_n w$ is the top-representative of the fiber of $x_n m$. Since $m' >_{\text{lex}} m$ in R_d, we have $w' >_{\text{lex}} w$ in S, and then $x_i w'$ is in the fiber of $x_n m$ such that $x_i w' >_{\text{lex}} x_n w$, which is a contradiction. So, $x_n m \notin R_1 L_d$. □

Definition 4.3.4. Let W be a d-monomial space spanned by some monomials $w_1, \ldots, w_s \in R_d$ with $0 = u(w_1) < \cdots < u(w_s)$. For $i \geq 0$, set

$$W(i) = \{w_j | \text{ the top representative of } w_j \text{ can be divided by } x_n^i \text{ but not by } x_n^{i+1}\}.$$

The set $W(i)$ is called *n-compressed* if $W(i) = \emptyset$ or $W(i) = \{w_{k_i}, w_{k_i+1}, \ldots, w_{k_i+t}\}$, for some $t \geq 0$ and $1 \leq k_i \leq s$, such that

$$u(w_{k_i}) = i(n-1+h), u(w_{k_i+1}) = i(n-1+h)+1, \ldots, u(w_{k_i+t}) = i(n-1+h)+t.$$

We say that a d-monomial space C is *n-compressed* if $C(i)$ is n-compressed for every $i \geq 0$.

Lemma 4.3.5. *Let m_1, m_2 be two monomials in R_d with $u(m_1) < u(m_2)$. Suppose that $u(m_1) = \alpha_1(n-1+h)+\beta_1$, and $u(m_2) = \alpha_2(n-1+h)+\beta_2$, where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are nonnegative integers and $\beta_1, \beta_2 < n-1+h$.*

(1) *If* $\alpha_1 = \alpha_2$, *then* $m_1 >_{lex} m_2$.

(2) *If* $\alpha_1 < \alpha_2$ *and* $\beta_1 - \beta_2 \leq (\alpha_2 - \alpha_1)(n-2)$, *then* $m_1 >_{lex} m_2$.

(3) *If* $\alpha_1 < \alpha_2$ *and* $\beta_1 - \beta_2 > (\alpha_2 - \alpha_1)(n-2)$, *then* $m_2 >_{lex} m_1$.

Proof. By Lemma 4.3.2, we can assume that $\alpha_1 = 0$.

(1) Now $u(m_1) = \beta_1, u(m_2) = \beta_2, 0 \leq \beta_1 < \beta_2 < n-1+h$, and we only need to prove the case $\beta_2 = \beta_1 + 1$. Suppose that $\beta_1 = \beta(n-2) + \gamma$, where β, γ are nonnegative integers and $\gamma < n-2$. If $\gamma = 0$, then $\beta_2 = \beta(n-2) + 1$, so that by Lemma 4.3.2, $x_1^{d-\beta} x_{n-1}^{\beta}$ and $x_1^{d-\beta-1} x_2 x_{n-1}^{\beta}$ are the top-representatives of the fibers of m_1 and m_2 respectively, thus $m_1 >_{\text{lex}} m_2$. If $\gamma > 0$, then $\beta_2 = \beta(n-2) + \gamma + 1$, so that by Lemma 4.3.2, $x_1^{d-\beta-1} x_{\gamma+1} x_{n-1}^{\beta}$ and $x_1^{d-\beta-1} x_{\gamma+2} x_{n-1}^{\beta}$ are the top-representatives of the fibers of m_1 and m_2 respectively, thus $m_1 >_{\text{lex}} m_2$.

(2) Suppose that $\beta_1 = \beta(n-2) + \gamma$, and $\beta_2 = \beta'(n-2) + \gamma'$, where $\beta, \beta', \gamma, \gamma'$ are nonnegative integers and $\gamma, \gamma' < n-2$. Then

$$\beta_1 - \beta_2 = (\beta - \beta')(n-2) + \gamma - \gamma' \leq \alpha_2(n-2),$$

that is,

$$(\beta - (\beta' + \alpha_2))(n-2) \leq \gamma' - \gamma. \tag{*}$$

If $\gamma = \gamma' = 0$, then by (*), we have $\beta \leq \beta' + \alpha_2$; by Lemma 4.3.2, we see that $x_1^{d-\beta} x_{n-1}^{\beta}$ and $x_1^{d-(\beta'+\alpha_2)} x_{n-1}^{\beta'} x_n^{\alpha_2}$ are the top-representatives of the fibers of m_1 and m_2 respectively, so that $m_1 >_{\text{lex}} m_2$. If $\gamma = 0$ and $\gamma' > 0$, then $\gamma' - \gamma < n-2$, hence by (*) we have $\beta \leq \beta' + \alpha_2$; by Lemma 4.3.2, we see that $x_1^{d-\beta} x_{n-1}^{\beta}$ and $x_1^{d-(\beta'+\alpha_2)-1} x_{\gamma'+1} x_{n-1}^{\beta'} x_n^{\alpha_2}$ are the top-representatives of the fibers of m_1 and m_2 respectively, so that $m_1 >_{\text{lex}} m_2$. If $\gamma > 0$ and $\gamma' = 0$, then $\gamma' - \gamma < 0$, hence by (*) we have $\beta < \beta' + \alpha_2$; by Lemma 4.3.2, we see that $x_1^{d-\beta-1} x_{\gamma+1} x_{n-1}^{\beta}$ and $x_1^{d-(\beta'+\alpha_2)} x_{n-1}^{\beta'} x_n^{\alpha_2}$ are the top-representatives of the fibers of m_1 and m_2 respectively, so that $m_1 >_{\text{lex}} m_2$. If $\gamma > 0$ and $\gamma' > 0$, then by Lemma 4.3.2, we see that $x_1^{d-\beta-1}x_{\gamma+1}x_{n-1}^{\beta}$ and $x_1^{d-(\beta'+\alpha_2)-1}x_{\gamma'+1}x_{n-1}^{\beta'}x_n^{\alpha_2}$ are the top-representatives of the fibers of m_1 and m_2 respectively; and by (*), we have either $\gamma' \geq \gamma, \beta \leq \beta' + \alpha_2$ or $\gamma' < \gamma, \beta < \beta' + \alpha_2$, then it follows that $m_1 >_{\text{lex}} m_2$.

(3) We use the notations in the proof of (2). Now $(\beta - (\beta' + \alpha_2))(n-2) > \gamma' - \gamma$. If $\gamma' \geq \gamma$, then $\beta > \beta' + \alpha_2$, and similar to the proof of (2), it is easy to check that $m_2 >_{\text{lex}} m_1$; if $\gamma' < \gamma$, then $\gamma' - \gamma > -(n-2)$, hence $\beta \geq \beta' + \alpha_2$, so that similar to the proof of (2), we get $m_2 >_{\text{lex}} m_1$. □

Remark 4.3.6. By Lemma 4.3.5 we make the following remarks.

(1) By Lemma 4.3.5, we see that the lex order $>_{\text{lex}}$ induces a total oder on the set of nonnegative integers.

(2) If L_d is a lex d-monomail spce, then by Lemma 4.3.5, it is easy to see that L_d is n-compressed and $|L_d(0)| \geq |L_d(1)| \geq |L_d(2)| \geq \cdots$.

(3) If L_d is a lex d-monomail spce and $|L_d(i)| < n - 1 + h$ for some $i \geq 0$, then by Lemma 4.3.5, one sees easily that $|L_d(i+1)| \leq \max\{0, |L_d(i)| - (n-2)\}$.

(4) If L_d is a lex d-monomail spce, then $|L_d(i+j)| \geq (|L_d(i)| - 1) - j(n-2)$ for $i, j \geq 0$. Indeed, if $|L_d(i)| - (|L_d(i+j)| + 1) > j(n-2)$, then by Lemma 4.3.5 (3), it is easy to see that L_d is not lex, which is a contradiction.

(5) Let L_d be a lex d-monomail space spanned by monomials $m_1, \cdots, m_s \in R_d$ with $0 = u(m_1) < \cdots < u(m_s)$, and $L'_{d'}$ a lex d'-monomail space spaned by monomials $m'_1, \cdots, m'_s \in R_{d'}$ with $0 = u(m'_1) < \cdots < u(m'_s)$; then by Lemma 4.3.5, we have $u(m_i) = u(m'_i)$ for $1 \leq i \leq s$. In particular, by Lemma 4.2.1 we have $\dim_k R_1 L_d = \dim_k R_1 L'_{d'}$.

(6) Let W be a d-monomial space spanned by monomials $w_1, \ldots, w_s \in R_d$ with $u(w_1) < \ldots < u(w_s)$. If $u(w_s) > d$, setting $\alpha = u(w_s) - d$ and $W' = \operatorname{span}\{x_1^\alpha w_1, \ldots, x_1^\alpha w_s\} \subseteq R_{d+\alpha}$, we have that $u(x_1^\alpha w_i) = u(w_i)$, $u(x_1^\alpha w_s) = d + \alpha$, and Lemma 4.2.1 implies that $\dim_k R_1 W = \dim_k R_1 W'$. So, by (5) and the above observation, to prove Lemma 4.1.1, we can always assume that $u(w_s) \leq d$, and then for any $0 \leq j \leq u(w_s)$, there exists $m = x_1^{d-j} x_2^j$ in R_d such that $u(m) = j$. Furthermore, there exists $\widehat{w_i} \in R_d$ such that $u(\widehat{w_i}) = u(w_i) - u(w_1)$. Let $\widehat{W} = \operatorname{span}\{\widehat{w_1}, \ldots, \widehat{w_s}\} \subseteq R_d$; then by Lemma 4.2.1, we have $\dim_k R_1 W = \dim_k R_1 \widehat{W}$, so that to prove Lemma 4.1.1, we can also assume that $u(w_1) = 0$.

Lemma 4.3.7. *Let L_d be a lex d-monomial space in R_d such that $L_d \neq R_d$, and m the first monomial in $R_d \backslash L_d$. Then*

$$\dim_k R_1(L_d + km) - \dim_k R_1 L_d = \begin{cases} n, & \text{if } u(m) = 0 \\ 2, & \text{if } 1 \leq u(m) \leq h \\ 1, & \text{if } u(m) > h. \end{cases}$$

Proof. Let $a_m = \dim_k R_1(L_d + km) - \dim_k R_1 L_d$; by Lemma 4.2.1 and Remark 4.3.6 (5), we see that a_m depends only on $u(m)$ and does not depend on d. If $u(m) = 0$, then it is clear that $a_m = n$. If $u(m) > h$, then by Lemma 4.3.3 (2), we see that $a_m \geq 1$.

If $1 \leq u(m) \leq h$, then $a_m \geq 2$. Indeed, if $x_{n-1}m \in R_1 L_d$, then $x_{n-1}m = x_j m'$ in R_d for some $j \neq n-1$ and $m' \in L_d$. Since $u(x_{n-1}m) = u(x_{n-1}) + u(m) \leq n - 2 + h$, it follows that $u(m') \leq n - 2 + h$. Note that $m' >_{\text{lex}} m$, then by Lemma 4.3.5 (1), we see that $u(m') < u(m)$, hence $x_j = x_n$, and then $u(x_{n-1}m) = u(x_n m') \geq n - 1 + h$, which is a contradiction. Thus, $x_{n-1}m \notin R_1 L_d$. By Lemma 4.3.3 (2), we see that $x_n m$ is also not in $R_1 L_d$, so $a_m \geq 2$.

Next we set $d = n + h$ and cosider R_{n+h}. By Lemma 4.3.2, it is easy to see that for any monomial $m \in R_{n+h}$, $u(m) \geq n - 1 + h$ if and only if $m = x_n m'$ for some monomial $m' \in R_{n-1+h}$, so that

$$R_{n+h} = x_n R_{n-1+h} \bigoplus \left(\bigoplus_{i=0}^{n-2+h} k m_i \right),$$

where $m_i = x_1^{n+h-i} x_2^i$ in R_{n+h} is such that $u(m_i) = i$, thus we have

$$\dim_k R_{n+h} - \dim_k R_{n-1+h} = n - 1 + h.$$

On the other hand, since R_{n-1+h} is a lex $(n-1+h)$-monomial space and $R_{n+h} = R_1 R_{n-1+h}$, it follows that

$$\begin{aligned} \dim_k R_{n+h} - \dim_k R_{n-1+h} &= (n-1) + \sum_{1 \leq u(m) \leq h} (a_m - 1) + \sum_{u(m) > h} (a_m - 1) \\ &\geq n - 1 + h. \end{aligned}$$

Since the equality holds, we must have that $a_m = 2$ if $1 \leq u(m) \leq h$ and $a_m = 1$ if $u(m) > h$. □

Lemma 4.3.8. *Let C be an n-compressed d-monomial space.*

(1) $R_1 C$ is an n-compressed $(d+1)$-monomial space.

(2) If C is spanned by monomials $c_1, \ldots, c_s \in R_d$ with $u(c_i) = i - 1$ and $s \leq h + 1$, then $|R_1 C(0)| = n - 2 + s$, $|R_1 C(1)| = s$, $|R_1 C(j)| = 0$ for $j \geq 2$, and $\dim_k R_1 C = n + 2(s - 1)$.

(3) If C is spanned by monomials $c_1, \ldots, c_s \in R_d$ with $u(c_i) = i - 1$ and $h + 2 \leq s \leq n - 1 + h$, then $|R_1 C(0)| = n - 1 + h$, $|R_1 C(1)| = s$, $|R_1 C(j)| = 0$ for $j \geq 2$, and $\dim_k R_1 C = n - 1 + h + s$.

Proof. (1) Let m be a monomial in $R_1 C$ such that $u(m) = p(n - 1 + h) + q$ for some $p \geq 0$ and $1 \leq q < n - 1 + h$; then $m = x_j m'$ for some j and $m' \in C$. If $n - 1 + h$ divides

$u(m')$ then $j \neq 1$ or n, so that $x_{j-1}m' \in R_1C$ and $u(x_{j-1}m') = u(x_jm') - 1 = u(m) - 1$; if $n-1+h$ does not divide $u(m')$, then since C is n-compressed, we have a monomial $m'' \in C$ such that $u(m'') = u(m') - 1$, so that $x_jm'' \in R_1C$ and $u(x_jm'') = u(x_jm') - 1 = u(m) - 1$. So R_1C is an n-compressed $(d+1)$-monomial space.

(2) It is clear that $|R_1C(j)| = 0$ for $j \geq 2$. By Lemma 4.2.1, we have

$$\begin{aligned} \dim_k R_1C &= sn - \sum_{1 \leq i \leq s-1} \lambda(c_i, c_{i+1}) \\ &= sn - (s-1)(n-2) \\ &= n + 2(s-1). \end{aligned}$$

Thus, $|R_1C(0)| + |R_1C(1)| = n + 2(s-1)$. By (1), we know that R_1C is n-compressed, so that $u(x_{n-1}c_s) = n - 2 + s - 1 < n - 1 + h$ and $u(x_nc_s) = n - 1 + h + s - 1$ imply that $|R_1C(0)| \geq n - 2 + s$ and $|R_1C(1)| \geq s$. Thus, $|R_1C(0)| = n - 2 + s$ and $|R_1C(1)| = s$.

(3) It is clear that $|R_1C(j)| = 0$ for $j \geq 2$. By Lemma 4.2.1, we have

$$\begin{aligned} \dim_k R_1C &= sn - \sum_{1 \leq i \leq s-1} \lambda(c_i, c_{i+1}) - \sum_{1 \leq i \leq s-h-1} \lambda(c_i, c_{i+h+1}) \\ &= sn - (s-1)(n-2) - (s-h-1) \\ &= n - 1 + h + s. \end{aligned}$$

Thus, $|R_1C(0)| + |R_1C(1)| = n - 1 + h + s$. By (1), we know that R_1C is n-compressed, so that $u(x_{n+h-s}c_s) = n - 2 + h < n - 1 + h$ and $u(x_nc_s) = n - 1 + h + s - 1$ imply that $|R_1C(0)| \geq n - 1 + h$ and $|R_1C(1)| \geq s$. Thus, $|R_1C(0)| = n - 1 + h$ and $|R_1C(1)| = s$. □

Lemma 4.3.9. *Let W be a d-monomial space spanned by monomials $w_1, \ldots, w_s \in R_d$ with $u(w_1) < \cdots < u(w_s) \leq d$, and $u(w_s) - u(w_1) < n - 1 + h$. Let C be the n-compressed d-monomial space spanned by monomials $c_1, \ldots, c_s \in R_d$ with $u(c_i) = i - 1$ for $1 \leq i \leq s$, and set $\widehat{W} = \{\text{monomial } m \in R_1W \mid u(w_1) \leq u(m) < u(w_1) + n - 1 + h\}$. Then $|\widehat{W}| \geq |R_1C(0)|$ and $\dim_k R_1W \geq \dim_k R_1C$.*

Proof. By Remark 4.3.6 (6), we can assume that $u(w_1) = 0$, then $u(w_s) < n - 1 + h$, and $\widehat{W} = R_1W(0)$. By Lemma 4.3.8, we see that $|R_1C(1)| = s$, hence $|R_1W(1)| \geq s = |R_1C(1)|$. Note that $\dim_k R_1W = |R_1W(0)| + |R_1W(1)|$ and $\dim_k R_1C = |R_1C(0)| + |R_1C(1)|$, thus we only need to prove that $|R_1W(0)| \geq |R_1C(0)|$.

First we suppose $s \leq h + 1$, then by Lemma 4.3.8 we have $|R_1C(0)| = n - 2 + s$. If there exist w_i, w_{i+1} such that $u(w_{i+1}) - u(w_i) > n - 2$, then $0 = u(x_1w_1) < u(x_1w_2) < \cdots < u(x_1w_i) < u(x_2w_i) < \cdots < u(x_{n-1}w_i) < u(x_1w_{i+1}) < \cdots < u(x_1w_s) < n - 1 + h$, which implies that $|R_1W(0)| \geq s + n - 2 = |R_1C(0)|$. So we can assume that $u(w_{i+1}) - u(w_i) \leq n - 2$ for $1 \leq i \leq s - 1$. For any non-negtive integer $l \leq u(x_{n-1}w_s)$, there exists w_i such that $u(w_i)$ is maximal with respect to the property that $u(w_i) \leq l$, then it is easy to see that $0 \leq l - u(w_i) \leq n - 3$ and $u(x_{l-u(w_i)+1}w_i) = l$. Therefore, if $u(x_{n-1}w_s) \geq n - 1 + h$, then

$$|R_1W(0)| = n - 1 + h \geq n - 2 + s = |R_1C(0)|;$$

if $u(x_{n-1}w_s) < n - 1 + h$, then

$$|R_1W(0)| = u(x_{n-1}w_s) + 1 \geq (n - 2) + (s - 1) + 1 = |R_1C(0)|.$$

Next we suppose $h+2 \leq s \leq n-1+h$, then by Lemma 4.3.8 we have $|R_1C(0)| = n - 1 + h$, and it is easy to see that $u(w_{i+1}) - u(w_i) \leq n - 2$ for $1 \leq i \leq s - 1$, and $u(x_{n-1}w_s) \geq n - 1 + h$; therefore, similar to the above argument, we have $|R_1W(0)| = n - 1 + h = |R_1C(0)|$. □

Lemma 4.3.10. *Let W be a d-monomial space spanned by monomials $w_1, \ldots, w_s \in R_d$ with $u(w_1) < \cdots < u(w_s) \leq d$. If there exists $1 \leq i < j \leq s$ such that $j - i \geq h$ and $u(w_j) - u(w_i) < n - 1 + h$, then*

$$\dim_k R_1L_W \leq \dim_k R_1W,$$

where L_W is the lex d-monomial space in R_d such that $\dim_k L_W = \dim_k W$.

Proof. By Lemma 4.3.7, we have that $\dim_k R_1 L_W \leq \dim_k L_W + (n-1) + h = \dim_k W + n - 1 + h = s + n - 1 + h$. On the other hand, it is easy to check that if $1 \leq p < i$, then $x_1 w_p \notin R_1\text{span}\{w_{p+1}, \ldots, w_i, \ldots, w_j\}$; if $j < q \leq s$, then $x_n w_q \notin R_1\text{span}\{w_1, \ldots, w_j, \ldots, w_{q-1}\}$. Thus, we have

$$\dim_k R_1 W \geq \dim_k R_1\text{span}\{w_i, \ldots, w_j\} + (i-1) + (s-j).$$

By Lemma 4.3.8 and 4.3.9, it is easy to see that

$$\dim_k R_1\text{span}\{w_i, \ldots, w_j\} \geq n - 1 + h + (j - i + 1).$$

Therefore, we have

$$\begin{aligned}\dim_k R_1 W &\geq n - 1 + h + (j - i + 1) + (i - 1) + (s - j)\\ &= n - 1 + h + s\\ &\geq \dim_k R_1 L_W.\end{aligned}$$

□

Lemma 4.3.11. *Let C be an n-compressed d-monomial space in R_d, and suppose that there exists $t \geq 0$ such that $0 < |C(i)| \leq h$ for $i = 0, \ldots, t$ and $|C(i)| = 0$ for $i > t$. Then*

$$\dim_k R_1 L_C \leq \dim_k R_1 C,$$

where L_C is the lex d-monomial space in R_d such that $\dim_k L_C = \dim_k C$.

Proof. If $|C(j)| < |C(j+1)| + (n-2)$ for some $0 \leq j \leq t-1$, then we consider the n-compressed d-monomial space C' such that

$$\begin{aligned}|C'(j)| &= |C(j)| + 1,\\ |C'(t)| &= |C(t)| - 1,\\ |C'(i)| &= |C(i)| \text{ if } i \neq j, t.\end{aligned}$$

By Lemma 4.3.8, one sees easily that

$$|R_1C(0)| = |C(0)| + (n-2),$$
$$|R_1C(i)| = \max\{|C(i)| + (n-2), |C(i-1)|\} \text{ for } 1 \leq i \leq t,$$
$$|R_1C(t+1)| = |C(t)|,$$
$$|R_1C(i)| = 0 \text{ for } i > t+1.$$

and we have similar formulas for C'. Then it is easy to check that

$$|R_1C'(j)| \leq |R_1C(j)| + 1,$$
$$|R_1C'(t)| \leq |R_1C(t)|,$$
$$|R_1C'(t+1)| = |R_1C(t+1)| - 1,$$
$$|R_1C'(i)| = |R_1C(i)| \text{ for } i \neq j, t, t+1.$$

Therefore, we have that $\dim_k C' = \dim_k C$ and $\dim_k R_1C' \leq \dim_k R_1C$. If $|C'(j)| = h+1$, then by Lemma 4.3.10, $\dim_k R_1L_C \leq \dim_k R_1C'$, and then $\dim_k R_1L_C \leq \dim_k R_1C$. So we can assume that $|C'(j)| \leq h$, that is, C' satisfies the assumption of the Lemma.

By the above observation, we can assume that C is an n-compressed d-monomial space in R_d and there exists $t \geq 0$, such that $0 < |C(i)| \leq h$ for $0 \leq i \leq t$, $|C(i)| \geq |C(i+1)| + (n-2)$ for $0 \leq i \leq t-1$, and $|C(i)| = 0$ for $i > t$. Then by Lemma 4.3.8, it is easy to see that

$$\begin{aligned} \dim_k R_1C &= |C(0)| + (n-2) + |C(0)| + |C(1)| + \cdots + |C(t)| \\ &= |C(0)| + n - 2 + \dim_k C. \end{aligned}$$

If $|L_C(0)| > |C(0)|$, then by Remark 4.3.6 (4), we have that for $1 \leq i \leq t$,

$$|L_C(i)| \geq |L_C(0)| - 1 - i(n-2) \geq |C(0)| - i(n-2) \geq |C(i)|,$$

and then

$$\begin{aligned}\dim_k L_C &\geq |L_C(0)| + |L_C(1)| + \cdots + |L_C(t)| \\ &> |C(0)| + |C(1)| + \cdots + |C(t)| \\ &= \dim_k C,\end{aligned}$$

which is a contradiction. So we have $|L_C(0)| \leq |C(0)| \leq h$. By Remark 4.3.6 (2), we see that $|L_C(i)| \leq h$ for $i \geq 0$. Thus, by Remark 4.3.6 (3), one sees easily that there exists $t' \geq 0$ such that $|L_C(i)| \geq |L_C(i+1)| + (n-2)$ for $0 \leq i \leq t' - 1$, and $|L_C(i)| = 0$ for $i > t'$. Therefore, by Lemma 4.3.8, it is easy to see that

$$\begin{aligned}\dim_k R_1 L_C &= |L_C(0)| + (n-2) + |L_C(0)| + |L_C(1)| + \cdots + |L_C(t')| \\ &= |L_C(0)| + (n-2) + \dim_k L_C \\ &\leq |C(0)| + n - 2 + \dim_k C \\ &= \dim_k R_1 C.\end{aligned}$$

□

Proof of Theorem 4.3.1. Let W be a d-monomial space spanned by monomials w_1, ..., w_s in R_d with $u(w_1) < \cdots < u(w_s)$; by Lemma 4.1.1, we only need to prove that

$$\dim_k R_1 L_W \leq \dim_k R_1 W,$$

where L_W is the lex d-monomial space in R_d such that $\dim_k L_W = \dim_k W$.

By Remark 4.3.6 (6), we can assume that $u(w_1) = 0$ and $u(w_s) \leq d$. Note that there exist $1 = i_0 < i_1 < \cdots < i_t \leq s$ for some $t \geq 0$ such that $u(w_s) - u(w_{i_t}) < n - 1 + h$, and for $1 \leq j \leq t$, $u(w_{i_j-1}) - u(w_{i_{j-1}}) < n - 1 + h$ and $u(w_{i_j}) - u(w_{i_{j-1}}) \geq n - 1 + h$. Set

$$W[0] = \{w_{i_0}, \ldots, w_{i_1-1}\},$$

$$W[1] = \{w_{i_1}, \ldots, w_{i_2-1}\},$$

$$\ldots\ldots,$$

$$W[t] = \{w_{i_t}, \ldots, w_s\},$$

then by Lemma 4.3.10, we can assume that $|W[j]| \leq h$ for $0 \leq j \leq t$.

Let C be the n-compressed d-monomial space such that $|C(j)| = |W[j]|$ for $0 \leq j \leq t$ and $|C(j)| = 0$ for $j \geq t+1$, then $\dim_k C = \dim_k W$ and it is easy to see that

$$\dim_k R_1C = |R_1C(0)| + |R_1C(1)| + \cdots + |R_1C(t)| + |R_1C(t+1)|,$$

$$\dim_k R_1W = |(R_1W)[0]| + |(R_1W)[1]| + \cdots + |(R_1W)[t]| + |(R_1W)[t+1]|,$$

where $(R_1W)[0] = R_1W(0)$, $(R_1W)[t+1]$ is the set of monomails $m \in R_1W$ such that $u(m) \geq u(w_{i_t}) + n - 1 + h$, and for $1 \leq j \leq t$, $(R_1W)[j]$ is the set of monomials $m \in R_1W$ such that $u(w_{i_{j-1}}) + n - 1 + h \leq u(m) < u(w_{i_j}) + n - 1 + h$. First it is easy to see that

$$|(R_1W)[t+1]| \geq |W[t]| = |C(t)| = |R_1C(t+1)|.$$

Then By Lemma 4.3.9, we get

$$|R_1W(0)| \geq |R_1C(0)|.$$

Finally, by Lemma 4.3.8 it is easy to see that for $1 \leq j \leq t$,

$$|R_1C(j)| = \max\{|C(j-1)|, |C(j)| + (n-2)\};$$

if $|R_1C(j)| = |C(j-1)|$, then we have

$$|(R_1W)[j]| \geq |W[j-1]| = |C(j-1)| = |R_1C(j)|;$$

if $|R_1C(j)| = |C(j)| + (n-2)$, then by Lemma 4.3.9, we also have

$$|(R_1W)[j]| \geq |R_1C(j)|.$$

So, we get $\dim_k R_1W \geq \dim_k R_1C$. By Lemma 4.3.11, we know that $\dim_k R_1C \geq \dim_k R_1L_C$, where L_C is the lex d-monomail space such that $\dim_k L_C = \dim_k C$. Note that $L_C = L_W$, so $\dim_k R_1W \geq \dim_k R_1L_W$. □

4.4 Two other classes of projective monomial curves

The main results of this section are Theorem 4.4.1 and Theorem 4.4.5.

Theorem 4.4.1. *Let*

$$A = \begin{pmatrix} 0 & 1+h & 2+h & \cdots & n-1+h \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}, \text{ where } n \geq 3, h \in \mathbb{Z}^+.$$

Let R be the toric ring associated to A.

(1) If $h = 1$, then Macaulay's Theorem holds over R.

(2) If $n = 3$, then Macaulay's Theorem holds over R.

(3) If $h \geq 2$ and $n \geq 4$, then Macaulay's Theorem does not hold over R.

In order to prove Theorem 4.4.1, we need the following lemmas 4.4.2, 4.4.3, 4.4.4.

Lemma 4.4.2. *Let R be the toric ring defined in Theorem 4.4.1 and R' the toric ring defined in section 4.3 such that R and R' satisfy the assumptions of Lemma 4.2.2; then we have an isomorphism $\hat{f} : S = k[x_1, \ldots, x_n] \longrightarrow S' = k[y_1, \ldots, y_n]$ with $\hat{f}(x_i) = y_{n+1-i}$, which induces an isomorphism f from R to R'. Setting $x_1 > \cdots > x_n$ and $y_1 > \cdots > y_n$ as usual, by definition 2.1.14 we have the lex orders $>_{lex}$, $>_{lex'}$ in R and R'.*

(1) *Let m be a monomial in R_d such that $y_1^{\alpha_1}\cdots y_n^{\alpha_n}$ is the top representative of the fiber of the monomail $f(m) \in R'_d$, then $\hat{f}^{-1}(y_1^{\alpha_1}\cdots y_n^{\alpha_n}) = x_1^{\alpha_n}\cdots x_n^{\alpha_1}$ is the top-representative of the fiber of m.*

(2) *Let m and m' be two monomials in R_d such that $u(m) < u(m')$, then $m >_{lex} m'$ in R_d, so that the lex order $>_{lex}$ in R_d is the same as the natural order $>_u$ definde in Remark 4.1.3.*

Proof. (1)Suppose that $x_1^{\beta_n}\cdots x_n^{\beta_1}$ is the top representative of the fiber of m, then $\beta_n \geq \alpha_n$ and $\hat{f}(x_1^{\beta_n}\cdots x_n^{\beta_1}) = y_1^{\beta_1}\cdots y_n^{\beta_n}$ is a monomial in the fiber of $f(m)$. Since $y_1^{\alpha_1}\cdots y_n^{\alpha_n}$ is the top representative of the fiber of $f(m)$, by Lemma 4.3.2 we have $\beta_n \leq \alpha_n$, so that $\beta_n = \alpha_n$, and then $\beta_{n-1} \geq \alpha_{n-1}$, but by Lemma 4.3.2 we have $\beta_{n-1} \leq \alpha_{n-1}$, so that $\beta_{n-1} = \alpha_{n-1}$. If there exits $2 \leq i \leq n-2$ such that $\beta_i > \alpha_i$ and $\beta_j = \alpha_j$ for $j > i$, then the monomial $y_1^{\beta_1}\cdots y_i^{\beta_i}y_{i+1}^{\alpha_{i+1}}\cdots y_n^{\alpha_n}$ is in the fiber of $f(m)$, by Lemma 4.3.2 on sees easily that $\beta_i \leq \alpha_i$, which is a contradiction, so we have $\beta_i = \alpha_i$ for $i = 2, \ldots, n-2$. Since $\deg(m) = \beta_1 + \cdots + \beta_n = \alpha_1 + \cdots + \alpha_n$, it follows that $\beta_1 = \alpha_1$, and then $x_1^{\alpha_n}\cdots x_n^{\alpha_1} = x_1^{\beta_n}\cdots x_n^{\beta_1}$ is the top-representative of the fiber of m.

(2)Let $y_1^{\alpha_1}\cdots y_n^{\alpha_n}$, $y_1^{\beta_1}\cdots y_n^{\beta_n}$ be the top-representatives of the fibers of $f(m)$ and $f(m')$, then (1) implies that $x_1^{\alpha_n}\cdots x_n^{\alpha_1}$, $x_1^{\beta_n}\cdots x_n^{\beta_1}$ are the top-representatives of the fibers of m and m'. Since $u(m) < u(m')$, by Lemma 4.2.3 (1), we have $u(f(m)) > u(f(m'))$, so that Lemma 4.3.2 implies $\alpha_n \geq \beta_n$. If $\alpha_n > \beta_n$, then $m >_{lex} m'$ and we are done. So we may assume $\alpha_n = \beta_n$. Then similarly, by Lemma 4.3.2 we have $\alpha_{n-1} \geq \beta_{n-1}$, and if $\alpha_{n-1} > \beta_{n-1}$, we are done. So we can also assume that $\alpha_{n-1} = \beta_{n-1}$. Then applying Lemma 4.3.2 again, we see that there exist $2 \leq r \leq n-2$,

$1 \leq r' \leq r-1$ such that

$$y_1^{\alpha_1} \cdots y_n^{\alpha_n} = y_1^{d-1-\alpha_{n-1}-\alpha_n} y_r y_{n-1}^{\alpha_{n-1}} y_n^{\alpha_n},$$
$$y_1^{\beta_1} \cdots y_n^{\beta_n} = y_1^{d-1-\alpha_{n-1}-\alpha_n} y_{r'} y_{n-1}^{\alpha_{n-1}} y_n^{\alpha_n},$$

and then we have that

$$\begin{aligned} x_1^{\alpha_n} \cdots x_n^{\alpha_1} &= x_1^{\alpha_n} x_2^{\alpha_{n-1}} x_{n+1-r} x_n^{d-1-\alpha_{n-1}-\alpha_n} \\ &>_{lex} x_1^{\alpha_n} x_2^{\alpha_{n-1}} x_{n+1-r'} x_n^{d-1-\alpha_{n-1}-\alpha_n} \\ &= x_1^{\beta_n} \cdots x_n^{\beta_1}, \end{aligned}$$

which implies $m >_{lex} m'$. □

Lemma 4.4.3. *Let R be the toric ring defined in Theorem 4.4.1 and suppose $h = 1$. Let L_d be an r dimensional lex d-monomial space in R_d with $0 \leq r < \dim_k R_d$, and m the first monomial in $R_d \backslash L_d$. If we set*

$$a_r = \dim_k R_1(L_d + km) - \dim_k R_1 L_d,$$

then $a_0 = n$, $a_1 = 2$ and $a_r = 1$ for $1 < r < \dim_k R_d$.

Proof. Without the loss of generality, we can assume $d \geq 1$. It is clear that $a_0 = n$. If $r = 1$, then it is easy to see that $L_d = \text{span}\{x_1^d\}$ and $m = x_1^{d-1}x_2$ in R_d, so that by Lemma 4.2.1,

$$\dim_k R_1(L_d + km) = 2n - \lambda(x_1^d, x_1^{d-1}x_2) = 2n - (n-2) = n+2,$$

hence $a_0 + a_1 = n + 2$, and then $a_1 = 2$. If $1 < r < \dim_k R_d$, by Lemma 4.4.2, we see that $u(x_n m) > u(x_j m')$ for any $1 \leq j \leq n$ and any monomail $m' \in L_d$, hence $x_n m \notin R_1 L_d$, and then $a_r \geq 1$ for $1 < r < \dim_k R_d$. Note that $\dim_k R_1 R_d = \dim_k R_{d+1}$, and it is easy to see that

$$\dim_k R_{d+1} - \dim_k R_d = \dim_k R'_{d+1} - \dim_k R'_d = n - 1 + h = n,$$

where R' is the toric ring defined in Lemma 4.4.2. Thus,

$$(a_0 - 1) + (a_1 - 1) + \sum_{1<r<\dim_k R_d} (a_r - 1) = n,$$

so that $\sum_{1<r<\dim_k R_d}(a_r - 1) = 0$, which implies $a_r = 1$ for $1 < r < \dim_k R_d$. □

Lemma 4.4.4. *Let R and R' be the toric rings defined in Lemma 4.4.2 and suppose $n = 3$. If L_d, L'_d are lex d-monomial spaces in R_d and R'_d such that* $\dim_k L_d = \dim_k L'_d$, *then* $\dim_k R_1 L_d = \dim_k R'_1 L'_d$.

Proof. Since the toric ring R is defined by the matrix $A = \begin{pmatrix} 0 & 1+h & 2+h \\ 1 & 1 & 1 \end{pmatrix}$ and Ker$A$ has dimension 1, one sees easily that the toric ideal $I_{\mathcal{A}}$ is generated by the binomial $x_2^{2+h} - x_1 x_3^{1+h}$, so that we have $R = k[x_1, x_2, x_3]/(x_2^{2+h} - x_1 x_3^{1+h})$, and similarly, $R' = k[y_1, y_2, y_3]/(y_2^{2+h} - y_1^{1+h} y_3)$.

Let T_d be the set of monomials in $k[x_1, x_2, x_3]_d$ which can not be divided by x_2^{2+h} and T'_d the set of monomials in $k[y_1, y_2, y_3]_d$ which can not be divided by y_2^{2+h}. It is easy to see that for any monomial $m \in R_d$ there is one and only one monomial in the fiber of m that can not be divided by x_2^{2+h}, then it follows that the monomials in R_d are in one-to-one correspondence with the monomials in T_d. Furthermore, if $\dim_k L_d = r$ and L_d is spanned by the monomials $m_1, \ldots, m_r \in R_d$ with $u(m_1) < \cdots < u(m_r)$, then $m_1, \ldots, m_r$ have top-representatives $w_1, \ldots, w_r \in T_d$ that are the first r monomials in T_d. Similarly, if $\dim_k L'_d = r$ and L'_d is spanned by monomials $m'_1, \ldots, m'_r \in R'_d$, then $m'_1, \ldots, m'_r$ have top-representatives $w'_1, \ldots, w'_r \in T'_d$ that are the first r monomials in T'_d.

Note that the natural isomorphism $g : S = k[x_1, x_2, x_3] \longrightarrow S' = k[y_1, y_2, y_3]$ with $g(x_j) = y_j$ for $j = 1, 2, 3$ induces an order-preserving bijection between T_d and T'_d, then $g(w_i) = w'_i$ for $1 \le i \le r$. Setting $W = \text{span}\{w_1, \ldots, w_r\} \subseteq S_d$ and

$W' = \text{span}\{w'_1, \ldots, w'_r\} \subseteq S'_d$, one sees easily that $\dim_k S_1 W = \dim_k S'_1 W'$. Let p be the number of monomials in $S_1 W$ that can be divided by x_2^{2+h} and p' the number of monomials in $S'_1 W'$ that can be divided by y_2^{2+h}; then we have $p = p'$. Note that if $x_2 w_i$ can be divided by x_2^{2+h} for some i, then $x_2 w_i = x_3(x_1 x_3^h w_i / x_2^{1+h})$ in R_{d+1} and $x_1 x_3^h w_i / x_2^{1+h} = w_j$ for some $j < i$. Therefore, the monomials in the lex (d+1)-monomial space $R_1 L_d$ are in one-to-one correspondence with the monomials in $S_1 W$ that can not be divided by x_2^{2+h}, so that we have

$$\dim_k R_1 L_d = \dim_k S_1 W - p.$$

Similarly, we have

$$\dim_k R'_1 L'_d = \dim_k S'_1 W - p',$$

and so $\dim_k R_1 L_d = \dim_k R'_1 L'_d$. □

Proof of Theorem 4.4.1. (1) Let W be a d-monomial space spanned by monomials $w_1, \ldots, w_r \in R_d$ with $u(w_1) < \cdots < u(w_r)$. By Lemma 4.1.1, it suffices to prove that $\dim_k R_1 L_W \leq \dim_k R_1 W$, where L_W is the lex d-monomial space in R_d such that $\dim_k L_W = \dim_k W = r$.

We prove by induction on r. If $r = 1$, then $\dim_k R_1 L_W = \dim_k R_1 W = n$. If $r = 2$, then by Lemma 4.4.3, $\dim_k R_1 L_W = a_0 + a_1 = n + 2$, and by Lemma 4.2.1, $\dim_k R_1 W = 2n - \lambda(w_1, w_2)$. It is easy to see that $\lambda(w_1, w_2) \leq n - 2$, thus we have

$$\dim_k R_1 W \geq 2n - (n-2) = n + 2 = \dim_k R_1 L_W.$$

If $r > 2$,let $\widehat{W}$ be the d-monomial space spanned by monomials $w_1, \ldots, w_{r-1} \in R_d$ and $L_{\widehat{W}}$ the lex d-monomial space in R_d such that $\dim_k L_{\widehat{W}} = \dim_k \widehat{W} = r - 1$, then by induction we have $\dim_k R_1 L_{\widehat{W}} \leq \dim_k R_1 \widehat{W}$. By Lemma 4.4.3, we see that $\dim_k R_1 L_W = \dim_k R_l L_{\widehat{W}} + 1$. On the other hand, since $u(x_n w_r) > u(x_j w_i)$ for any

$1 \le j \le n$ and any $1 \le i \le r-1$, we have $x_n w_r \notin R_1\widehat{W}$, and then $\dim_k R_1 W \ge \dim_k R_1\widehat{W} + 1$. Therefore,

$$\dim_k R_1 W \ge \dim_k R_1 \widehat{W} + 1 \ge \dim_k R_1 L_{\widehat{W}} + 1 = \dim_k R_1 L_W,$$

and we are done.

(2)Let W be an r-dimendsional d-monomial space in R_d. By Lemma 4.1.1, it suffices to prove that $\dim_k R_1 L_W \le \dim_k R_1 W$ where L_W is the lex d-monomial space in R_d such that $\dim_k L_W = r$.

Let f and R' be as in Lemma 4.4.2, then by Lemma 4.2.3 (1), we see that $f(W)$ is an r-dimensional d-monomial space in R'_d and $\dim_k R_1 W = \dim_k R'_1 f(W)$. Let $L'_{f(W)}$ be the lex d-monomial space in R'_d such that $\dim_k L'_{f(W)} = r$, then by Lemma 4.4.4, we have $\dim_k R_1 L_W = \dim_k R'_1 L'_{f(W)}$. By Theorem 4.3.1, we see that R' satisfies Macaulay's Theorem, hence $\dim_k R'_1 L'_{f(W)} \le \dim_k R'_1 f(W)$. So, $\dim_k R_1 L_W \le \dim_k R_1 W$, and we are done.

(3)Considering the 1-monomial space $W = \text{span}\{x_2, x_3\}$ and the lex 1-monomial space $L_W = \text{span}\{x_1, x_2\}$ in R_1, we have $\dim_k W = \dim_k L_W = 2$. However, by lemma 4.2.1, it is easy to see that

$$\dim_k R_1 W = 2n - \lambda(x_2, x_3) = 2n - (n-2) = n+2,$$

and

$$\dim_k R_1 L_W = 2n - \lambda(x_1, x_2) = \begin{cases} 2n-1, & \text{if } n \le h+2 \\ 2n - (1+n-h-2) = n+h+1, & \text{if } n \ge h+3. \end{cases}$$

Since $h \ge 2$ and $n \ge 4$, one can check easily that $\dim_k R_1 L_W > \dim_k R_1 W$. So by Lemma 4.1.1, Macaulay's Theorem does not hold over R. □

Theorem 4.4.5. *Let*

$$A = \begin{pmatrix} 0 & 1 & \cdots & m-1 & m+h & \cdots & n-1+h \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $n \geq 4$, $2 \leq m \leq n-2$ and $h \in \mathbb{Z}^+$. Let R be the toric ring associated to A. Then Macaulay's Theorem does not hold over R.

Proof. We have three cases.

Case 1: $h \leq m-1$. Let $W = \operatorname{span}\{x_1^2, x_1x_2, \ldots, x_1x_m, x_2x_m\} \subseteq R_2$ and $L_W = \operatorname{span}\{x_1^2, x_1x_2, \ldots, x_1x_m, x_1x_{m+1}\} \subseteq R_2$, then W is a 2-monomail space in R_2 and L_W is a lex 2-monomial space in R_2 such that $\dim_k W = \dim_k L_W = m+1$. By Lemma 4.2.1, we have

$$\dim_k R_1W = (m+1)n - \sum_{1\leq i<j\leq m} \lambda(x_1x_i, x_1x_j) - \sum_{1\leq i\leq m} \lambda(x_1x_i, x_2x_m),$$
$$\dim_k R_1L_W = (m+1)n - \sum_{1\leq i<j\leq m} \lambda(x_1x_i, x_1x_j) - \sum_{1\leq i\leq m} \lambda(x_1x_i, x_1x_{m+1}),$$

so that we get

$$\dim_k R_1L_W - \dim_k R_1W = \sum_{1\leq i\leq m} \lambda(x_1x_i, x_2x_m) - \sum_{1\leq i\leq m} \lambda(x_1x_i, x_1x_{m+1}).$$

It is easy to see that

$$\lambda(x_1x_m, x_2x_m) = n-2, \quad \lambda(x_1x_{m-h}, x_2x_m) = 1,$$

and

$$\lambda(x_1x_i, x_2x_m) = 0 \text{ for } 1 \leq i \leq m-1 \text{ and } i \neq m-h.$$

Thus, we have

$$\sum_{1\leq i\leq m} \lambda(x_1x_i, x_2x_m) = n-2+1 = n-1.$$

On the other hand, one sees easily that

$$\lambda(x_1x_i, x_1x_{m+1}) = \begin{cases} 1, & \text{if } m-h \le i \le m-1; \\ 0, & \text{if } i < m-h. \end{cases}$$

If $n-m-1 \ge h+1$, then it is easy to check that

$$\begin{aligned} \lambda(x_1x_m, x_1x_{m+1}) &= 1 + ((m-1)-(h+1)+1) + ((n-m-1)-(h+1)+1) \\ &= n-2h-1, \end{aligned}$$

so that we have

$$\sum_{1\le i\le m} \lambda(x_1x_i, x_1x_{m+1}) = h + n - 2h - 1 = n - h - 1,$$

and then

$$\dim_k R_1L_W - \dim_k R_1W = n-1-(n-h-1) = h \ge 1 > 0,$$

therefore, by Lemma 4.1.1 we see that Macaulay's Theorem does not hold over R. If $n-m-1 < h+1$, then it is easy to check that

$$\lambda(x_1x_m, x_1x_{m+1}) = 1 + ((m-1)-(h+1)+1) = m-h,$$

so that we have

$$\sum_{1\le i\le m} \lambda(x_1x_i, x_1x_{m+1}) = h + m - h = m,$$

and then

$$\dim_k R_1L_W - \dim_k R_1W = n-1-m \ge n-1-(n-2) = 1 > 0,$$

so by Lemma 4.1.1 we see that Macaulay's Theorem does not hold over R.

Case 2: $h \ge m$ and $m < n-2$. Let W and L_W be the same 2-monomial spaces as in Case 1, then

$$\dim_k R_1L_W - \dim_k R_1W = \sum_{1\le i\le m} \lambda(x_1x_i, x_2x_m) - \sum_{1\le i\le m} \lambda(x_1x_i, x_1x_{m+1}).$$

It is easy to see that

$$\lambda(x_1x_m, x_2x_m) = n-2, \text{ and } \lambda(x_1x_i, x_2x_m) = 0 \text{ for } 1 \le i \le m-1.$$

Thus, we have

$$\sum_{1\le i\le m} \lambda(x_1x_i, x_2x_m) = n-2.$$

On the other hand, one sees easily that

$$\lambda(x_1x_i, x_1x_{m+1}) = 1 \text{ for } 1 \le i \le m-1.$$

If $n-m-1 \ge h+1$, then it is easy to check that

$$\lambda(x_1x_m, x_1x_{m+1}) = 1 + ((n-m-1)-(h+1)+1) = n-m-h,$$

so that we have

$$\sum_{1\le i\le m} \lambda(x_1x_i, x_1x_{m+1}) = m-1+n-m-h = n-h-1,$$

and then

$$\dim_k R_1L_W - \dim_k R_1W = n-2-(n-h-1) = h-1 \ge m-1 \ge 1 > 0,$$

therefore, by Lemma 4.1.1 we see that Macaulay's Theorem does not hold over R. If $n-m-1 < h+1$, then it is easy to check that $\lambda(x_1x_m, x_1x_{m+1}) = 1$, so that we have

$$\sum_{1\le i\le m} \lambda(x_1x_i, x_1x_{m+1}) = m-1+1 = m,$$

and then

$$\dim_k R_1L_W - \dim_k R_1W = n-2-m > n-2-(n-2) = 0,$$

so by Lemma 4.1.1 we see that Macaulay's Theorem does not hold over R.

Case 3: $h \ge m$ and $m = n-2$. Let p be the maximal integer such that $p \le (h-1)/(m-1)$, then $p \ge 1$. Considering R_{p+1}, we see that for any monomial

$w \in R_{p+1}$, $0 \leq u(w) \leq (p+1)(n-1+h)$. More precisely, one can check easily that there are $(n-1)+(p-i)(m-1)+i$ monomials $w \in R_{p+1}$ such that $i(n-1+h) \leq u(w) < (i+1)(n-1+h)$ for $0 \leq i \leq p$, so that

$$\dim_k R_{p+1} = 1 + \sum_{i=0}^{p}(n-1)+(p-i)(m-1)+i = 1+(p+1)(n+\frac{pm}{2}-1).$$

Similarly, we have

$$\begin{aligned}\dim_k R_{p+2} &= (n-1+h)+1+\sum_{i=0}^{p}(n-1)+(p-i)(m-1)+(i+1)\\ &= n+h+p+1+(p+1)(n+\frac{pm}{2}-1).\end{aligned}$$

Setting $l = 1+(p+1)(n+\frac{pm}{2}-1)$ we have that

$$\dim_k R_{p+1} = l \quad \text{and} \quad \dim_k R_1 R_{p+1} = \dim_k R_{p+2} = n+h+p+l.$$

Let W be the l-monomial space spanned by the monomials $w_1, \ldots, w_l \in R_l$ such that $u(w_i) = i-1$ for $1 \leq i \leq l$. Let monomials $w'_1, \ldots, w'_l$ be a basis of R_{p+1}, and let L_W be the l-monomial space spanned by the monomials $x_1^{l-p-1}w'_1, \ldots, x_1^{l-p-1}w'_l \in R_l$, then it is easy to see that L_W is a lex l-monomial space such that

$$\dim_k L_W = \dim_k W = l \text{ and } \dim_k R_1 L_W = \dim_k R_1 R_{p+1} = n+h+p+l.$$

However, by Lemma 4.2.1, one can check easily that

$$\dim_k R_1 W = ln-(l-1)(n-2)-((l-1)-(h+1)+1) = n+h-1+l,$$

so that

$$\dim_k R_l L_W - \dim_k R_1 W = (n+h+p+l)-(n+h-1+l) = p+1 \geq 2 > 0,$$

so by Lemma 4.1.1 we see that Macaulay's Theorem does not hold over R. □

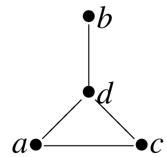
CHAPTER 5

MINIMAL FREE RESOLUTIONS OF LINEAR EDGE IDEALS

5.1 Introduction

In this chapter we consider minimal free resolutions of quadratic monomial ideals in $S = k[x_1, \ldots, x_n]$. By polarization, the study of such resolutions is equivalent to the study of the resolutions of squarefree quadratic monomial ideals, that is, edge ideals. Such an ideal can be easily encoded in a graph as follows: let G be a simple graph with vertices $x_1, \ldots, x_n$, then the *edge ideal* I_G of the graph G is the monomial ideal in S generated by $\{x_i x_j \mid \{x_i, x_j\}$ is an edge of $G\}$. The general goal is to relate the properties of the minimal free resolution of I_G and the combinatorial properties of the graph G. In 1990, Fröberg [Fro] proved that I_G has a linear free resolution if and only if the complement graph $\overline{G}$ is chordal (see Definition 5.2.1). Because of this, I_G is called a linear edge ideal if $\overline{G}$ is chordal.

Minimal free resolutions were constructed for the following two classes of linear edge ideals. In [CN], Corso and Nagel used cellular resolutions to get the minimal free resolutions of the linear edge ideals I_G where G is a Ferrers graph. In [Ho], Horwitz constructed the minimal free resolutions of the linear edge ideals I_G provided that G does not contain an ordered subgraph as in Figure 5.1, which is called the pattern Γ in [Ho]. However, from Example 3.18 in [Ho], we see that if $\overline{G}$ is complicated, then it may be impossible to satisfy the Γ avoidance condition. In Construction 5.3.4 and Theorem 5.3.7 we provide the minimal free resolutions of *all* linear edge ideals. The construction is different than the one in [Ho] and the following paragraph explains the difference.



with $\quad a < b < c < d,$

Figure 5.1: Patten Γ

In 1990, Eliahou and Kervaire (See Construction 2.2.9) constructed the minimal free resolutions of Borel ideals. In 1995, Charalambous and Evans [CE] noted that the Eliahou-Kervaire resolution can be obtained by using iterated mapping cones (See Construction 2.2.13). Then in 2002, Herzog and Takayama [HT] used the iterated mapping cone construction to obtain the minimal free resolutions of monomial ideals which have linear quotients and satisfy some regularity condition. Following this idea, in 2007, Horwitz [Ho] constructed the minimal free resolutions of a class of linear edge ideals. In [HT] and [Ho], the constructions are based on induction on the number of generators of the monomial ideal and the resolutions are similar to the Eliahou-Kervaire resolution. In this chapter we will use the mapping cone construction in a new way: (1) we use induction on the number of variables, that is the number of vertices of G; (2) in each induction step, we use the mapping cone construction twice. Consequently, the minimal free resolution in this chapter is very different from the Eliahou-Kervaire resolution and is not a modification of the resolution obtained in [Ho] (See Remark 5.3.12).

Another thing that plays an important role in our construction is the notion of a perfect elimination order (See Definition 5.2.1) of a chordal graph. From [Di] and [HHZ], we know that every chordal graph has a perfect elimination order on the set of vertices; conversely, it is easy to see that if a simple graph has a perfect elimination order then it is chordal. Therefore, a simple graph is chordal if and only if it has a perfect elimination order. In general, given a chordal graph, there are many perfect elimination orders. In section 5.2 we give an algorithm (Algorithm 5.2.2) to produce a special perfect elimination order on the vertices of a chordal graph. This special perfect elimination order has a nice property (Lemma 5.3.2) and will be used in the construction of the minimal free resolutions of linear edge ideals.

In section 5.3 we construct the minimal free resolutions of linear edge ideals and Theorem 5.3.7 is the main result of this paper.

In section 5.4 we prove $d^2 = 0$ case by case, where d is the differential defined in Construction 5.3.4. The proof is not difficult but very long.

Section 5.5 gives a nice formula (Corollary 5.5.2) for calculating the Betti numbers of linear edge ideals and the formula works for any perfect elimination order of $\overline{G}$. Finally, in Corollary 5.5.4, we use our method to prove another Betti number formula obtained by Roth and Van Tuyl in [RV] (see also [HV]).

5.2 Perfect elimination orders

In this section we use H to denote a chordal graph. In the other sections of this paper, we have $H = \overline{G}$.

Definition 5.2.1. Let H be a simple graph with vertices $x_1, \dots, x_n$. We write $x_i x_j \in H$ if $\{x_i, x_j\}$ is an edge of H. We say that $C = (x_{j_1} x_{j_2} \dots x_{j_r})$ is a *cycle* of H of length r if $x_{j_i} \neq x_{j_l}$ for all $1 \leq i < l \leq r$ and $x_{j_i} x_{j_{i+1}} \in H$ for all $1 \leq i \leq r$(where $x_{j_{r+1}} = x_{j_1}$). A *chord* in the cycle C is an edge between two non-consecutive vertices in the cycle. We say that H is a *chordal graph* if every cycle of length > 3 in H has a chord. The order $x_1, \ldots, x_n$ on the vertices of H is called a *perfect elimination order* if the following condition is satisfied: for any $1 \leq i < j < l \leq n$, if $x_i x_j \in H$ and $x_i x_l \in H$, then $x_j x_l \in H$.

The perfect elimination orders we will use in sections 5.3 and 5.4 are given by the following algorithm.

Algorithm 5.2.2. *Let H be a chordal graph with vertices $x_1, \ldots, x_n$. Let Σ be a set containing a sequence of sets.*

Input: $\Sigma = \{\{x_1, \ldots, x_n\}\}$, $i = n + 1$.

Step 1: Choose and remove a vertex v from the first set in Σ. Set $i := i - 1$ and $v_i := v$. If the first set in Σ is now empty, remove it from Σ. Go to setp 2.

Step 2: If $\Sigma = \emptyset$, stop. If $\Sigma \neq \emptyset$, suppose $\Sigma = \{S_1, S_2, \ldots, S_r\}$. For any $1 \leq j \leq r$, replace the set S_j by two sets T_j and T'_j such that $S_j = T_j \cup T'_j$, $T_j \cap T'_j = \emptyset$, $v_i w \in H$ for any $w \in T_j$ and $v_i w' \notin H$ for any $w' \in T'_j$. Now we set

$$\Sigma := \{T_1, T_2, \ldots, T_r, T'_1, T'_2, \ldots, T'_r\}.$$

Remove all the empty sets from Σ. Go back to step 1.

Output: $v_1, \ldots, v_n$.

Remark 5.2.3. The above algorithm is a modification of an algorithm of Rose-Tarjan-Lueker. In section 5.2 of [RTL], they set

$$\Sigma := \{T_1, T'_1, T_2, T'_2, \ldots, T_r, T'_r\}.$$

The reason we difine Σ differently in Algorithm 5.2.2 is illustrated in Example 5.2.6 and Lemma 5.3.2.

Before proving Theorem 5.2.5, we make the following observation.

Lemma 5.2.4. *Let $v_1, \dots, v_n$ be an output of Algorithm 5.2.2. If $v_iv_l \in H$, $v_jv_l \notin H$ and $i < j < l$, then there exists λ with $j < \lambda < l$ such that $v_iv_\lambda \notin H$ and $v_jv_\lambda \in H$.*

Proof. Since $v_iv_l \in H$, $v_jv_l \notin H$ and $i < j < l$, it follows from the algorithm that after v_l is taken from the first set of Σ, v_i and v_j will be in different sets of Σ and the set containing v_i is before the set containing v_j. If there does not exist $j < \lambda < l$ such that $v_iv_\lambda \notin H$ and $v_jv_\lambda \in H$, then after v_{j+1} is taken from the first set of Σ, the set containing v_i is still before the set containing v_j and in particular, v_j is not in the first set of the new Σ. So after removing v_{j+1} we need to remove a vertex different from v_j, which is a contradiction. So there must exist $j < \lambda < l$ such that $v_iv_\lambda \notin H$ and $v_jv_\lambda \in H$. □

Theorem 5.2.5. *The output of Algorithm 5.2.2 is a perfect elimination order of the chordal graph H.*

Proof. First, we see that $v_1, \dots, v_n$ is a reordering of the vertices $x_1, \dots, x_n$ of H. To show that $v_1, \dots, v_n$ is a perfect elimination order, we need only show that for any $1 \le i < j < l \le n$, if $v_iv_j \in H$ and $v_iv_l \in H$, then $v_jv_l \in H$. Assume to the contrary that $v_jv_l \notin H$.

Since $v_iv_l \in H$, $v_jv_l \notin H$ and $i < j < l$, Lemma 5.2.4 implies that there exists $j < \lambda_1 < l$ such that $v_iv_{\lambda_1} \notin H$ and $v_jv_{\lambda_1} \in H$. And we choose the largest λ_1 which satisfies this property. If $v_{\lambda_1}v_l \in H$, then $(v_iv_jv_{\lambda_1}v_l)$ is a cycle of length 4 with no chord, which contradicts to the assumption that H is chordal. So $v_{\lambda_1}v_l \notin H$.

Since $v_iv_l \in H$, $v_{\lambda_1}v_l \notin H$ and $i < \lambda_1 < l$, Lemma 5.2.4 implies that there exists $\lambda_1 < \lambda_2 < l$ such that $v_iv_{\lambda_2} \notin H$ and $v_{\lambda_1}v_{\lambda_2} \in H$. And we choose the largest λ_2 which satisfies this property. Note that by the choice of λ_1, we have that $v_j v_{\lambda_2} \notin H$. If $v_{\lambda_2} v_l \in H$, then $(v_i v_j v_{\lambda_1} v_{\lambda_2} v_l)$ is a cycle of length 5 with no chord, which contradicts to the assumption that H is chordal. So $v_{\lambda_2} v_l \notin H$.

Since $v_i v_l \in H$, $v_{\lambda_2} v_l \notin H$ and $i < \lambda_2 < l$, Lemma 5.2.4 implies that there exists $\lambda_2 < \lambda_3 < l$ such that $v_i v_{\lambda_3} \notin H$ and $v_{\lambda_2} v_{\lambda_3} \in H$. And we choose the largest λ_3 which satisfies this property. Note that by the choices of λ_1 and λ_2, we have that $v_j v_{\lambda_3} \notin H$ and $v_{\lambda_1} v_{\lambda_3} \notin H$. If $v_{\lambda_3} v_l \in H$, then $(v_i v_j v_{\lambda_1} v_{\lambda_2} v_{\lambda_3} v_l)$ is a cycle of length 6 with no chord, which contradicts to the assumption that H is chordal. So $v_{\lambda_3} v_l \notin H$.

Proceeding in the same way, we get an infinite sequence of vertices $v_{\lambda_1}, v_{\lambda_2}, v_{\lambda_3}, \ldots$ such that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. This is a contradiction because there are only finitely many vertices. So $v_j v_l \in H$ and we are done. □

The following example illustrates the difference among different perfect elimination orders.

Example 5.2.6. Let H be the following chordal graph. Then $x_7, x_6, x_5, x_1, x_4, x_2, x_3$

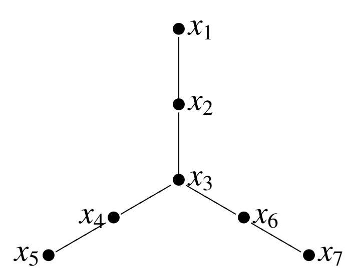


Figure 5.2: Different perfect elimination orders

is a perfect elimination order of H, but it can not be produced by Algorithm 5.2.2 or the algorithm in [RTL]; $x_7, x_5, x_6, x_4, x_3, x_2, x_1$ is a perfect elimination order which can be produced by the algorithm in [RTL] ; $x_7, x_6, x_5, x_4, x_3, x_2, x_1$ is a perfect elimination order which is produced by Algorithm 5.2.2.

If we compare these three perfect elimination orders, the third one looks nicer in the sense that there is no unnecessary "jump" in the perfect elimination order. Here, "jump" means going from one branch of the star-shaped graph H to another branch. For example, in the first perfect elimination order, x_5 is followed by x_1 instead of x_4; in the second perfect elimination order, x_7 is followed by x_5 instead of x_6. However, in the third perfect elimination order, this kind of "jump" does not happen unless it is necessary, say, x_6 is followed by x_5. This nice property of the perfect elimination orders produced by Algorithm 5.2.2 is reflected in Lemma 5.3.2 .

5.3 Construction of the resolution

Let G be a simple graph with vertices $x_1, \ldots, x_n$. The *complement graph* $\overline{G}$ of G is the simple graph with the same vertex set whose edges are the non-edges of G. The *subgraph* of G *induced by vertices* $x_{i_1}, \ldots, x_{i_r}$ for some $1 \leq i_1 < \cdots < i_r \leq n$ is the simple graph with the vertices $x_{i_1}, \ldots, x_{i_r}$ and the edges that connect them in G. We define the *preneighborhood* of a vertex x_j in G to be the set

$$pnbhd(x_j) = \{x_i \mid i < j, x_i x_j \in G\}.$$

The following two lemmas will be important in section 5.3 and section 5.4.

Lemma 5.3.1. *Let G be a simple graph with vertices $x_1, \ldots, x_n$ such that $\overline{G}$ is chordal. Let $x_1, \ldots, x_n$ be in the reverse order of a perfect elimination order of $\overline{G}$. For any $1 \leq i < j < l \leq n$, if $x_i x_j \in G$, then $x_i x_l \in G$ or $x_j x_l \in G$. In particular, if $pnbhd(x_i) \not\subseteq pnbhd(x_j)$ for some $1 \leq i < j \leq n$ then $x_i x_j \in G$.*

Proof. Assume to the contrary that $x_ix_l \notin G$ and $x_jx_l \notin G$, then $x_ix_l \in \overline{G}$ and $x_jx_l \in \overline{G}$. Since $x_1, \ldots, x_n$ is in the reverse order of a perfect elimination order of $\overline{G}$, we have $x_ix_j \in \overline{G}$, and hence $x_ix_j \notin G$, which is a contradiction. □

Lemma 5.3.2. *Let G be a simple graph with vertices $x_1, \ldots, x_n$ such that $\overline{G}$ is chordal. Let $x_1, \ldots, x_n$ be in the reverse order of a perfect elimination order of $\overline{G}$ produced by Algorithm 5.2.2.*

(1) If $x_ix_j \in \overline{G}$ for some $i < j$, then for any $i < t \leq j$ we have $pnbhd(x_i) \subseteq pnbhd(x_t)$ in G.

(2) If $pnbhd(x_i) \nsubseteq pnbhd(x_t)$ in G for some $i < t$, then $x_ix_j \in G$ for all $j \geq t$.

Proof. Note that part (1) and part (2) are equivalent, so we only need to prove part (1). Assume to the contrary that there exists $i < t \leq j$ such that $\text{pnbhd}(x_i) \nsubseteq \text{pnbhd}(x_t)$ in G. We choose the minimal t which satisfies this property. Then there exists $l < i$ such that $x_lx_i \notin \overline{G}$, $x_lx_t \in \overline{G}$. Since $x_1, \ldots, x_n$ is in the reverse order of a perfect elimination order of $\overline{G}$, we must have that $x_ix_t \notin \overline{G}$ and in particular $t \neq j$. Now since $x_ix_t \notin \overline{G}$, $x_ix_j \in \overline{G}$ and $i < t < j$, Lemma 5.2.4 implies that there exists $i < m < t$ such that $x_mx_t \in \overline{G}$, $x_mx_j \notin \overline{G}$. However, $x_mx_t \in \overline{G}$, $x_lx_t \in \overline{G}$ and $l < m < t$ imply that $x_lx_m \in \overline{G}$, so that $\text{pnbhd}(x_i) \nsubseteq \text{pnbhd}(x_m)$ and $i < m < t < j$, which contradicts to the minimality of t. So for all $i < t \leq j$, $\text{pnbhd}(x_i) \subseteq \text{pnbhd}(x_t)$ in G. □

Let G be a simple graph with vertices $x_1, \ldots, x_n$. The *edge ideal* I_G of the graph G is the monomial ideal in the polynomial ring $S = k[x_1, \ldots, x_n]$ with the minimal generating set $\{x_ix_j \mid x_ix_j \in G\}$. An important result about edge ideals was obtained by Fröberg in [Fro].

Theorem 5.3.3 (Fröberg)**.** *Let I_G be the edge ideal of a simple graph G. Then I_G has a linear free resolution if and only if $\overline{G}$ is chordal.*

By the above theorem, the edge ideal I_G of a simple graph G is called a *linear edge ideal* if $\overline{G}$ is chordal. The goal of this section is to construct the minimal free resolution of S/I_G where I_G is a linear edge ideal.

Construction 5.3.4. Let G be a simple graph with vertices $x_1, \ldots, x_n$ such that $\overline{G}$ is chordal. Let $x_1, \ldots, x_n$ be in the reverse order of a perfect elimination order of $\overline{G}$ produced by Algorithm 5.2.2.

If $p \geq 1, q \geq 1, 1 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_q \leq n$ and $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_1})$, then the symbol $(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q})$ will be used to denote the generator of the free S-module $S(-x_{i_1} \cdots x_{i_p} x_{j_1} \cdots x_{j_q})$ in homological degree $p+q-1$ and multidegree $x_{i_1} \cdots x_{i_p} x_{j_1} \cdots x_{j_q}$. We set

$$\mathcal{B} = \{1\} \cup \bigcup_{p \geq 1, q \geq 1} \left\{ (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q}) : \begin{array}{l} 1 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_q \leq n \\ \{x_{i_1}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_1}) \end{array} \right\}.$$

We define the map d on the set $\mathcal{B}$ by $d(1) = 1$, $d(x_{i_1} | x_{j_1}) = x_{i_1} x_{j_1}$, and for $p + q \geq 3$,

$$\begin{aligned}
& d(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q}) \\
&= \sum_{s=1}^{p} (-1)^{s+1} x_{i_s} (x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q}) \\
&+ \sum_{t=1}^{q} (-1)^{t+p} x_{j_t} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q}) \\
&+ \sum_{s=1}^{p} (-1)^{s+1+\beta} x_{i_s} (x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_\beta}, \ldots, x_{j_q}) \\
&+ (-1)^{p} x_{j_\beta} (x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \ldots, x_{j_q}),
\end{aligned}$$

where $\beta = \min\{t \mid 2 \le t \le q, \{x_{i_1}, \dots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_t})\}$.

Note that if $\{x_{i_1}, \dots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_t})$ for all $1 \le t \le q$, then β does not exist and there are no β terms in the above formula. Also, if $p+q \ge 3$, then the formula of d may yield symbols which are not in $\mathcal{B}$ and we will regard them as zeros. And Lemma 5.3.2 implies that for any $1 \le t \le \beta - 1$ and $\beta \le t' \le q$, we have $x_{j_t} x_{j_{t'}} \in G$.

Example 5.3.5. The following are some examples for the formula of d.

(1). If $p \ge 2$ and $q = 1$, then just like the Koszul complex, we have that

$$d(x_{i_1}, \dots, x_{i_p} | x_{j_1}) = \sum_{s=1}^{p} (-1)^{s+1} x_{i_s} (x_{i_1}, \dots, \widehat{x_{i_s}}, \dots, x_{i_p} | x_{j_1}).$$

(2). If $p \ge 2$, $q = 3$, $\{x_{i_1}, \dots, x_{i_p}\} \backslash \text{pnbhd}(x_{j_2}) = \{x_{i_1}\}$ and $\{x_{i_1}, \dots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_3})$, then $\beta = 2$ and a computation will reveal that

$$\begin{aligned}
&d(x_{i_1}, \dots, x_{i_p} | x_{j_1}, x_{j_2}, x_{j_3}) \\
&= x_{i_1} [(x_{i_2}, \dots, x_{i_p} | x_{j_1}, x_{j_2}, x_{j_3}) + (x_{i_2}, \dots, x_{i_p}, x_{j_1} | x_{j_2}, x_{j_3})] \\
&+ \sum_{s=2}^{p} (-1)^{s+1} x_{i_s} (x_{i_1}, \dots, \widehat{x_{i_s}}, \dots, x_{i_p} | x_{j_1}, x_{j_2}, x_{j_3}) \\
&+ (-1)^{2+p} x_{j_2} [(x_{i_1}, \dots, x_{i_p} | x_{j_1}, x_{j_3}) + (x_{i_1}, \dots, x_{i_p}, x_{j_1} | x_{j_3})] \\
&+ (-1)^{3+p} x_{j_3} (x_{i_1}, \dots, x_{i_p} | x_{j_1}, x_{j_2}).
\end{aligned}$$

(3). If $p \ge 2$, $q \ge 4$, $\beta = 3$, $\{x_{i_1}, \dots, x_{i_p}\} \backslash \text{pnbhd}(x_{j_3}) = \{x_{i_1}, x_{i_2}\}$ and $\{x_{i_1}, \dots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_4})$, then a computation will reveal that

$$\begin{aligned}
d(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}) &= \sum_{s=1}^{p} (-1)^{s+1} x_{i_s} (x_{i_1}, \dots, \widehat{x_{i_s}}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}) \\
&+ \sum_{t=1}^{q} (-1)^{t+p} x_{j_t} (x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q}).
\end{aligned}$$

Lemma 5.3.6. *Let d be the map defined in Construction 5.3.4. Then $d^2 = 0$.*

The proof of the above lemma is very long and is given in section 5.4. The next theorem is the main result of this chapter.

Theorem 5.3.7. *Let $\mathbf{F}$ be the multigraded complex of free S-modules with basis $\mathcal{B}$ and differential d as defined in Construction 5.3.4. Then $\mathbf{F}$ is the minimal free resolution of S/I_G.*

Proof. We prove by induction on the number of vertices of the graph G. If G has one or two vertices then it is clear. Now as in Construction 5.3.4, let G have vertices $x_1, \dots, x_n$ with $n \geq 3$.

If $\operatorname{pnbhd}(x_n) = \emptyset$ in G, then $x_i x_n \in \overline{G}$ for all $1 \leq i \leq n-1$. Since $x_1, \dots, x_n$ is in the reverse order of a perfect elimination order of $\overline{G}$, it follows that $\overline{G}$ is a complete graph, so that G has no edges. Hence $I_G = (0)$ and there is nothing to prove. Next we will assume that $\operatorname{pnbhd}(x_n) = \{x_{\lambda_1}, \dots, x_{\lambda_r}\}$ for some $1 \leq \lambda_1 < \cdots < \lambda_r \leq n-1$.

Let G' be the graph obtained from G by deleting the edges $x_{\lambda_1} x_n, \dots, x_{\lambda_r} x_n$. Then I_G and $I_{G'}$ are both edge ideals in S. Note that $\overline{G'}$ is chordal. Indeed, it is easy to see that $x_n, x_1, x_2, \dots, x_{n-1}$ is in the reverse order of a perfect elimination order of $\overline{G'}$ produced by Algorithm 5.2.2. Setting $J = (x_{\lambda_1}, \dots, x_{\lambda_r}) \subseteq S$, we have $I_G = I_{G'} + x_n J$ and a natural short exat sequence

$$0 \longrightarrow \frac{I_{G'} + x_n J}{I_{G'}} \longrightarrow \frac{S}{I_{G'}} \longrightarrow \frac{S}{I_G} = \frac{S}{I_{G'} + x_n J} \longrightarrow 0.$$

Note that $x_n J \cap I_{G'} = x_n I_{G'}$: indeed, by Lemma 5.3.1 we see that $I_{G'} \subseteq J$ and hence $x_n I_{G'} \subseteq x_n J \cap I_{G'}$; on the other hand, if $x_n m \in I_{G'}$ for some monomial $m \in J$, then $m \in I_{G'}$, and hence $x_n J \cap I_{G'} \subseteq x_n I_{G'}$. Therefore,

$$\frac{I_{G'} + x_n J}{I_{G'}} \cong \frac{x_n J}{x_n J \cap I_{G'}} = \frac{x_n J}{x_n I_{G'}}.$$

Let G'' be the subgraph of G induced by the vertices $x_1, \ldots, x_{n-1}$. Then $\overline{G''}$ is chordal and $x_1, \ldots, x_{n-1}$ is in the reverse order of a perfect elimination order of $\overline{G''}$ produced by Algorithm 5.2.2. Let $S' = k[x_1, \ldots, x_{n-1}] \subseteq S$. Then $I_{G''}$ is an edge ideal in the polynomial ring S' and $I_{G''}S = I_{G'}$. Set

$$\mathcal{B}' = \{1\} \cup \bigcup_{p \geq 1, q \geq 1} \left\{ (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q}) : \begin{array}{l} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q}) \in \mathcal{B} \\ j_q \leq n-1 \end{array} \right\}.$$

Suppose that $\mathbf{L}$ is the multigraded complex of free S'-modules with basis $\mathcal{B}'$ and differential $d_{\mathbf{L}} = d$ as defined in Construction 5.3.4, then by the induction hypothesis, $\mathbf{L}$ is the minimal free resolution of $S'/I_{G''}$. Let $\mathbf{F}' = \mathbf{L} \bigotimes S$. Since $S = S'[x_n]$ is a flat S'-module, it follows that $\mathbf{F}'$ is the multigraded minimal free resolution of the S-module $S'/I_{G''} \bigotimes S = S/(I_{G''}S) = S/I_{G'}$, and $\mathbf{F}'$ has basis $\mathcal{B}'$ and differential $d' = d_{\mathbf{L}} = d$ as in Construction 5.3.4. Setting

$$\mathcal{A} = \{(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q}, x_n) : (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q}) \in \mathcal{B}'\},$$

$$\mathcal{T} = \{(x_{i_1}, \ldots, x_{i_p} | x_n) : p \geq 1, \{x_{i_1}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_n)\},$$

we have the disjoint union

$$\mathcal{B} = \mathcal{B}' \cup \mathcal{A} \cup \mathcal{T}.$$

Let $\mathbf{E} : \cdots \to E_1 \to E_0 \to x_n I_{G'}$ be the multigraded minimal free resolution of $x_n I_{G'}$ induced naturally by the minimal free resolution $\mathbf{F}'$ of $S/I_{G'}$. Then $\mathbf{E}$ has basis $\mathcal{A}$ and the basis element $(x_{i_1}, \ldots, x_{i_p} \mid x_{j_1}, \ldots, x_{j_q}, x_n)$ is in homological degree $p + q - 2$ in $\mathbf{E}$. We denote the differential of $\mathbf{E}$ by $d_{\mathbf{E}}$. Note that $d_{\mathbf{E}}(x_{i_1} \mid x_{j_1}, x_n) = x_{i_1} x_{j_1} x_n$. Let $\mathbf{K}$ be the multigraded complex of free S-modules with basis $\mathcal{T}$ and differential $-\partial = -d$ where d is as in Construction 5.3.4. Note that the basis element $(x_{i_1}, \ldots, x_{i_p} \mid x_n)$ is in homological degree $p - 1$ in $\mathbf{K}$. And it is easy to see that $\mathbf{K}$ is the minimal free resolution of $x_n J$.

For any $(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n)\in\mathcal{A}$, we have that

$$\begin{aligned}d(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n) &= \mu_1(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n)\\ &+\mu_2(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n)\\ &+\mu_3(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n),\end{aligned}$$

where $\mu_1(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n)$ is the sum of the terms of $d(x_{i_1},\ldots,x_{i_p} \mid x_{j_1},\ldots,x_{j_q},x_n)$ that contain basis elements in $\mathcal{A}$, $\mu_2(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n)$ is the sum of the terms that contain basis elements in $\mathcal{T}$ and $\mu_3(x_{i_1},\ldots,x_{i_p} \mid x_{j_1},\ldots,x_{j_q},x_n)$ is the sum of the terms that contain basis elements in $\mathcal{B}'$. Note that $\mu_3(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n) = (-1)^{q+1+p}x_n(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})$. And by the definition of d, we can check that if $p+q\geq 3$, then

$$\mu_1(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n) = d_{\mathbf{E}}(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n).$$

We claim that $-\mu_2:\mathbf{E}\to\mathbf{K}$ is a multigraded complex map of degree 0 lifting the inclusion map $\phi: x_nI_{G'}\to x_nJ$. Indeed, $\phi d_{\mathbf{E}}(x_{i_1}|x_{j_1},x_n) = x_{i_1}x_{j_1}x_n$, and

$$\begin{aligned}(-\partial)(-\mu_2)(x_{i_1}|x_{j_1},x_n) &= \begin{cases}\partial(x_{j_1}(x_{i_1}|x_n)), & \text{if } x_{i_1}x_n\in G\\ \partial(x_{i_1}(x_{j_1}|x_n)), & \text{if } x_{i_1}x_n\notin G\end{cases}\\ &= x_{i_1}x_{j_1}x_n.\end{aligned}$$

Hence, $\phi d_{\mathbf{E}}(x_{i_1}|x_{j_1},x_n) = (-\partial)(-\mu_2)(x_{i_1}|x_{j_1},x_n)$. Then we need to show that for $p+q\geq 3$,

$$(-\mu_2)d_{\mathbf{E}}(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n) = (-\partial)(-\mu_2)(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q},x_n).$$

By Lemma 5.3.6, we have that

$$\begin{aligned} 0 &= d^2(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n) \\ &= \mu_1\mu_1(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n) + \mu_2\mu_1(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n) \\ &+ \mu_3\mu_1(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n) + \partial\mu_2(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n) \\ &+ d\mu_3(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n). \end{aligned} \tag{5.1}$$

In the above formula, collecting the terms which contain basis elements in $\mathcal{T}$, we get

$$\mu_2\mu_1(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n) + \partial\mu_2(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n) = 0.$$

Since $\mu_1 = d_{\mathbf{E}}$ for $p + q \geq 3$, it follows that

$$(-\mu_2)d_{\mathbf{E}}(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n) = (-\partial)(-\mu_2)(x_{i_1}, \dots, x_{i_p} | x_{j_1}, \dots, x_{j_q}, x_n),$$

and the claim is proved.

Let $\mathbf{F}''$ be the mapping cone $\mathrm{MC}(-\mu_2)$. Then $\mathbf{F}'' : \cdots \to F_1'' \to F_0'' \to x_nJ/x_nI_{G'}$ is a multigraded free resolution of $x_nJ/x_nI_{G'}$. Note that $F_0'' = K_0$ and $F_i'' = E_{i-1} \bigoplus K_i$ for $i \geq 1$. If we denote the differential of $\mathbf{F}''$ by d'', then $d_0''(x_{i_1}|x_n) = -\partial(x_{i_1}|x_n) = -x_{i_1}x_n$, $d_1''(x_{i_1}|x_{j_1}, x_n) = -\mu_2(x_{i_1}|x_{j_1}, x_n)$, $d_1''(x_{i_1}, x_{i_2}|x_n) = -\partial(x_{i_1}, x_{i_2}|x_n)$, that is, $d_1'' = (-\mu_2, -\partial)$, and for $i \geq 2$,

$$d_i'' = \begin{pmatrix} -d_{\mathbf{E}} & 0 \\ -\mu_2 & -\partial \end{pmatrix} = \begin{pmatrix} -\mu_1 & 0 \\ -\mu_2 & -\partial \end{pmatrix}.$$

Since the differential matrices of $\mathbf{F}''$ have monomial entries, $\mathbf{F}''$ is the minimal free resolution of $x_nJ/x_nI_{G'} \cong (I_{G'} + x_nJ)/I_{G'}$.

Next we define a map $\mu : \mathbf{F}'' \to \mathbf{F}'$ such that $\mu : F_0'' = K_0 \to F_0' = S$ is given by $\mu(x_{i_1}|x_n) = x_{i_1}x_n$ and for $i \geq 1$, $\mu : F_i'' = E_{i-1} \bigoplus K_i \to F_i'$ is given by

$\mu = (\mu_3, 0)$. We claim that $-\mu$ is a multigraded complex map of degree 0 lifting the inclusion map $\psi : (I_{G'} + x_n J)/I_{G'} \to S/I_{G'}$. Indeed, if $i = 0$ then $\psi d_0''(x_{i_1}|x_n) = -x_{i_1}x_n$, $d_0'(-\mu)(x_{i_1}|x_n) = -x_{i_1}x_n$, and hence $\psi d_0'' = d_0'(-\mu)$. If $i = 1$ then

$$\begin{aligned}
(-\mu)d_1''(x_{i_1}|x_{j_1}, x_n) &= (-\mu)(-\mu_2)(x_{i_1}|x_{j_1}, x_n) \\
&= \begin{cases} \mu(x_{j_1}(x_{i_1}|x_n)), & \text{if } x_{i_1}x_n \in G \\ \mu(x_{i_1}(x_{j_1}|x_n)), & \text{if } x_{i_1}x_n \notin G \end{cases} \\
&= x_{i_1}x_{j_1}x_n, \\
d_1'(-\mu)(x_{i_1}|x_{j_1}, x_n) &= d_1'(x_n(x_{i_1}|x_{j_1})) \\
&= x_{i_1}x_{j_1}x_n, \\
(-\mu)d_1''(x_{i_1}, x_{i_2}|x_n) &= (-\mu)(-\partial)(x_{i_1}, x_{i_2}|x_n) \\
&= \mu(x_{i_1}(x_{i_2}|x_n) - x_{i_2}(x_{i_1}|x_n)) \\
&= x_{i_1}x_{i_2}x_n - x_{i_2}x_{i_1}x_n = 0, \\
d_1'(-\mu)(x_{i_1}, x_{i_2}|x_n) &= d_1'(0) = 0,
\end{aligned}$$

and hence $(-\mu)d_1'' = d_1'(-\mu)$. If $i \geq 2$ then it is easy to see that for $p \geq 3$,

$$(-\mu)d_i''(x_{i_1}, \ldots, x_{i_p}|x_n) = d_i'(-\mu)(x_{i_1}, \ldots, x_{i_p}|x_n) = 0,$$

so we need only to prove that for $p + q = i + 1 \geq 3$,

$$(-\mu)d_i''(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = d_i'(-\mu)(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n),$$

that is,

$$\mu(-\mu_1 - \mu_2)(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = d\mu_3(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n).$$

Since $\mu\mu_2(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = 0$, it suffices to prove that

$$-\mu_3\mu_1(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = d\mu_3(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n).$$

However, in formula (5.1), collecting the terms which contain basis elements in $\mathcal{B}'$, we see that

$$\mu_3\mu_1(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) + d\mu_3(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}, x_n) = 0,$$

and the claim is proved. So $\mu : \mathbf{F}'' \to \mathbf{F}'$ is a complex map lifting $-\psi : (I_{G'} + x_n J)/I_{G'} \to S/I_{G'}$, and it is eay to see that μ is multigraded of degree 0.

Let $\mathbf{F}^*$ be the mapping cone $\mathrm{MC}(\mu)$. Then $\mathbf{F}^* : \cdots \to F_1^* \to F_0^* \to \mathrm{coker}(-\psi)$ gives a multigraded free resolution of $\mathrm{coker}(-\psi) = S/I_G$. Note that $F_0^* = S$, $F_1^* = F_0'' \bigoplus F_1' = K_0 \bigoplus F_1'$ and for $i \geq 2$, $F_i^* = F_{i-1}'' \bigoplus F_i' = E_{i-2} \bigoplus K_{i-1} \bigoplus F_i'$. If we denote the differential of $\mathbf{F}^*$ by d^*, then $d_0^*(1) = 1$, $d_1^* = (\mu, d_1')$,

$$d_2^* = \begin{pmatrix} -d_1'' & 0 \\ \mu & d_2' \end{pmatrix} = \begin{pmatrix} \mu_2 & \partial & 0 \\ \mu_3 & 0 & d \end{pmatrix},$$

and for $i \geq 3$,

$$d_i^* = \begin{pmatrix} -d_{i-1}'' & 0 \\ \mu & d_i' \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 & 0 \\ \mu_2 & \partial & 0 \\ \mu_3 & 0 & d \end{pmatrix}.$$

Note that $\mathbf{F}^*$ and $\mathbf{F}$ have the same basis and the same differential. So $\mathbf{F}^* = \mathbf{F}$, and then $\mathbf{F}$ is a multigraded free resolution of S/I_G. Since $d_i(F_i) \subseteq (x_1, \ldots, x_n)F_{i-1}$ for all $i \geq 1$, the resolution $\mathbf{F}$ is minimal, and we are done. □

Example 5.3.8. Let G be the following graph. Then $\overline{G}$ is chordal and x_1, x_2, x_3, x_4

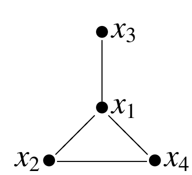


Figure 5.3: A resolution of pattern Γ

is in the reverse order of a perfect elimination order of $\overline{G}$ produced by Algorithm 5.2.2. Note that

$$S = k[x_1, x_2, x_3, x_4], \quad I_G = (x_1x_2, x_1x_3, x_1x_4, x_2x_4),$$

$$\text{pnbhd}(x_1) = \emptyset,\ \text{pnbhd}(x_2) = \{x_1\},\ \text{pnbhd}(x_3) = \{x_1\},\ \text{pnbhd}(x_4) = \{x_1, x_2\}.$$

By Construction 5.3.4, the minimal free resolution of S/I_G has basis

$$1; (x_1|x_2, x_3, x_4), (x_1|x_2, x_3), (x_1|x_2, x_4), (x_1|x_2);$$

$$(x_1|x_3, x_4), (x_1|x_3); (x_1, x_2|x_4), (x_1|x_4), (x_2|x_4).$$

And we have the map d such that

$$d(x_1|x_2) = x_1x_2, \quad d(x_1|x_3) = x_1x_3,$$

$$d(x_1|x_4) = x_1x_4, \quad d(x_2|x_4) = x_2x_4,$$

$$d(x_1|x_2, x_3) = x_2(x_1|x_3) - x_3(x_1|x_2),$$

$$d(x_1|x_2, x_4) = x_2(x_1|x_4) - x_4(x_1|x_2),$$

$$d(x_1|x_3, x_4) = x_3(x_1|x_4) - x_4(x_1|x_3),$$

$$d(x_1, x_2|x_4) = x_1(x_2|x_4) - x_2(x_1|x_4),$$

$$d(x_1|x_2, x_3, x_4) = x_2(x_1|x_3, x_4) - x_3(x_1|x_2, x_4) + x_4(x_1|x_2, x_3).$$

Therefore, the minimal free resolution of S/I_G is

$$0 \to S(-x_1x_2x_3x_4) \xrightarrow{d_3} S(-x_1x_2x_3) \oplus S(-x_1x_2x_4) \oplus S(-x_1x_3x_4) \oplus S(-x_1x_2x_4)$$

$$\xrightarrow{d_2} S(-x_1x_2) \oplus S(-x_1x_3) \oplus S(-x_1x_4) \oplus S(-x_2x_4) \xrightarrow{d_1} S \to S/I_G,$$

where

$$d_3 = \begin{pmatrix} x_4 \\ -x_3 \\ x_2 \\ 0 \end{pmatrix}, d_2 = \begin{pmatrix} -x_3 & -x_4 & 0 & 0 \\ x_2 & 0 & -x_4 & 0 \\ 0 & x_2 & x_3 & -x_2 \\ 0 & 0 & 0 & x_1 \end{pmatrix}, d_1 = \begin{pmatrix} x_1x_2 & x_1x_3 & x_1x_4 & x_2x_4 \end{pmatrix}.$$

Remark 5.3.9. In the above example, we have that $\text{pnbhd}(x_1) \subseteq \text{pnbhd}(x_2) \subseteq \text{pnbhd}(x_3) \subseteq \text{pnbhd}(x_4)$. But in general, given a linear edge ideal I_G, there may not exist a perfect elimination order of $\overline{G}$ such that its reverse order $x_1, \ldots, x_n$ satisfies $\text{pnbhd}(x_i) \subseteq \text{pnbhd}(x_{i+1})$ in G for $i = 1, \ldots, n-1$. For example, if $\overline{G}$ is the star-shaped chordal graph in Example 5.2.6, then we can check that $\overline{G}$ has no perfect elimination order satisfying the above property. However, the following proposition says that if the above property is satisfied then the perfect elimination order of $\overline{G}$ can be produced by Algorithm 5.2.2.

Proposition 5.3.10. *Let G be a simple graph with vertices $x_1, \ldots, x_n$ such that $\overline{G}$ is chordal. Let $x_1, \ldots, x_n$ be in the reverse order of a perfect elimination order of $\overline{G}$ such that $pnbhd(x_i) \subseteq pnbhd(x_{i+1})$ in G for $i = 1, \ldots, n-1$. Then the perfect elimination order $x_n, \ldots, x_1$ of $\overline{G}$ can be produced by Algorithm 5.2.2.*

Proof. First we choose $v_n = x_1$ in Algorithm 5.2.2. Since $\text{pnbhd}(x_2) \subseteq \text{pnbhd}(x_j)$ in G for any $2 < j \leq n$, it follows that if $x_1x_2 \notin \overline{G}$ then $x_1x_j \notin \overline{G}$ for all $2 < j \leq n$, so that in Algorithm 5.2.2 we can choose $v_{n-1} = x_2$. Now suppose that we have chosen $v_n = x_1, v_{n-1} = x_2, \ldots, v_{n-(i-2)} = x_{i-1}$ for some $3 \leq i \leq n$. Since $\text{pnbhd}(x_i) \subseteq \text{pnbhd}(x_j)$ in G for any $i < j \leq n$, it follows that for any $1 \leq l \leq i-1$, if $x_lx_i \notin \overline{G}$ then $x_lx_j \notin \overline{G}$ for all $i < j \leq n$, so that in Algorithm 5.2.2 we can choose $v_{n-(i-1)} = x_i$. So by using induction we see that $x_n, \ldots, x_1$ can be the output of Algorithm 5.2.2 and we are done. □

Remark 5.3.11. If the conditions in the above proposition are satisfied, then there will be no β terms in the differential formula. However, as we have seen in Remark 5.3.9, the conditions in the above proposition can not always be satisfied, especially when $\overline{G}$ is a complicated chordal graph. So in general, the β terms in the differential formula can not be avoided.

Remark 5.3.12. Let $G = K_n$ be the complete graph with n vertices $x_1, \ldots, x_n$. Then we have the Eliahou-Kervaire resolution of S/I_G. It is easy to see that the basis element $(x_i x_j; i_1, \ldots, i_p, j_1, \ldots, j_q)$ with $i_1 < \cdots < i_p < i < j_1 < \cdots < j_q < j$ in the Eliahou-Kervaire resolution corresponds naturally to the basis element $(x_{i_1}, \ldots, x_{i_p}, x_i | x_{j_1}, \ldots, x_{j_q}, x_j)$ in Construction 5.3.4. But the differential maps defined on them are different. For example, if $G = K_3$, then $d(x_2x_3; 1) = x_1(x_2x_3; \emptyset) - x_3(x_1x_2; \emptyset)$, but $d(x_1, x_2|x_3) = x_1(x_2|x_3) - x_2(x_1|x_3)$. So in the case of complete graphs, the resolution defined in Construction 5.3.4 does not recover the Eliahou-Kervaire resolution. By contrast, the resolution in [Ho] recovers the Eliahou-Kervaire resolution in the case of complete graphs.

5.4 The proof of $d^2 = 0$

Before proving Lemma 5.3.6, we look at the following example.

Example 5.4.1. Let G be the graph such that $\overline{G}$ is the chordal graph given in Example 5.2.6. Then $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ is in the reverse order of a perfect elimination order of $\overline{G}$ produced by Algorithm 5.2.2. Note that in G,

$$\text{pnbhd}(x_5) = \{x_1, x_2, x_3\} \nsubseteq \text{pnbhd}(x_6) = \{x_1, x_2, x_4, x_5\}.$$

Next we check that $d^2(x_1, x_2, x_3|x_5, x_6) = 0$. In fact, by the definition of d, we have that

$$
\begin{aligned}
d(x_1, x_2, x_3|x_5, x_6) =& x_1(x_2, x_3|x_5, x_6) - x_2(x_1, x_3|x_5, x_6) \\
& + x_3[(x_1, x_2|x_5, x_6) + (x_1, x_2, x_5|x_6)] - x_6(x_1, x_2, x_3|x_5), \\
d(x_1(x_2, x_3|x_5, x_6)) =& x_1x_2(x_3|x_5, x_6) - x_1x_3[(x_2|x_5, x_6) + (x_2, x_5|x_6)] \\
& + x_1x_6(x_2, x_3|x_5), \\
d(-x_2(x_1, x_3|x_5, x_6)) =& - x_2x_1(x_3|x_5, x_6) + x_2x_3[(x_1|x_5, x_6) + (x_1, x_5|x_6)] \\
& - x_2x_6(x_1, x_3|x_5), \\
d(x_3(x_1, x_2|x_5, x_6)) =& x_3x_1(x_2|x_5, x_6) - x_3x_2(x_1|x_5, x_6) \\
& - x_3x_5(x_1, x_2|x_6) + x_3x_6(x_1, x_2|x_5), \\
d(x_3(x_1, x_2, x_5|x_6)) =& x_3x_1(x_2, x_5|x_6) - x_3x_2(x_1, x_5|x_6) + x_3x_5(x_1, x_2|x_6), \\
d(-x_6(x_1, x_2, x_3|x_5)) =& - x_6x_1(x_2, x_3|x_5) + x_6x_2(x_1, x_3|x_5) - x_6x_3(x_1, x_2|x_5).
\end{aligned}
$$

So the sum of the terms in $d^2(x_1, x_2, x_3|x_5, x_6)$ containing x_1x_2 is

$$x_1x_2(x_3|x_5, x_6) - x_2x_1(x_3|x_5, x_6) = 0;$$

the sum of the terms in $d^2(x_1, x_2, x_3|x_5, x_6)$ containing x_1x_3 is

$$-x_1x_3[(x_2|x_5, x_6) + (x_2, x_5|x_6)] + x_3x_1(x_2|x_5, x_6) + x_3x_1(x_2, x_5|x_6) = 0;$$

and similarly, we have

$$x_2x_3[(x_1|x_5, x_6) + (x_1, x_5|x_6)] - x_3x_2(x_1|x_5, x_6) - x_3x_2(x_1, x_5|x_6) = 0,$$

$$-x_3x_5(x_1, x_2|x_6) + x_3x_5(x_1, x_2|x_6) = 0,$$

$$x_1x_6(x_2, x_3|x_5) - x_6x_1(x_2, x_3|x_5) = 0,$$

$$-x_2x_6(x_1, x_3|x_5) + x_6x_2(x_1, x_3|x_5) = 0,$$

$$x_3x_6(x_1, x_2|x_5) - x_6x_3(x_1, x_2|x_5) = 0.$$

Therefore, $d^2(x_1, x_2, x_3|x_5, x_6) = 0$.

Proof of Lemma 5.3.6. First we have that

$$
\begin{aligned}
d^2(x_{i_1}|x_{j_1}) &= d(x_{i_1}x_{j_1}) = x_{i_1}x_{j_1} = 0 \text{ in } S/I_G,\\
d^2(x_{i_1}, x_{i_2}|x_{j_1}) &= d(x_{i_1}(x_{i_2}|x_{j_1}) - x_{i_2}(x_{i_1}|x_{j_1}))\\
&= x_{i_1}x_{i_2}x_{j_1} - x_{i_2}x_{i_1}x_{j_1} = 0,\\
d^2(x_{i_1}|x_{j_1}, x_{j_2}) &= \begin{cases} d(x_{j_1}(x_{i_1}|x_{j_2}) - x_{j_2}(x_{i_1}|x_{j_1})), & \text{if } x_{i_1}x_{j_2} \in G\\ d(x_{i_1}(x_{j_1}|x_{j_2}) - x_{j_2}(x_{i_1}|x_{j_1})), & \text{if } x_{i_1}x_{j_2} \notin G \end{cases}\\
&= \begin{cases} x_{j_1}x_{i_1}x_{j_2} - x_{j_2}x_{i_1}x_{j_1}, & \text{if } x_{i_1}x_{j_2} \in G\\ x_{i_1}x_{j_1}x_{j_2} - x_{j_2}x_{i_1}x_{j_1}, & \text{if } x_{i_1}x_{j_2} \notin G \end{cases}\\
&= 0.
\end{aligned}
$$

Next we need only to prove that $d^2(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}) = 0$ for $p + q \geq 4$. Just as in Example 5.4.1, it suffices to prove that if we write out all the terms of $d^2(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})$, then given any $\lambda, \lambda' \in \{i_1, \ldots, i_p, j_1, \ldots, j_q\}$, the sum of the terms containing $x_\lambda x_{\lambda'}$ is zero, that is all the terms containing $x_\lambda x_{\lambda'}$ cancel. Hence, a computation will reveal that if β does not exist, that is $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_t})$ for all $1 \leq t \leq q$, then $d^2(x_{i_1}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}) = 0$. So we will assume that $q \geq 2$ and β exists. The proof is case by case and there are five main cases.

[Case A]: $\lambda, \lambda' \in \{i_1, \ldots, i_p\}$.

[Case A-a]: if $1 \leq s < s' \leq p$ such that $x_{i_s}x_{j_\beta} \in G$ and $x_{i_{s'}}x_{j_\beta} \in G$, then the sum of the terms containing $x_{i_s}x_{i_{s'}}$ is

$$
\begin{aligned}
&(-1)^{s+1}x_{i_s}(-1)^{s'}x_{i_{s'}}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_{s'}}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})\\
&+ (-1)^{s'+1}x_{i_{s'}}(-1)^{s+1}x_{i_s}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_{s'}}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}) = 0.
\end{aligned}
$$

[Case A-b]: suppose that there is a term containing $x_{i_s}x_{i_\alpha}$ for some $1 \leq s, \alpha \leq p$

such that $x_{i_s}x_{j_\beta} \in G$ and $x_{i_\alpha}x_{j_\beta} \notin G$. Without the loss of generality, we assume $s < \alpha$.

Subcase (i): if $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_s}x_{i_\alpha}$ is

$$\begin{aligned}&(-1)^{s+1}x_{i_s}(-1)^{\alpha}x_{i_\alpha}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})\\ &+ (-1)^{\alpha+1}x_{i_\alpha}(-1)^{s+1}x_{i_s}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}) = 0.\end{aligned}$$

Subcase (ii): if $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_\beta})$, then we set

$$\beta' = \min\{t \mid \beta < t \leq q, \{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_t})\}.$$

Lemma 5.3.2 implies that for any $\beta \leq t \leq q$, $x_{j_1}x_{j_t}, \ldots, x_{j_{\beta-1}}x_{j_t} \in G$, so we have

$$\beta' = \min\{t \mid \beta < t \leq q, \{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}\} \nsubseteq \text{pnbhd}(x_{j_t})\}.$$

Subsubcase (ii)(a): if one of the following conditions is satisfied:

1) β' does not exist,

2) $x_{i_s}x_{j_{\beta'}} \in G$,

3) $x_{i_s}x_{j_{\beta'}} \notin G$ and $\{x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_{\beta'}})$,

then the sum of the terms containing $x_{i_s}x_{i_\alpha}$ is

$$\begin{aligned}&(-1)^{s+1}x_{i_s}(-1)^{\alpha}x_{i_\alpha}[(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})\\ &+ (-1)^{\beta}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, x_{j_q})]\\ &+ (-1)^{\alpha+1}x_{i_\alpha}[(-1)^{s+1}x_{i_s}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})\\ &+ (-1)^{\beta}(-1)^{s+1}x_{i_s}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, x_{j_q})] = 0.\end{aligned}$$

Subsubcase (ii)(b): if $x_{i_s}x_{j_{\beta'}} \notin G$, $\{x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta'}})$, then the sum of the terms containing $x_{i_s}x_{i_\alpha}$ is

$$\begin{aligned}
&(-1)^{s+1}x_{i_s}(-1)^{\alpha}x_{i_\alpha}[(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})\\
&+(-1)^{\beta}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, x_{j_q})]\\
&+(-1)^{\alpha+1}x_{i_\alpha}\{(-1)^{s+1}x_{i_s}[(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})\\
&+(-1)^{\beta'}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta'-1}}|x_{j_{\beta'}}, \ldots, x_{j_q})]\\
&+(-1)^{\beta}(-1)^{s+1}x_{i_s}[(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, x_{j_q})\\
&+(-1)^{\beta'-\beta+1}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta'-1}}|x_{j_{\beta'}}, \ldots, x_{j_q})]\} = 0.
\end{aligned}$$

Note that in the above two subsubcases, if $s = 1$ and $\alpha = p = 2$ then the terms containing $(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})$ are zeros.

[Case A-c]: suppose that there is a term containing $x_{i_\alpha}x_{i_{\alpha'}}$ for some $1 \leq \alpha < \alpha' \leq p$ such that $x_{i_\alpha}x_{j_\beta} \notin G$ and $x_{i_{\alpha'}}x_{j_\beta} \notin G$.

Subcase (i): if $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, \widehat{x_{i_{\alpha'}}}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha}x_{i_{\alpha'}}$ is

$$\begin{aligned}
&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{\alpha'}x_{i_{\alpha'}}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, \widehat{x_{i_{\alpha'}}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})\\
&+(-1)^{\alpha'+1}x_{i_{\alpha'}}(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, \widehat{x_{i_{\alpha'}}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q}) = 0.
\end{aligned}$$

Subcase (ii): if $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, \widehat{x_{i_{\alpha'}}}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha}x_{i_{\alpha'}}$ is

$$\begin{aligned}
&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{\alpha'}x_{i_{\alpha'}}[(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, \widehat{x_{i_{\alpha'}}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})\\
&+(-1)^{\beta}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, \widehat{x_{i_{\alpha'}}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, x_{j_q})]\\
&+(-1)^{\alpha'+1}x_{i_{\alpha'}}(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, \widehat{x_{i_{\alpha'}}}, \ldots, x_{i_p}|x_{j_1}, \ldots, x_{j_q})\\
&+(-1)^{\beta}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, \widehat{x_{i_{\alpha'}}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, x_{j_q})] = 0.
\end{aligned}$$

Note that if $\alpha = 1$ and $\alpha' = p = 2$, then in the above formula, the two terms containing $(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, \widehat{x_{i_{\alpha'}}}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q})$ are zeros.

[Case B]: $\lambda \in \{i_1, \ldots, i_p\}$ and $\lambda' = j_1$.

[Case B-a]: suppose that there is a term containing $x_{i_s} x_{j_1}$ for some $1 \leq s \leq p$ such that $x_{i_s} x_{j_\beta} \in G$, then it is easy to see that $\beta \neq 2$ and the sum of the terms containing $x_{i_s} x_{j_1}$ is

$$(-1)^{s+1} x_{i_s} (-1)^{1+(p-1)} x_{j_1} (x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p} | x_{j_2}, \ldots, x_{j_q}) \\ + (-1)^{p+1} x_{j_1} (-1)^{s+1} x_{i_s} (x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p} | x_{j_2}, \ldots, x_{j_q}) = 0.$$

[Case B-b]: suppose that there is a term containing $x_{i_\alpha} x_{j_1}$ for some $1 \leq \alpha \leq p$ such that $x_{i_\alpha} x_{j_\beta} \notin G$.

Subcase (i): $\beta = 2$. If we have $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \nsubseteq \mathrm{pnbhd}(x_{j_\beta})$, then it is easy to see that there is no term containing $x_{i_\alpha} x_{j_1}$, hence we must have $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_\beta})$ and the sum of the terms containing $x_{i_\alpha} x_{j_1}$ is

$$(-1)^{\alpha+1} x_{i_\alpha} [(-1)^p x_{j_1} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | \widehat{x_{j_1}}, x_{j_2}, \ldots, x_{j_q}) \\ + (-1)^{\beta} (-1)^{p+1} x_{j_1} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, \widehat{x_{j_1}} | x_{j_2}, \ldots, x_{j_q})] = 0.$$

Subcase (ii): if $\beta > 2$ and $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \nsubseteq \mathrm{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha} x_{j_1}$ is

$$(-1)^{\alpha+1} x_{i_\alpha} (-1)^p x_{j_1} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | \widehat{x_{j_1}}, x_{j_2}, \ldots, x_{j_q}) \\ + (-1)^{p+1} x_{j_1} (-1)^{\alpha+1} x_{i_\alpha} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | \widehat{x_{j_1}}, x_{j_2}, \ldots, x_{j_q}) = 0.$$

Subcase (iii): if $\beta > 2$ and $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha}x_{j_1}$ is

$$\begin{aligned}
&(-1)^{\alpha+1}x_{i_\alpha}[(-1)^p x_{j_1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|\widehat{x_{j_1}},x_{j_2},\dots,x_{j_q})\\
&+(-1)^{\beta}(-1)^{p+1}x_{j_1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},\widehat{x_{j_1}},x_{j_2},\dots,x_{j_{\beta-1}}|x_{j_\beta},\dots,x_{j_q})]\\
&+(-1)^{p+1}x_{j_1}(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|\widehat{x_{j_1}},x_{j_2},\dots,x_{j_q})\\
&+(-1)^{\beta-1}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},\widehat{x_{j_1}},x_{j_2},\dots,x_{j_{\beta-1}}|x_{j_\beta},\dots,x_{j_q})]=0.
\end{aligned}$$

[Case C]: $\lambda \in \{i_1,\dots,i_p\}$ and $\lambda' \in \{j_2,\dots,j_q\}$.

[Case C-a]: if $1 \le s \le p$, $2 \le t \le q$ such that $x_{i_s}x_{j_\beta} \in G$ and $t \neq \beta$, then the sum of the terms containing $x_{i_s}x_{j_t}$ is

$$\begin{aligned}
&(-1)^{s+1}x_{i_s}(-1)^{t+(p-1)}x_{j_t}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\
&+(-1)^{t+p}x_{j_t}(-1)^{s+1}x_{i_s}(x_{i_1},\dots,\widehat{x_{i_s}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q})=0.
\end{aligned}$$

[Case C-b]: suppose that there is a term containing $x_{i_\alpha}x_{j_t}$ for some $1 \le \alpha \le p$, $2 \le t \le q$ such that $x_{i_\alpha}x_{j_\beta} \notin G$ and $t \neq \beta$.

Subcase (i): if $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \nsubseteq \mathrm{pnbhd}(x_{j_\beta})$, then the sum of the terms containing $x_{i_\alpha}x_{j_t}$ is

$$\begin{aligned}
&(-1)^{\alpha+1}x_{i_\alpha}(-1)^{t+(p-1)}x_{j_t}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q})\\
&+(-1)^{t+p}x_{j_t}(-1)^{\alpha+1}x_{i_\alpha}(x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}|x_{j_1},\dots,\widehat{x_{j_t}},\dots,x_{j_q})=0.
\end{aligned}$$

Subcase (ii): if $\{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \subseteq \mathrm{pnbhd}(x_{j_\beta})$, then as in subcase (ii) of [Case A-b], we set

$$\begin{aligned}
\beta' &= \min\{t \mid \beta < t \le q, \{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p}\} \nsubseteq \mathrm{pnbhd}(x_{j_t})\}\\
&= \min\{t \mid \beta < t \le q, \{x_{i_1},\dots,\widehat{x_{i_\alpha}},\dots,x_{i_p},x_{j_1},\dots,x_{j_{\beta-1}}\} \nsubseteq \mathrm{pnbhd}(x_{j_t})\}.
\end{aligned}$$

Subsubcase (ii)(a): if $t < \beta$, then the sum of the terms containing $x_{i_\alpha} x_{j_t}$ is

$$(-1)^{\alpha+1} x_{i_\alpha}[(-1)^{t+(p-1)} x_{j_t}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})$$
$$+ (-1)^{\beta}(-1)^{t+(p-1)+1} x_{j_t}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, x_{j_q})]$$
$$+ (-1)^{t+p} x_{j_t}(-1)^{\alpha+1} x_{i_\alpha}[(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})$$
$$+ (-1)^{\beta-1}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, x_{j_q})] = 0.$$

Subsubcase (ii)(b): if one of the following conditions is satisfied:

1) $t > \beta$ and β' does not exist,
2) $t > \beta$ and $t \neq \beta'$,
3) $t = \beta' = q$,
4) $t = \beta'$ and $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_{\beta'+1}})$,

then the sum of the terms containing $x_{i_\alpha} x_{j_t}$ is

$$(-1)^{\alpha+1} x_{i_\alpha}[(-1)^{t+(p-1)} x_{j_t}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})$$
$$+ (-1)^{\beta}(-1)^{t+p-1} x_{j_t}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})]$$
$$+ (-1)^{t+p} x_{j_t}(-1)^{\alpha+1} x_{i_\alpha}[(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})$$
$$+ (-1)^{\beta}(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_\beta}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})] = 0.$$

Note that in the above two subsubcases, if $\alpha = p = 1$ then the terms containing $(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}|x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})$ are zeros and β' does not exist.

Subsubcase (ii)(c): if $t = \beta'$ and $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta'+1}})$, then the sum of the terms containing $x_{i_\alpha}x_{j_t}$ is

$$(-1)^{\alpha+1}x_{i_\alpha}\{(-1)^{t+(p-1)}x_{j_t}[(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})$$
$$+(-1)^t(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{t-1}}|x_{j_{t+1}},\ldots,x_{j_q})]$$
$$+(-1)^\beta(-1)^{t+p-1}x_{j_t}[(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_\beta},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})$$
$$(-1)^{t-\beta+1}(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{t-1}}|x_{j_{t+1}},\ldots,x_{j_q})]\}$$
$$+(-1)^{t+p}x_{j_t}(-1)^{\alpha+1}x_{i_\alpha}[(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})$$
$$+(-1)^\beta(x_{i_1},\ldots,\widehat{x_{i_\alpha}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_\beta},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})]=0.$$

[Case C-c]: suppose that there is a term containing $x_{i_s}x_{j_\beta}$ for some $1\le s\le p$ such that $x_{i_s}x_{j_\beta}\in G$. We set

$$\beta''=\min\{t\mid \beta<t\le q,\{x_{i_1},\ldots,x_{i_p}\}\nsubseteq \mathrm{pnbhd}(x_{j_t})\}.$$

Lemma 5.3.2 implies that for any $\beta\le t\le q$, $x_{j_1}x_{j_t},\ldots,x_{j_{\beta-1}}x_{j_t}\in G$, so we have

$$\beta''=\min\{t\mid \beta<t\le q,\{x_{i_1},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}\}\nsubseteq \mathrm{pnbhd}(x_{j_t})\}.$$

Subcase (i): if $\beta=q$ or $\{x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}\}\nsubseteq \mathrm{pnbhd}(x_{j_{\beta+1}})$, then the sum of the terms containing $x_{i_s}x_{j_\beta}$ is

$$(-1)^{s+1}x_{i_s}(-1)^{\beta+(p-1)}x_{j_\beta}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})$$
$$+(-1)^{\beta+p}x_{j_\beta}(-1)^{s+1}x_{i_s}(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})=0.$$

Subcase (ii): if $\{x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}\}\subseteq \mathrm{pnbhd}(x_{j_{\beta+1}})$ and $x_{i_s}x_{j_{\beta+1}}\notin G$, then $\beta''=\beta+1$ and the sum of the terms containing $x_{i_s}x_{j_\beta}$ is

$$(-1)^{s+1}x_{i_s}(-1)^{\beta+(p-1)}x_{j_\beta}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})$$
$$+(-1)^\beta(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]$$
$$+(-1)^{\beta+p}x_{j_\beta}(-1)^{s+1}x_{i_s}[(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})$$
$$+(-1)^\beta(x_{i_1},\ldots,\widehat{x_{i_s}},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]=0.$$

Subcase (iii): if one of the following conditions is satisfied:

1) $\beta < q$ and β'' does not exist,

2) $\beta'' > \beta + 1$ and $x_{i_s}x_{j_{\beta''}} \in G$,

3) $\beta'' > \beta + 1$, $x_{i_s}x_{j_{\beta''}} \notin G$ and $\{x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_{\beta''}})$

then the sum of the terms containing $x_{i_s}x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{s+1}x_{i_s}(-1)^{\beta+(p-1)}x_{j_\beta}[(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}|x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})\\
&+ (-1)^{\beta}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_{\beta+1}}, \ldots, x_{j_q})]\\
&+ (-1)^{\beta+p}x_{j_\beta}[(-1)^{s+1}x_{i_s}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}|x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})\\
&+ (-1)^{\beta}(-1)^{s+1}x_{i_s}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_{\beta+1}}, \ldots, x_{j_q})] = 0.
\end{aligned}$$

Subcase (iv): if $\beta'' > \beta + 1$, $x_{i_s}x_{j_{\beta''}} \notin G$, $\{x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta''}})$, then the sum of the terms containing $x_{i_s}x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{s+1}x_{i_s}(-1)^{\beta+(p-1)}x_{j_\beta}[(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}|x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})\\
&+ (-1)^{\beta}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_{\beta+1}}, \ldots, x_{j_q})]\\
&+ (-1)^{\beta+p}x_{j_\beta}\{(-1)^{s+1}x_{i_s}[(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}|x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})\\
&+ (-1)^{\beta''-1}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}, x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_{\beta''-1}}|x_{j_{\beta''}}, \ldots, x_{j_q})]\\
&+ (-1)^{\beta}(-1)^{s+1}x_{i_s}[(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}|x_{j_{\beta+1}}, \ldots, x_{j_q})\\
&+ (-1)^{\beta''-\beta}(x_{i_1}, \ldots, \widehat{x_{i_s}}, \ldots, x_{i_p}, x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_{\beta''-1}}|x_{j_{\beta''}}, \ldots, x_{j_q})]\} = 0.
\end{aligned}$$

[Case C-d]: suppose that there is a term containing $x_{i_\alpha}x_{j_\beta}$ for some $1 \le \alpha \le p$ such that $x_{i_\alpha}x_{j_\beta} \notin G$. As in [Case C-c], we set

$$\begin{aligned}
\beta'' &= \min\{t \mid \beta < t \le q, \{x_{i_1}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_t})\}\\
&= \min\{t \mid \beta < t \le q, \{x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}\} \nsubseteq \text{pnbhd}(x_{j_t})\}.
\end{aligned}$$

Subcase (i): if $\beta = q$ or $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_{\beta+1}})$, then the sum of the terms containing $x_{i_\alpha} x_{j_\beta}$ is

$$(-1)^{\alpha+1} x_{i_\alpha} (-1)^{\beta+(p-1)} x_{j_\beta} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})$$
$$+ (-1)^{\beta+p} x_{j_\beta} (-1)^{\alpha+1} x_{i_\alpha} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q}) = 0.$$

Subcase (ii): if $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_\beta})$, then we have the following three subsubcases.

Subsubcase (ii)(a): if $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta+1}})$ and $x_{i_\alpha} x_{j_{\beta+1}} \notin G$, then $\beta'' = \beta + 1$ and the sum of the terms containing $x_{i_\alpha} x_{j_\beta}$ is

$$(-1)^{\alpha+1} x_{i_\alpha} [(-1)^{\beta+(p-1)} x_{j_\beta} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})$$
$$+ (-1)^{\beta} (-1)^{\beta+p-1} x_{j_\beta} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \ldots, x_{j_q})]$$
$$+ (-1)^{\beta+p} x_{j_\beta} (-1)^{\alpha+1} x_{i_\alpha} [(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})$$
$$+ (-1)^{\beta} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \ldots, x_{j_q})] = 0.$$

Subsubcase (ii)(b): if $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta+1}})$ and one of the following conditions is satisfied:

1) β'' does not exist,

2) $x_{i_\alpha} x_{j_{\beta''}} \in G$,

3) $x_{i_\alpha} x_{j_{\beta''}} \notin G$ and $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_{\beta''}})$,

then the sum of the terms containing $x_{i_\alpha} x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{\alpha+1} x_{i_\alpha}[(-1)^{\beta+(p-1)} x_{j_\beta}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_q}) \\
&+ (-1)^{\beta}(-1)^{\beta+p-1} x_{j_\beta}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \dots, x_{j_q})] \\
&+ (-1)^{\beta+p} x_{j_\beta}[(-1)^{\alpha+1} x_{i_\alpha}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_q}) \\
&+ (-1)^{\beta}(-1)^{\alpha+1} x_{i_\alpha}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \dots, x_{j_q})] = 0.
\end{aligned}$$

Subsubcase (ii)(c): if $\beta'' \geq \beta + 2$, $x_{i_\alpha} x_{j_{\beta''}} \notin G$ and $\{x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta''}})$, then the sum of the terms containing $x_{i_\alpha} x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{\alpha+1} x_{i_\alpha}[(-1)^{\beta+(p-1)} x_{j_\beta}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_q}) \\
&+ (-1)^{\beta}(-1)^{\beta+p-1} x_{j_\beta}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \dots, x_{j_q})] \\
&+ (-1)^{\beta+p} x_{j_\beta}\{(-1)^{\alpha+1} x_{i_\alpha}[(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_q}) \\
&+ (-1)^{\beta''-1}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_{\beta''-1}} | x_{j_{\beta''}}, \dots, x_{j_q})] \\
&+ (-1)^{\beta}(-1)^{\alpha+1} x_{i_\alpha}[(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \dots, x_{j_q}) \\
&+ (-1)^{\beta''-\beta}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_{\beta''-1}} | x_{j_{\beta''}}, \dots, x_{j_q})]\} = 0.
\end{aligned}$$

Subcase (iii): if $\{x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_\beta})$, then just as in subcase (ii), we have the following three subsubcases.

Subsubcase (iii)(a): if $\{x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta+1}})$ and $x_{i_\alpha} x_{j_{\beta+1}} \notin G$, then $\beta'' = \beta + 1$ and the sum of the terms containing $x_{i_\alpha} x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{\alpha+1} x_{i_\alpha}(-1)^{\beta+(p-1)} x_{j_\beta}[(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_q}) \\
&+ (-1)^{\beta}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \dots, x_{j_q})] \\
&+ (-1)^{\beta+p} x_{j_\beta}(-1)^{\alpha+1} x_{i_\alpha}[(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p} | x_{j_1}, \dots, \widehat{x_{j_\beta}}, \dots, x_{j_q}) \\
&+ (-1)^{\beta}(x_{i_1}, \dots, \widehat{x_{i_\alpha}}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \dots, x_{j_q})] = 0.
\end{aligned}$$

Subsubcase (iii)(b): if $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta+1}})$ and one of the following conditions is satisfied:

1) β'' does not exist,

2) $x_{i_\alpha} x_{j_{\beta''}} \in G$,

3) $x_{i_\alpha} x_{j_{\beta''}} \notin G$ and $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_{\beta''}})$,

then the sum of the terms containing $x_{i_\alpha} x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{\alpha+1} x_{i_\alpha} (-1)^{\beta+(p-1)} x_{j_\beta} [(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q}) \\
&+ (-1)^{\beta} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \ldots, x_{j_q})] \\
&+ (-1)^{\beta+p} x_{j_\beta} [(-1)^{\alpha+1} x_{i_\alpha} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q}) \\
&+ (-1)^{\beta} (-1)^{\alpha+1} x_{i_\alpha} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \ldots, x_{j_q})] = 0.
\end{aligned}$$

Subsubcase (iii)(c): if $\beta'' \geq \beta + 2$, $x_{i_\alpha} x_{j_{\beta''}} \notin G$ and $\{x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta''}})$, then the sum of the terms containing $x_{i_\alpha} x_{j_\beta}$ is

$$\begin{aligned}
&(-1)^{\alpha+1} x_{i_\alpha} (-1)^{\beta+(p-1)} x_{j_\beta} [(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q}) \\
&+ (-1)^{\beta} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \ldots, x_{j_q})] \\
&+ (-1)^{\beta+p} x_{j_\beta} \{(-1)^{\alpha+1} x_{i_\alpha} [(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q}) \\
&+ (-1)^{\beta''-1} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_{\beta''-1}} | x_{j_{\beta''}}, \ldots, x_{j_q})] \\
&+ (-1)^{\beta} (-1)^{\alpha+1} x_{i_\alpha} [(x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+1}}, \ldots, x_{j_q}) \\
&+ (-1)^{\beta''-\beta} (x_{i_1}, \ldots, \widehat{x_{i_\alpha}}, \ldots, x_{i_p}, x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_{\beta''-1}} | x_{j_{\beta''}}, \ldots, x_{j_q})]\} = 0.
\end{aligned}$$

[Case D]: $\lambda = j_1$ and $\lambda' \in \{j_2, \ldots, j_q\}$.

[Case D-a]: suppose that there is a term containing $x_{j_1} x_{j_t}$ for some $2 \leq t \leq q$ such that $t \neq \beta$, then $\beta \neq 2$ and if $t = 2$ then $\beta \neq 3$. Hence, the sum of the terms containing $x_{j_1}x_{j_t}$ is

$$(-1)^{1+p}x_{j_1}(-1)^{(t-1)+p}x_{j_t}(x_{i_1},\ldots,x_{i_p}|\widehat{x_{j_1}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})$$
$$+(-1)^{t+p}x_{j_t}(-1)^{1+p}x_{j_1}(x_{i_1},\ldots,x_{i_p}|\widehat{x_{j_1}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})=0.$$

[Case D-b]: suppose that there is a term containing $x_{j_1}x_{j_\beta}$.

Subcase (i): $\beta = 2$. Assume that $\{x_{i_1},\ldots,x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_3})$, then there is no term containing $x_{j_1}x_{j_\beta}$, hence we must have $\{x_{i_1},\ldots,x_{i_p}\} \subseteq \text{pnbhd}(x_{j_3})$ and the sum of the terms containing $x_{j_1}x_{j_\beta}$ is

$$(-1)^{\beta+p}x_{j_\beta}[(-1)^{1+p}x_{j_1}(x_{i_1},\ldots,x_{i_p}|\widehat{x_{j_1}},x_{j_3},\ldots,x_{j_q})$$
$$+(-1)^{\beta}(-1)^{p+2}x_{j_1}(x_{i_1},\ldots,x_{i_p},\widehat{x_{j_1}}|x_{j_3},\ldots,x_{j_q})]=0.$$

Subcase (ii): if $\beta > 2$ such that $\beta = q$ or $\{x_{i_1},\ldots,x_{i_p}\} \nsubseteq \text{pnbhd}(x_{j_{\beta+1}})$, then the sum of the terms containing $x_{j_1}x_{j_\beta}$ is

$$(-1)^{1+p}x_{j_1}(-1)^{(\beta-1)+p}x_{j_\beta}(x_{i_1},\ldots,x_{i_p}|x_{j_2},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})$$
$$+(-1)^{\beta+p}x_{j_\beta}(-1)^{1+p}x_{j_1}(x_{i_1},\ldots,x_{i_p}|x_{j_2},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})=0.$$

Subcase (iii): if $\beta > 2$ and $\{x_{i_1},\ldots,x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta+1}})$, then the sum of the terms containing $x_{j_1}x_{j_\beta}$ is

$$(-1)^{1+p}x_{j_1}(-1)^{(\beta-1)+p}x_{j_\beta}[(x_{i_1},\ldots,x_{i_p}|x_{j_2},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})$$
$$+(-1)^{\beta-1}(x_{i_1},\ldots,x_{i_p},x_{j_2},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]$$
$$+(-1)^{\beta+p}x_{j_\beta}[(-1)^{1+p}x_{j_1}(x_{i_1},\ldots,x_{i_p}|x_{j_2},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_q})$$
$$+(-1)^{\beta}(-1)^{p+2}x_{j_1}(x_{i_1},\ldots,x_{i_p},x_{j_2},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,x_{j_q})]=0.$$

[Case E]: $\lambda, \lambda' \in \{j_2,\ldots,j_q\}$.

[Case E-a]: if $2 \le t < t' \le q$ such that $t \neq \beta$ and $t' \neq \beta$, then the sum of the terms containing $x_{j_t} x_{j_{t'}}$ is

$$(-1)^{t+p} x_{j_t} (-1)^{(t'-1)+p} x_{j_{t'}} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_{t'}}}, \ldots, x_{j_q})$$
$$+ (-1)^{t'+p} x_{j_{t'}} (-1)^{t+p} x_{j_t} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_{t'}}}, \ldots, x_{j_q}) = 0.$$

[Case E-b]: suppose that there is a term containing $x_{j_t} x_{j_\beta}$ for some $2 \le t \le q$ with $t \neq \beta$. As in [Case C-c], we set

$$\beta'' = \min\{t \mid \beta < t \le q, \{x_{i_1}, \ldots, x_{i_p}\} \not\subseteq \mathrm{pnbhd}(x_{j_t})\}$$
$$= \min\{t \mid \beta < t \le q, \{x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}}\} \not\subseteq \mathrm{pnbhd}(x_{j_t})\}.$$

Subcase (i): if one of the following conditions is satisfied:

1) $\beta = q$,

2) $\beta = q - 1$ and $t = q$,

3) $\beta'' = \beta + 1$ and $t \neq \beta''$,

4) $\beta'' = \beta + 1$, $t = \beta''$ and $\{x_{i_1}, \ldots, x_{i_p}\} \not\subseteq \mathrm{pnbhd}(x_{j_{\beta+2}})$,

5) $\beta'' = \beta + 2$ and $t = \beta + 1$,

then the sum of the terms containing $x_{j_t} x_{j_\beta}$ is

$$(-1)^{t+p} x_{j_t} (-1)^{(\beta-1)+p} x_{j_\beta} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})$$
$$+ (-1)^{\beta+p} x_{j_\beta} (-1)^{t+p} x_{j_t} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q})$$
$$= 0, \quad \text{for } t < \beta;$$
$$(-1)^{t+p} x_{j_t} (-1)^{\beta+p} x_{j_\beta} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})$$
$$+ (-1)^{\beta+p} x_{j_\beta} (-1)^{(t-1)+p} x_{j_t} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \ldots, \widehat{x_{j_t}}, \ldots, x_{j_q})$$
$$= 0, \quad \text{for } t > \beta.$$

Subcase (ii): if $\beta'' = \beta + 1$, $t = \beta''$ and $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_{\beta+2}})$, then the sum of the terms containing $x_{j_t} x_{j_\beta}$ is

$$
\begin{aligned}
&(-1)^{t+p} x_{j_t} (-1)^{\beta+p} x_{j_\beta} [(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \widehat{x_{j_t}}, \ldots, x_{j_q}) \\
&+ (-1)^{\beta} (x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+2}}, \ldots, x_{j_q})] \\
&+ (-1)^{\beta+p} x_{j_\beta} (-1)^{(t-1)+p} x_{j_t} [(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_\beta}}, \widehat{x_{j_t}}, \ldots, x_{j_q}) \\
&+ (-1)^{\beta} (x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_{\beta-1}} | x_{j_{\beta+2}}, \ldots, x_{j_q})] = 0.
\end{aligned}
$$

Subcase (iii): if one of the following conditions is satisfied:

1) $\beta = q - 1$, $t < \beta$ and $\{x_{i_1}, \ldots, x_{i_p}\} \subseteq \text{pnbhd}(x_{j_q})$,
2) $\beta \leq q - 2$ and β'' does not exist,
3) $\beta'' > \beta + 1$, $t \neq \beta''$ such that $t \neq \beta + 1$ or $\beta'' \neq \beta + 2$,
4) $\beta'' > \beta + 1$ and $t = \beta'' = q$,
5) $\beta'' > \beta + 1$, $t = \beta''$ and $\{x_{i_1}, \ldots, x_{i_p}\} \not\subseteq \text{pnbhd}(x_{j_{\beta''+1}})$,

then the sum of the terms containing $x_{j_t} x_{j_\beta}$ is

$$
\begin{aligned}
&(-1)^{t+p} x_{j_t} (-1)^{(\beta-1)+p} x_{j_\beta} [(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q}) \\
&+ (-1)^{\beta-1} (x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_\beta}} | x_{j_{\beta+1}}, \ldots, x_{j_q})] \\
&+ (-1)^{\beta+p} x_{j_\beta} [(-1)^{t+p} x_{j_t} (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_\beta}}, \ldots, x_{j_q}) \\
&+ (-1)^{\beta} (-1)^{t+p+1} x_{j_t} (x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, \widehat{x_{j_t}}, \ldots, \widehat{x_{j_\beta}} | x_{j_{\beta+1}}, \ldots, x_{j_q})] \\
&= 0, \quad \text{for } t < \beta;
\end{aligned}
$$

$$(-1)^{t+p}x_{j_t}(-1)^{\beta+p}x_{j_\beta}[(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})$$
$$+(-1)^{\beta}(x_{i_1},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})]$$
$$+(-1)^{\beta+p}x_{j_\beta}[(-1)^{(t-1)+p}x_{j_t}(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})$$
$$+(-1)^{\beta}(-1)^{t-1+p}x_{j_t}(x_{i_1},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})]$$
$$=0,\quad \text{for } t>\beta.$$

Subcase (iv): if $\beta'' > \beta+1$, $t=\beta''$ and $\{x_{i_1},\ldots,x_{i_p}\}\subseteq \text{pnbhd}(x_{j_{\beta''+1}})$, then the sum of the terms containing $x_{j_t}x_{j_\beta}$ is

$$(-1)^{t+p}x_{j_t}(-1)^{\beta+p}x_{j_\beta}[(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})$$
$$+(-1)^{\beta}(x_{i_1},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})]$$
$$+(-1)^{\beta+p}x_{j_\beta}\{(-1)^{(t-1)+p}x_{j_t}[(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})$$
$$+(-1)^{t-1}(x_{i_1},\ldots,x_{i_p},x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_{t-1}}|x_{j_{t+1}},\ldots,x_{j_q})]$$
$$+(-1)^{\beta}(-1)^{t-1+p}x_{j_t}[(x_{i_1},\ldots,x_{i_p},x_{j_1},\ldots,x_{j_{\beta-1}}|x_{j_{\beta+1}},\ldots,\widehat{x_{j_t}},\ldots,x_{j_q})$$
$$(-1)^{t-\beta}(x_{i_1},\ldots,x_{i_p},x_{j_1},\ldots,\widehat{x_{j_\beta}},\ldots,x_{j_{t-1}}|x_{j_{t+1}},\ldots,x_{j_q})]\}=0.$$

Since the above five main cases have included all the possible terms, it follows that $d^2(x_{i_1},\ldots,x_{i_p}|x_{j_1},\ldots,x_{j_q})=0$ and we are done. □

5.5 Betti numbers

In Section 5.3, to construct the differential maps of the minimal free resolution of S/I_G, we need to assume that $x_n,\ldots,x_1$ is a perfect elimination order of $\overline{G}$ produced by Algorithm 5.2.2. However, to get a nice formula for Betti numbers (Corollary 5.5.2), we only need to know a basis for the minimal free resolution.

Therefore, we have the following theorem which does not require that the perfect elimination order $x_n, \ldots, x_1$ of $\overline{G}$ is produced by Algorithm 5.2.2.

Theorem 5.5.1. *Let G be a simple graph with vertices $x_1, \ldots, x_n$ such that $\overline{G}$ is chordal and $x_1, \ldots, x_n$ is in the reverse order of a perfect elimination order of $\overline{G}$. Then in the polynomial ring $S = k[x_1, \ldots, x_n]$ we have the linear edge ideal I_G of the graph G. Let the symbol $(x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q})$ be as defined in Construction 5.3.4. And we set*

$$\mathcal{B} = \{1\} \cup \bigcup_{p \geq 1, q \geq 1} \left\{ (x_{i_1}, \ldots, x_{i_p} | x_{j_1}, \ldots, x_{j_q}) : \begin{array}{c} 1 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_q \leq n \\ \{x_{i_1}, \ldots, x_{i_p}\} \subseteq pnbhd(x_{j_1}) \end{array} \right\}.$$

Then there exists a multigraded minimal free resolution $\mathbf{F}$ of S/I_G such that $\mathbf{F}$ has basis $\mathcal{B}$.

We will not prove Theorem 5.5.1 because the proof is very similar to the proof of Theorem 5.3.7. The only difference is that in the proof of Theorem 5.3.7 we know the complex maps $-\mu_2 : \mathbf{E} \to \mathbf{K}$ and $\mu : \mathbf{F}'' \to \mathbf{F}'$ explicitly, while in the proof of Theorem 5.5.1 we only know their existence. However, we can still use the mapping cones to show the existence of the multigraded minimal free resolution with the desired basis $\mathcal{B}$.

Now Theorem 5.5.1 imply immediately the following corollary about Betti numbers and the projective dimension of S/I_G.

Corollary 5.5.2. *Let I_G be a linear edge ideal as defined in Theorem 5.5.1. For $2 \leq i \leq n$, we set $\lambda_i = |pnbhd(x_i)|$. Then for $i \geq 1$, the Betti numbers of S/I_G are*

$$\beta_{i,j}(S/I_G) = \begin{cases} \displaystyle\sum_{l=2}^{n} \left(\sum_{p=1}^{\lambda_l} \binom{\lambda_l}{p} \binom{n-l}{i-p} \right), & \text{if } j = i+1, \\ 0, & \text{if } j \neq i+1, \end{cases}$$

and the projective dimension of S/I_G is

$$projdim(S/I_G) = n - \min\{i - \lambda_i : 2 \leq i \leq n \text{ and } \lambda_i \neq 0\} \leq n-1.$$

Proof. The formula for Betti numbers follows from counting the number of basis elements of homological degree i and degree $i+1$ in $\mathcal{B}$. The projective dimension formula also follows easily by looking at the basis elements in $\mathcal{B}$. Since $\lambda_i \leq i-1$ for $2 \leq i \leq n$, it follows that $\operatorname{projdim}(S/I_G) \leq n-1$. □

Example 5.5.3. Let G be the graph such that $\overline{G}$ is the chordal graph given in Example 5.2.6. Then $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ is in the reverse order of a perfect elimination order of $\overline{G}$ and we have that

$$\lambda_2 = 0,\ \lambda_3 = 1,\ \lambda_4 = 2,\ \lambda_5 = 3,\ \lambda_6 = 4,\ \lambda_7 = 5.$$

Therefore, by Corollary 5.5.2, we have $\operatorname{projdim}(S/I_G) = 5$ and a computation will reveal that the Betti numbers of S/I_G are

$$\beta_{1,2} = 15,\ \beta_{2,3} = 40,\ \beta_{3,4} = 45,\ \beta_{4,5} = 24,\ \beta_{5,6} = 5.$$

In [RV] and [HV], the following formula for the Betti numbers is proved by using Hochster's formula. Now we prove the formula by using Theorem 5.5.1.

Corollary 5.5.4. *Let I_G be the linear edge ideal of a graph G with vertices $x_1, \ldots, x_n$. For any nonempty subset σ of $\{x_1, \ldots, x_n\}$, let $\overline{G}_\sigma$ be the subgraph of $\overline{G}$ induced by σ and let $\#(\overline{G}_\sigma)$ be the number of connected components of $\overline{G}_\sigma$. Then for $i \geq 1$, we have*

$$\beta_{i,j}(S/I_G) = \begin{cases} \displaystyle\sum_{\sigma \subseteq \{x_1,\ldots,x_n\}, |\sigma| = i+1} \left(\#(\overline{G}_\sigma) - 1\right), & \text{if } j = i+1, \\ 0, & \text{if } j \neq i+1. \end{cases}$$

Proof. Without the loss of generality, we can assume that $x_n, \ldots, x_1$ is a perfect elimination order of the chordal graph $\overline{G}$. Let $\mathcal{B}$ be as defined in Theorem 5.5.1. We say that the vertex x_s is smaller than the vertex x_t if $s < t$. For any $i \geq 1$, let $\sigma = \{x_{\alpha_1}, \ldots, x_{\alpha_{i+1}}\}$ be a subset of $\{x_1, \ldots, x_n\}$ for some $1 \leq \alpha_1 < \cdots < \alpha_{i+1} \leq n$.

We claim that $(x_{\alpha_1}, \dots, x_{\alpha_{p-1}} | x_{\alpha_p}, \dots, x_{\alpha_{i+1}}) \in \mathcal{B}$ if and only if $p \neq 1$ and x_{α_p} is the smallest vertex in the connected component of $\overline{G}_\sigma$ containing x_{α_p}. Indeed, if $p \geq 2$ and x_{α_p} is the smallest vertex in the connected component of $\overline{G}_\sigma$ containing x_{α_p}, then $x_{\alpha_s} x_{\alpha_p} \in G$ for all $1 \leq s \leq p-1$, so that $(x_{\alpha_1}, \dots, x_{\alpha_{p-1}} | x_{\alpha_p}, \dots, x_{\alpha_{i+1}}) \in \mathcal{B}$. On the other hand, assume that $p \geq 2$ and there exists $1 \leq s \leq p-1$ such that x_{α_s} and x_{α_p} are in the same connected component of $\overline{G}_\sigma$. Set $\sigma' = \{x_{\alpha_1}, \dots, x_{\alpha_p}\} \subseteq \sigma$. Since $x_{\alpha_{i+1}}, \dots, x_{\alpha_1}$ is a perfect elimination order of $\overline{G}_\sigma$, it is easy to see that x_{α_s} and x_{α_p} are still in the same connected component of $\overline{G}_{\sigma'}$. Therefore, there exists $1 \leq s' \leq p-1$ such that $x_{\alpha_{s'}} x_{\alpha_p} \in \overline{G}_{\sigma'}$, and hence $x_{\alpha_{s'}} x_{\alpha_p} \notin G$, which implies $(x_{\alpha_1}, \dots, x_{\alpha_{p-1}} | x_{\alpha_p}, \dots, x_{\alpha_{i+1}}) \notin \mathcal{B}$. So the claim is proved. It follows that there are $\#(\overline{G}_\sigma) - 1$ basis elements in $\mathcal{B}$ with multidegree $x_{\alpha_1} \cdots x_{\alpha_{i+1}}$ and we are done. □

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