REFERENCE-DEPENDENT AMBIGUITY AVERSION

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This dissertation contributes to the growing literature in economics on ambiguity aversion. I identify an implicit reference-point assumption in the multiple priors model of Gilboa and Schmeidler (1989), generalize their decision theory to allow for stochastic reference-points, and study the market implications of endowment-dependent ambiguity aversion. Chapter 2 identifies the implicit reference-point assumption in the multiple priors model and provides an axiomatic characterization of a reference-dependent multiple priors model. I also provide an axiomatic characterization of a reference-dependent version of the Choquet Expected Utility model of Schmeidler (1989), which can accommodate different attitudes towards ambiguity. Chapter 3 studies the implications of reference-dependent ambiguity aversion when reference points are given by the endowment in an Arrow-Debreu exchange economy. I illustrate that no-trade and underinsurance are robust implications of ambiguity aversion when investors view ambiguity from the perspective of their endowments. Chapter 4 extends the decision model to intertemporal choice problems and studies the effects of reference-dependent ambiguity aversion in the context of a dynamic asset pricing model.

BIOGRAPHICAL SKETCH

Maximilian Mihm was born on October 1st, 1980, in Friedberg, Germany. He completed high school at Forres Academy in Scotland and then studied for 2 years at Heriot-Watt University in Edinburgh, Scotland. Maximilian moved to Heidelberg, Germany, in 2000 and was awarded a Diplom in Economics from the Ruprecht-Karls University in Heidelberg in 2004. He began his graduate studies in the PhD program at Cornell University in Fall 2005. Prior to the completion of his PhD, Maximilian received a Master of Arts degree in Economics from Cornell University in Fall 2009.

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TABLE OF CONTENTS

Biographical Sketch . . . iii
Dedication . . . iv
Acknowledgements . . . v
Table of Contents . . . vi
List of Figures . . . viii

1 Introduction 1
1.1 Related Literature . . . 5
1.2 Organization of the Dissertation . . . 10

2 Reference-Dependent Ambiguity Aversion 12
2.1 Introduction . . . 12
2.1.1 An insurance example . . . 15
2.1.2 An Ellsberg example . . . 19
2.2 A Model of Reference-Dependent Maxmin Expected Utility . . . 23
2.2.1 Decision environment . . . 23
2.2.2 Preferences . . . 23
2.2.3 Axioms . . . 24
2.2.4 Representation . . . 34
2.3 A Model of Reference-Dependent Choquet Expected Utility . . . 47
2.4 Unbounded Reference-Dependent Maxmin Expected Utility . . . 52
2.5 Conclusion . . . 54

3 Risk and Ambiguity Aversion in Markets 56
3.1 Introduction . . . 56
3.1.1 Examples from a simple investor choice problem . . . 60
3.1.2 Outline of the Chapter . . . 62
3.2 The Economy . . . 63
3.3 Market Collapse . . . 66
3.3.1 An example of market collapse with risk and ambiguity . . . 72
3.3.2 An example of non-participation . . . 76
3.4 Underinsurance . . . 78
3.4.1 An example of underinsurance with weak agreement and two states of the world . . . 82
3.4.2 An example of underinsurance with strong agreement and multiple states of the world . . . 84
3.4.3 Underinsurance with aggregate uncertainty . . . 86
3.5 Conclusion . . . 88

4 Recursive Reference-Dependent Ambiguity Aversion **90**
4.1 Introduction . . . 90
4.2 A Dynamic Decision Problem . . . 94
4.2.1 Axioms . . . 95
4.2.2 Representation . . . 100
4.3 An Intertemporal Asset Pricing Model . . . 103
4.3.1 The economy . . . 104
4.3.2 Equilibrium analysis . . . 107
4.4 Conclusion . . . 111

A Proofs for the results in the text **113**
A.1 Proofs for results in Chapter 2 . . . 113
A.1.1 Mathematical notation . . . 113
A.1.2 Proofs . . . 114
A.2 Proofs of results in Chapter 3 . . . 135
A.2.1 Superdifferential . . . 135
A.2.2 Characterization of a competitive equilibrium . . . 136
A.2.3 Proofs . . . 137
A.3 Proofs of results in Chapter 4 . . . 142

Bibliography **145**

LIST OF FIGURES

2.1 Risk premium in models of decision making under uncertainty 17
2.1.1 SEU 17
2.1.2 MEU 17
2.1.3 RMEU 17
2.2 Comparative statics with Reference-Dependent Maxmin Expected Utility 19
2.2.1 *Ex-post* RMEU 19
2.2.2 *Ex-ante* RMEU 19
2.3 The set of crisp acts with two states of the world 40

3.1 Motives for trade under SEU and RMEU 60
3.1.1 Speculation 60
3.1.2 Insurance 60
3.1.3 Hedging 60
3.2 No-trade and market collapse in an Edgeworth box economy 71
3.2.1 No-trade 71
3.2.2 Market collapse 71
3.3 Equilibrium insurance in an Edgeworth box economy 84
3.3.1 Without ambiguity 84
3.3.2 With ambiguity 84

CHAPTER 1

INTRODUCTION

There is a fundamental difference between the uncertainty involved in the roll of a fair die and the uncertainty involved in predicting the performance of a company stock over time. The roll of a die can be described with some confidence in terms of a probability law, while the uncertainty regarding stock performance is subjective and difficult to quantify. It is therefore common to distinguish between the objective uncertainty or *risk* involved in games of chance and the subjective uncertainty or *ambiguity* that is more pervasive in economic decision problems. But how do decision makers respond to ambiguity? In the classic subjective expected utility (SEU) theory of Savage (1954) and Anscombe and Aumann (1963), decision makers act *as if* the problem of predicting company stock performances is essentially analogous to betting on dice: They assign a unique subjective probability distribution (often called a prior) over events and maximize expected utility with respect to this unique prior regardless of the source of the uncertainty. As a theory of decision making under uncertainty, SEU is elegant and has found many applications in economics, finance and other social sciences. But a line of research following Knight (1921) has argued that the distinction between risk and ambiguity – ignored in SEU theory – has behavioral implications that are important for understanding choice behavior under uncertainty.

A powerful illustration of the Knightian perspective is provided by the well-known Ellsberg (1961) paradox. The Ellsberg paradox is based on the following thought experiment. Suppose a decision maker is confronted with two urns: Urn A contains 50 black and 50 white balls, and urn B contains 100 black and white balls but in an unspecified proportion. A ball will be drawn at random from an urn and the decision maker's payoffs are determined by the color that is drawn. The Ellsberg (1961) paradox is the finding that most individuals are indifferent about which should be the high payoff color in each urn, but strictly prefer a bet involving urn A over the same bet involving urn B. Such choices are inconsistent with the SEU hypothesis because they indicate a behaviorally meaningful distinction between risk (faced in bets on urn A) and ambiguity (faced in bets on urn B).

The Ellsberg paradox has motivated a large literature that studies decision making under ambiguity. A seminal contribution to the decision theoretic literature is the multiple priors or Maxmin Expected Utility (MEU) model of Gilboa and Schmeidler (1989). Gilboa and Schmeidler propose a decision model in which the unique prior of SEU is replaced with a set of priors (hence multiple priors), and the decision maker evaluates an uncertain alternative according to its expected utility under the worst-case prior (hence MEU). The set of priors reflects the perception of ambiguity and the minimization over priors reflects an aversion to ambiguity. The MEU model is the best-known model of ambiguity aversion in the literature. The axiomatic characterization provided by Gilboa and Schmeidler is simple and normatively appealing, the MEU decision rule is consistent with the Ellsberg paradox, and the model has been fruitfully applied to study the effects of ambiguity aversion in finance, macroeconomics, labor economics and political economy.

This dissertation presents a generalization of MEU. There is a sense in which MEU implicitly assumes that ambiguity is perceived from the perspective of a particular constant act (an alternative with payoffs that are not state-contingent). This implicit reference-point assumption is not readily transparent from the axioms in Gilboa and Schmeidler (1989) but becomes clear once these axioms are more carefully decomposed. While the other axioms that characterize the MEU decision rule are normatively appealing, the constant reference-point assumption is somewhat arbitrary. Moreover, experimental evidence extending on the Ellsberg paradox has re-affirmed a behaviorally meaningful distinction between risk and ambiguity aversion but explicitly casts doubt on the assumption that ambiguity is always viewed from the perspective of constant acts (Heath and Tversky, 1991; Fox and Tversky, 1995; Roca et al., 2006). In Chapter 2, I therefore model decision makers who view ambiguity relative to an arbitrary reference-point – not necessarily a constant act – and provide a representation theorem for this extension of MEU. I refer to this as a Reference-Dependent Maxmin Expected Utility (RMEU) decision model.

The influence of reference-points on decision making is well-documented in settings with and without uncertainty, but the RMEU model identifies a novel channel for reference-point effects. In contrast to cumulative prospect theory (Tversky and Kahneman, 1992), reference-effects in the RMEU model are *ex-ante* to the realization of uncertainty and are directly related to the source of uncertainty and the ambiguity aversion of the decision maker. As a result, the reference-effects in the RMEU model imply a trade-off between an insurance motive for trade (due to risk aversion) and a hedging motive for trade (due to ambiguity aversion). In existing models of ambiguity averse preferences, ambiguity and risk are both perceived from the perspective of constant acts and so aversion to ambiguity and aversion to risk both imply a motive to trade towards full insurance. However, when ambiguity is viewed relative to a non-constant reference-point – such as a status-quo, a particular contract, or a social convention – risk aversion induces a motive to trade towards full insurance, while ambiguity aversion implies a motive to trade towards the reference-point. RMEU therefore identifies a natural trade-off between hedging and insurance which is not captured by existing models of reference-point effects (such as cumulative prospect theory) or by existing models of ambiguity aversion (such as MEU).

In Chapter 3 of the Dissertation, I illustrate that the RMEU model can change substantially our view of how ambiguity affects markets. In particular, the trade-off between hedging and insurance motives for trade implied by RMEU can explain the partial participation, under-insurance and under-diversification of many potential investors in financial markets (see, e.g., Campbell, 2006). MEU has been used to model these features of modern asset markets, but it is still a matter of debate how general the explanations provided by MEU are (see, e.g., Rigotti and Shannon, 2008, for a general critique). For example, the first result on non-participation by MEU decision makers, due to Dow and Werlang (1992), hinges crucially on the assumption that the investor begins with a portfolio that is risk-free. Other applications of MEU, including Epstein and Wang (1994) and Billot et al. (2000), are similarly restricted by conditions on the status-quo allocation. It would be easy to conclude that ambiguity affects market allocations in a qualitative way only under knife-edge conditions (i.e. non-generically), but the RMEU model illustrates that this conclusion is misplaced. While models of financial markets with MEU investors must generally line up initial conditions with the specific constant reference-point embedded in MEU theory, RMEU illustrates that ambiguity can affect market outcomes for arbitrary initial allocations if investors use these allocations as reference-points.

The final chapter of the dissertation extends the static analysis of Chapters 2 and 3 to study reference-dependent ambiguity aversion in an intertemporal setting. One substantial advantage of SEU is that it comes equipped with an essentially in-built theory of dynamic behavior over time. This is not the case for models of ambiguity aversion (see, e.g., Epstein and LeBreton, 1993; Siniscalchi, 2009a). However, Epstein and Schneider (2003, 2007) have recently proposed extensions of MEU to intertemporal settings that are both tractable and based on solid axiomatic foundations. I build on their work and give an axiomatic characterization of a Recursive Reference-Dependent Maxmin

Expected Utility (RRMEU) model, which is dynamically consistent (in an acceptable sense) and embodies a natural theory of learning (prior-by-prior Bayesian updating). As an application, I extend the intertemporal asset pricing model of Epstein and Wang (1994) and illustrate robust price indeterminacy *for all* endowment processes. Epstein and Wang (1994) demonstrate the possibility of price indeterminacy with recursive multiple priors utility and relate this price indeterminacy to the excess price volatility puzzle. However, using an intertemporal extension of MEU, they are not able to provide a definitive answer regarding the "frequency" of such price indeterminacy, precisely because under MEU this depends on the initial conditions (i.e., the stochastic endowment process). As in the application of the static decision theory in Chapter 3, I illustrate that with RRMEU preferences, price indeterminacy is robust (and, in fact, ubiquitous).

1.1 Related Literature

The decision-theoretic literature on ambiguity is large and active (see, for example, Siniscalchi, 2008, and the references therein for an overview). My analysis builds on a number of contributions in the literature and in this section I discuss some of these connections. Further references are given in the body of the text.

Ghirardato et al. (2004) study a decision maker (DM) whose preferences satisfy the axioms in Gilboa and Schmeidler (1989) except (possibly) uncertainty aversion, and thereby attribute a perception of (subjective) ambiguity to a DM without attributing a particular attitude towards ambiguity. I perform essentially the opposite exercise. The RMEU model satisfies uncertainty aversion, and generalizes certainty independence to allow for the perception of ambiguity to be relative to a reference-point. Hence, a particular attitude towards ambiguity (namely, aversion) is fixed, but the DM regards ambiguity from the perspective of a reference-point which need not be constant.

The separation of ambiguity and ambiguity attitude in Ghirardato et al. (2004) is a key step also in the development of RMEU and is based on a model of ambiguity by Bewley (2002). Bewley (2002) models ambiguity through incomplete preferences in the Anscombe and Aumann (1963) framework, and gives a unanimity representation: The DM acts *as if* (s)he entertains a set of priors Π and has a von Neumann/Morgenstern payoff function u, and prefers f to g if and only if the expected utility of f is greater than the expected utility of g for all priors in Π. When there is ambiguity (Π is not a singleton) some alternatives are non-comparable. The model of preferences therefore captures ambiguity in an intuitively appealing way, but does not embody any particular ambiguity attitude. Bewley (2002) therefore introduces an inertia assumption to model aversion to ambiguity: A DM chooses an alternative f only if the expected utility of f is greater than the expected utility of the status-quo for all priors in Π. Under the inertia assumption ambiguity-aversion depends on a (potentially) non-constant status quo, and this naturally induces a status-quo bias. The same status-quo bias follows from RMEU preferences when the reference-point is given by the status-quo. However, the interpretation of the status-quo bias in Bewley (2002) is conceptually very different. In Bewley (2002) inertia is an assumption about how decision makers respond to ambiguity but is not related directly to the preferences of the individual. The RMEU model is a complete preference model in which status-quo bias emanates from a particular perspective on ambiguity (namely, status-quo-dependent-ambiguity) coupled with a particular response to ambiguity (namely, ambiguity aversion). Moreover, an important practical difference is that the choices made by the DM in Bewley's model remain indeterminate when there are alternatives in a choice set that dominate the status quo. In the RMEU model demand is always single valued (at least when the DM is also risk averse), and the RMEU model can therefore be used to model status-quo bias in a model with greater predictive power.

Ortoleva (2010) presents an alternative approach that can also provide revealed preference foundations for the inertia assumption in Bewley (2002).[1] Ortoleva (2010) models a DM who has a context-free preference relation, as well as a preference relation given any specific status-quo. Axioms describe preferences in the absence of a status quo, preferences given any particular status-quo, and relate preferences given different status-quibus to each other and to the status-quo-free preference. As a result, the axioms can be verified only by observing the choices of a DM given all possible status-quibus. In the RMEU model, the axioms describing behavior given a particular reference-point can be verified by observing the choices of the DM given that particular reference-point alone, and an axiom relating preferences given different reference-points is introduced only to allow for meaningful comparative statics. The revealed preference foundations in Ortoleva (2010) and in the RMEU model are therefore quite distinct. Moreover, the model of reference-dependent ambiguity aversion does not contain any specific assumption about what alternative will serve as a reference-point for a DM in a given context. Status-quo bias can be modeled if the RMEU model is augmented with the assumption that the reference-point is the status-quo. In this case, there is nevertheless a substantive difference between the model in Ortoleva (2010) and the RMEU model in terms of the effect that the status-quo has on choices. Ortoleva (2010) explicitly limits the status-quo effect to a desire to hold on to the status-quo: The preference between two alternatives is determined by a status-quo free preference relation, but alternatives are chosen from a choice set only if they are preferred to the status-quo according to a status-quo-dependent preference relation. However, a DM with reference-dependent ambiguity averse preferences (and a reference-point given by the status quo) also acts *as if* all available alternatives are related to the status-quo. The RMEU model therefore captures a reference-effect that is explicitly ruled out in Ortoleva (2010). As a result, preferences

[1] The approach in Ortoleva (2010) is based on the model of reference-dependent preferences studied in Masatlioglu and Ok (2005).

given a reference-point are continuous in the RMEU model, while the preferences given a status-quo are not continuous in the decision model studied in Ortoleva (2010). Also, the reference-effect in the RMEU model implies that preferences generalize MEU, and the comparison to existing models of ambiguity aversion is therefore more transparent. The relation to MEU is also convenient because it allows for an extension of the RMEU model to dynamic choice settings based on approaches to dynamic models of ambiguity aversion in the existing literature.[2] It is not clear how one would extend the model in Ortoleva (2010) to study dynamic choice problems.

Sagi (2006) also provides a representation in a similar spirit to the RMEU representation, but restricted to the context of choice under risk. Sagi (2006) provides an axiomatic characterization of reference-dependent preferences over lotteries having the following representation: For any two lotteries, p and q, and for any reference-lottery, e, $p \succeq_e q$ if and only if $\inf_{\chi \in Y} E[\chi(p) - \chi(e)] \geq \inf_{\chi \in Y} E[\chi(q) - \chi(e)]$, where Y is a (suitably restricted) set of payoff functions on prizes, and $E[\chi(z)]$ is the expected utility of z given payoff function χ (see Theorem 6 in Sagi, 2006). Sagi (2006) refers to preferences with this representation as *anchored preferences*. For anchored preferences, the reference-point effects the minimization over a set of payoff functions in a similar manner as the reference-point effects the minimization over a set of priors in the RMEU representation, although the axiomatic characterization of anchored preferences in Sagi (2006) is closer to the approach in Ortoleva (2010).[3]

[2]I illustrate one approach to modeling RMEU in dynamic choice settings in Chapter 4, where I give an axiomatic characterization of recursive reference-dependent ambiguity aversion preferences that generalize the recursive multiple-priors preferences in Epstein and Schneider (2003).

[3]Sagi (2006) also provides a detailed and relevant discussion of how anchored preferences relate to cumulative prospect theory (Tversky and Kahneman, 1992). In particular, Sagi (2006) proposes a normative criterion for models of reference-dependent choice (under risk), which requires that for any lotteries p and q, $p \succeq_q q$ if and only if $p \succeq_e q$ for all reference-points e (Axiom 1 in Sagi (2006)). This condition is necessary and sufficient to rule out cyclic behavior that Sagi (2006) argues is difficult to interpret normatively (the reader is referred to Sagi, 2006, for a more detailed discussion of the normative appeal of this axiom). Under mild conditions, prospect theory, cumulative prospect theory and other commonly used models of reference-dependent behavior violate this axiom, but it is satisfied by anchored preferences. An appropriate generalization of Axiom 1 in Sagi (2006) to the Anscombe and Aumann (1963) setting is

Finally, there is also a literature related to the equilibrium analysis of Chapter 3. In particular, Rigotti and Shannon (2005) present a theory of exchange for an economy in which consumers have the preferences introduced in Bewley (2002). They illustrate the possibility of no-trade and indeterminacy of the equilibrium price as a result of the inertia assumption in Bewley (2002), and relate their results to the possibility of endogenous market incompleteness. Easley and O'Hara (2010) also study market collapse using Bewley's decision model, and relate their analysis to the financial crisis of 2008. However, a theory of exchange based on the preference in Bewley (2002) is conceptually and substantively different from the theory of exchange based on RMEU preferences in Chapter 3. Intuitively, one can view market collapse under RMEU as a missing markets problem. The missing commodity that consumers with reference-dependent ambiguity aversion preferences would like to purchase is information regarding the likelihood of states, because ambiguity averse DM's *prefer* risk regarding the realization of states to the ambiguity they currently face. The same perspective does not follow from the decision model in Bewley (2002) because in Bewley's decision model there is no innate attitude towards sources of uncertainty. DM's have incomplete preferences, and there is no sense in which they prefer objective lotteries to subjective uncertainty in the economy. The inertia assumption introduced by Bewley (2002) is simply an assumption about how consumers act when alternatives are not comparable to the status-quo. It is therefore not exactly clear what the market collapse as a result of inertia means. The analysis under reference-dependent ambiguity aversion therefore allows for a more transparent interpretation in which market collapse is related to a particular attitude towards ambiguity, namely ambiguity aversion.

The indeterminacy of equilibrium due to incomplete preferences is also distinct from the indeterminacy of equilibrium due to reference-dependent ambiguity aversion. With

also satisfied by RMEU preferences, and in the axiomatic characterization in Section 2.2, I identify the axiom that guarantees this.

the decision model presented in Bewley (2002) there are two sources of indeterminacy: (1) indeterminacy due to status-quo bias is similar to the indeterminacy of the equilibrium price identified in Chapter 3 of this Dissertation; and (2) indeterminacy when consumers trade away from the status-quo and the decision model therefore does not provide a complete theory of how they choose amongst incomparable alternatives. For a given endowment and risk preferences, more ambiguity leads to indeterminacy due to a collapse of the market (as with RMEU). However, when there is less ambiguity the equilibrium price and the equilibrium allocation are indeterminate because of the incompleteness of the preferences. It is not clear what the economic interpretation for this latter indeterminacy is. Consider, for example, an economy with no aggregate uncertainty but an asymmetric distribution of endowments. With sufficient ambiguity, DM with incomplete preferences and inertia do not trade and the competitive equilibrium allocation is therefore unique and not a full insurance allocation (compare Figure 3.2.1). Now consider the comparative static exercise of decreasing the ambiguity perceived by consumers. Under general conditions, there will eventually be trade, and the set of equilibrium allocation will generally be uncountable. In particular, it will include both full-insurance and non-full-insurance allocations. It is therefore difficult to determine whether there is still a trade off between hedging and insurance.

1.2 Organization of the Dissertation

The dissertation is organized as follows. Chapter 2 presents the static decision theory. I decompose the axioms that characterize MEU in Gilboa and Schmeidler (1989) to identify explicitly the reference-point assumption in the MEU model, and provide a representation result for the generalization to arbitrary reference-point (the RMEU model). I also provide a representation result for reference-dependent Choquet Expected Utility (CEU), which generalizes the CEU model in Schmeidler (1989) to accommodate reference-dependent preferences with different attitudes to ambiguity (from ambiguity aversion to ambiguity loving behavior). Chapter 3 analyzes risk sharing under RMEU utility in an Arrow-Debreu exchange economy in which investors view ambiguity from the perspectives of their initial endowment. I illustrate robust market collapse, non-participation and underinsurance, and relate these to a trade-off between an insurance motive to trade (due to risk aversion) and hedging motive to trade (due to ambiguity aversion). Chapter 4 presents an axiomatic characterization of Recursive Reference-Dependent Maxmin Expected Utility and generalizes the price indeterminacy result of Epstein and Schneider (2003) in an intertemporal asset pricing model. All proofs are presented separately in an Appendix.

CHAPTER 2

REFERENCE-DEPENDENT AMBIGUITY AVERSION

2.1 Introduction

Following the seminal work of Frank Knight (1921), and in response to the Ellsberg (1961) paradox, a growing literature in economics has studied decision making under uncertainty when there is a behaviorally meaningful distinction between risk (uncertainty that is easily quantified in terms of a probability law) and ambiguity (uncertainty that is not easily quantified). In this Chapter, I introduce an axiomatic model of ambiguity averse preferences in the Knightian tradition, and illustrate some of its key implications in a number of simple consumer choice problems. As in other decision models that distinguish between ambiguity and risk, the perception of and response to ambiguity are subjective and characterized behaviorally by weakening the independence axiom that relates subjective uncertainty to objective randomization in the SEU theory of Anscombe and Aumann (1963). In departure from existing models of ambiguity aversion, I study decision makers who view ambiguity relative to a reference-point – some alternative in the choice space that is particularly focal or familiar to them – and model ambiguity aversion as a preference for hedging against this reference-dependent ambiguity.

Formally, the reference-dependent ambiguity aversion decision model generalizes the Maxmin Expected Utility (MEU) model of Gilboa and Schmeidler (1989) by incorporating into the multiple-priors framework the well-documented influence of reference-points. In the classic setting of Anscombe and Aumann (1963), I provide axioms on preferences that characterize a decision maker who acts *as if* an alternative f, mapping states of the world $s \in S$ into lotteries over prizes $f(s) \in \Delta(X)$, is evaluated according to a utility functional of the form

$$V(f) = \min_{\pi \in \Pi} \int_S [u(f(s)) - u(r(s))] \, \pi(ds) \,, \tag{2.1}$$

where X is an outcome space, $\Delta(X)$ are lotteries with finite support on X, $u : X \to \mathbb{R}$ is a mixture-linear von Neumann/Morgenstern payoff function on lotteries over outcomes, Π is a weak*-closed, convex set of priors, and $r : S \to \Delta(X)$ is a reference point that describes the perspective from which the decision maker (DM) views ambiguity. Due to the close relation to MEU, I refer to the decision model of (2.1) as a Reference-Dependent Maxmin Expected Utility (RMEU) decision model.[1] The existence of multiple-priors, expressed by the non-singleton set Π, reflects ambiguity about the likelihood of states and captures the idea that the decision maker is not confident in one particular probability law. Risk aversion is a property of the von Neumann/Morgenstern payoff function u, while the minimization over a set of priors implies a distinct role for ambiguity aversion. Such a distinction has no behavioral implications exactly when Π is a singleton, and in this case the RMEU model is equivalent to the Subjective Expected Utility (SEU) model of Anscombe and Aumann (1963). However, when Π is not a singleton, choice behavior under ambiguity differs from behavior under risk. In this case, RMEU preferences coincide with MEU preferences if the reference-point r is constant. But when Π is not a singleton and ambiguity is viewed from a non-constant reference-point, the RMEU model is behaviorally distinct from both MEU and SEU.

The behavioral distinction between RMEU and MEU preferences has substantive implications for exchange under uncertainty. Chapter 3 demonstrates implications for

[1]Note, however, that the name may be misleading. As I argue in detail in the remainder of this Chapter, MEU does *not* represent a reference-free benchmark of the RMEU decision rule. Rather, MEU is the special case of RMEU preferences in which the reference-point is not state-contingent. When there are multiple priors, preferences are by their nature reference-dependent and so the reference-free benchmark should be viewed as SEU which is the special case of RMEU preferences without ambiguity aversion (i.e., there is no reference-effect exactly when there is no ambiguity aversion).

markets in more detail, but some observations by way of motivation are useful at this stage. Under the SEU hypothesis, exchange under uncertainty is determined by a trade-off between a speculative motive for exchange (due to differences in subjective beliefs) and an insurance motive for exchange (due to risk aversion). While ambiguity aversion in the MEU model reinforces the insurance motive for trade, RMEU more generally implies a trade-off between ambiguity- and risk-aversion: Ambiguity aversion implies a motive to hedge against ambiguity and participate in trades that are "more easily compared" to the reference-point, while risk aversion implies a motive to trade for insurance. MEU is the special case where these two motives generally coincide. However, when reference-points are not constant across states, reference-dependent ambiguity aversion implies a trade-off between the insurance and hedging motives that is not captured by existing models of ambiguity averse preferences.

The potential trade-off between an insurance and hedging motive for trade is also not captured in reference-dependent decision models in the spirit of cumulative prospect theory (Tversky and Kahneman, 1992) in which reference-points affect payoffs *ex-post* to the realization of uncertainty. In a typical application of cumulative prospect theory, the reference-point may be some wealth level (such as current wealth) and outcomes are viewed as deviations from this wealth level. The reference-point therefore does not, in general, have the same structure as the objects of choice (which are mappings from a state space to an outcome space), but it can affect behavior if gains and losses with respect to the reference-point are evaluated with different *ex-post* utility functions or different belief measures (Wakker and Tversky, 1993). In the RMEU model the reference-point is an Anscombe and Aumann (1963) act with the same structure as the objects of choice, and the reference-effect is *ex-ante* to the realization of uncertainty. As a result, the model separates ambiguity aversion from risk aversion: Risk aversion is a property of *ex-post* utility on outcomes, while the reference-effect is *ex-ante* and related to the way in which the DM perceives and therefore responds to ambiguity. The following example helps to illustrate the differences between a model of reference-dependent ambiguity aversion, MEU or models of reference-dependent preferences in the spirit of cumulative prospect theory.

2.1.1 An insurance example

Consider a simple insurance problem. A risk averse decision maker (DM) with wealth w and utility of wealth $u(.)$ can purchase insurance against some loss L. What premium, P, would the DM be willing to pay for full insurance against the loss? Suppose first that the DM maximizes SEU with prior probability $\hat{\pi}$ on the state where no loss occurs (state 1). At the solution P^*, illustrated in Figure 2.1.1, $w - P^*$ is the certainty equivalent of the act $(w, w-L)$ given prior $\hat{\pi}$ and the payoff function u on *ex-post* wealth. Now suppose that the DM is not confident in the prior $\hat{\pi}$ and instead entertains the possibility that the probability of state 1 is between $\hat{\pi} - \epsilon$ and $\hat{\pi} + \epsilon$. Given any premium $0 \leq P \leq L$, a MEU DM will choose to insure if

$$\min_{\pi \in [\hat{\pi}-\epsilon, \hat{\pi}+\epsilon]} \{\pi u(w) + (1-\pi) u(w-L)\} \leq u(w-P) \,. \tag{2.2}$$

As a result, it is the prior under which loss is most likely to occur, $\hat{\pi}-\epsilon$, that is relevant for the DM in deciding whether to insure. Figure 2.1.2 illustrates the certainty equivalent of $(w, w-L)$ for the MEU DM. The diagram depicts two indifference curves of a SEU DM with priors $\hat{\pi} - \epsilon$ and $\hat{\pi} + \epsilon$, respectively, that intersect on the 45° line. The indifference curve of a MEU DM corresponding to the utility of $(w, w-L)$ is given by the upper envelope of the indifference curves of the SEU DMs. Ambiguity aversion therefore implies that the DM is willing to pay a higher premium, $P' > P^*$, for full insurance. We can think of this in the following way. Viewed from the perspective of the alternative

$(w-P', w-P')$, the DM is ambiguous about alternatives (w_1, w_2) for which $\pi u(w_1) + (1-\pi)u(w_2) = \pi u(w-P') + (1-\pi)u(w-P') = u(w-P')$ for some $\pi \in [\hat{\pi}-\epsilon, \hat{\pi}+\epsilon]$, because the DM is not confident about the precise probability of state 1. Being ambiguity averse, the DM prefers the full insurance outcome $(w-P', w-P')$ to each of the alternatives in the set of ambiguous alternatives. Since $(\hat{\pi}-\epsilon)u(w) + (\hat{\pi}+\epsilon)u(w-L) = u(w-P')$, the DM will purchase full insurance at P'. In some sense, this is a reference-point argument. Faced with ambiguity, the DM views ambiguity from the perspective of "purchase full insurance" and prefers this alternative to any alternative that is not unambiguously better (i.e., dominates the purchase of insurance for all priors in $[\hat{\pi}-\epsilon, \hat{\pi}+\epsilon]$).

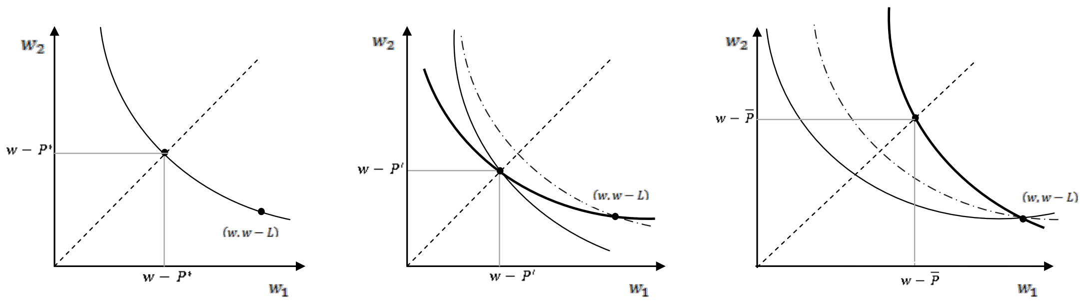
The RMEU model allows for a simple change in perspective. In many insurance problems not purchasing a policy is the status-quo, and a DM could therefore also view ambiguity from the perspective of not purchasing the policy.[2] With the reference-point $(w, w-L)$ the DM will purchase insurance at a premium P if

$$\min_{\pi\in[\hat{\pi}-\epsilon,\hat{\pi}+\epsilon]} \{\pi\,[u(w-P)-u(w)] + (1-\pi)\,[u(w-P)-u(w-L)]\} \geq 0\,. \qquad (2.3)$$

As a result, it is the prior under which loss is *least* likely to occur, $\hat{\pi}+\epsilon$, that is relevant for the DM in deciding whether to insure. Figure 2.1.3 illustrates the certainty equivalent of $(w, w-L)$ for an ambiguity averse DM who views ambiguity from the perspective of the status-quo $(w, w-L)$. The diagram depicts two indifference curves of a SEU DM with prior $\hat{\pi}-\epsilon$ and $\hat{\pi}+\epsilon$, respectively, that intersect at the point $(w, w-L)$. An indifference curve of the DM with RMEU preferences corresponding to the utility of $(w, w-L)$ is given by the upper envelope of these indifference curves. RMEU therefore implies that the DM is willing to pay a *lower* premium for full insurance. The reason is that, viewed from the perspective of the alternative $(w, w-L)$, the DM is ambiguous about

[2]For example, a literature in psychology demonstrates that DM's usually prefer errors of omission (where they do not act and this turns out to be a mistake) to errors of commission (where they act and this turns out to be a mistake, see, e.g., Kahneman and Tversky (1982)). In the case of insurance, not purchasing insurance and incurring a loss is an error of omission while purchasing the policy and then observing that no loss occurs is an error of commission.

alternatives (w_1, w_2) for which $\pi u(w_1) + (1 - \pi)u(w_2) = \pi u(w) + (1 - \pi)u(w - L)$ for some $\pi \in [\hat{\pi} - \epsilon, \hat{\pi} + \epsilon]$. Being ambiguity averse the DM prefers not to purchase insurance unless the purchase of insurance is unambiguously better (i.e., dominates the purchase of insurance for all priors in $[\hat{\pi} - \epsilon, \hat{\pi} + \epsilon]$). This occurs at the premium $\bar{P} < P^*$.



2.1.1: SEU　　2.1.2: MEU　　2.1.3: RMEU

Figure 2.1: *Certainty equivalents (and risk premia) for different decision models of decision making under uncertainty: (2.1.2) Subjective Expected Utility; (2.1.2) Maxmin Expected Utility; (2.1.3) Reference-Dependent Maxmin Expected Utility.*

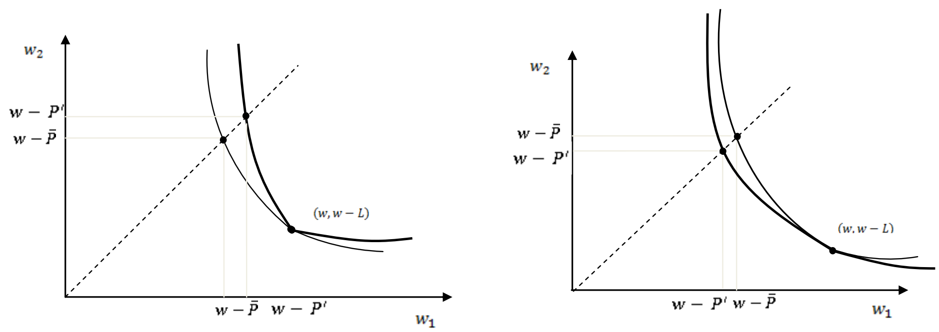
The RMEU decision maker is ambiguity averse, he just views ambiguity *relative* to the reference-point $(w, w - L)$, and is averse to this relative ambiguity. As a result, the DM faces a trade-off between insurance against risk and hedging against ambiguity. To illustrate, consider the effects of (1) an increase in risk aversion captured via a strictly concave transformation of the payoff function u, and (2) an increase in ambiguity captured by an increase in ϵ. It is straightforward to see that for a DM with MEU preferences both an increase in risk-aversion and an increase in ambiguity imply that the insurance premium P' that the DM is willing to pay increases. The reason is that an increase in risk-aversion increases the motive to insure under every prior the DM regards possible, while an increase in ambiguity "shifts" the prior that is relevant for the insurance problem to one where a loss is more likely to occur. Both effects increase the insurance motive and therefore the willingness-to-pay. However, if $(w, w - L)$ is the reference-point for an ambiguity averse DM, an increase in risk aversion leads to an increase in the insurance premium the DM is willing to pay, while an increase in ambiguity leads to a decrease in the willingness to pay. The increase in the willingness to pay as a result of greater risk-aversion is exactly as in the MEU case. However, as ambiguity increases, a DM who views ambiguity from the perspective of the status-quo evaluates insurance in terms of a probability distribution under which loss is *less* likely to occur. These effects therefore move in opposite directions.

The separation of ambiguity aversion from risk aversion in the RMEU model is crucial to capture this trade-off. Consider, for example, a DM with a reference-point $r : S \to X \subset \mathbb{R}$ who evaluates alternatives (on the restricted space of Savage (1954) acts mapping S to X) using the following utility functional

$$\hat{V}(f) = \min_{\pi \in \Pi} \int_S u\,(f(s) - r(s))\,\pi(ds)\,, \tag{2.4}$$

i.e., the DM maximizes the minimal expected utility over a set of priors but with payoffs determined on the deviations $(f(s) - r(s))$ rather than the outcomes $f(s)$ directly. This utility specification captures a version of reference-dependent MEU in the spirit of cumulative prospect theory, i.e., with *ex-post* reference-effects. The preferences represented by $\hat{V}$ are RMEU preferences if and only if the DM is risk-neutral (and u is therefore linear). More generally, for the preferences represented by $\hat{V}$ both ambiguity aversion and risk aversion are relative to the reference-point. As a result, ambiguity aversion and risk aversion both imply a motive to trade towards full insurance, except that now full insurance is given by constant deviations from the state contingent reference-point, r, rather than constant outcomes. However, in the RMEU model, ambiguity aversion is relative to the reference-point while risk aversion is a property of *ex-post* utility on prizes.

Some immediate implications of the difference between *ex-ante* and *ex-post* reference-effects can be illustrated in the context of the insurance example from the previous section. Figure 2.2.1 depicts the certainty equivalent of $(w, w - L)$ for a DM with utility function $\hat{V}$ and reference-point $(w, w-L)$. Both an increase in ambiguity and an increase in risk aversion now imply that the premium the DM is willing to pay decreases. Compare this to Figure 2.2.2, which illustrates that an increase in risk aversion in the RMEU model leads to an increase in the willingness to pay for insurance. This trade-off between reference-dependent ambiguity aversion and risk aversion requires an *ex-ante* perspective on reference-effects related directly to the response to ambiguity. The behavioral implications of an *ex-ante* reference-effect are studied formally in Chapter 2.2. I outline the motivation for the key axiom in the following example.



2.2.1: *Ex-post* RMEU 2.2.2: *Ex-ante* RMEU

Figure 2.2: *Certainty equivalents with varying risk-aversion: (2.2.1) Ex-post Reference-Dependent Maxmin Expected Utility model of Eq. 2.4; (2.2.2) Reference-Dependent Maxmin Expected Utility model of Eq. 2.1.*

2.1.2 An Ellsberg example

Consider the classic Ellsberg (1961) problem. Suppose that a DM is confronted with two urns: Urn A contains 50 black and 50 white balls, and urn B contains 100 black and white balls but in an unspecified proportion. A ball will be drawn at random from an urn and the DM's payoffs are determined by the color. The Ellsberg (1961) paradox is the finding that most individuals are indifferent about which should be the high payoff color in each urn, but strictly prefer a bet involving urn A over the same bet involving urn B. Such choices are inconsistent with the SEU hypothesis because they indicate a behaviorally meaningful distinction between risk (faced in bets on urn A) and ambiguity (faced in bets on urn B). However, Heath and Tversky (1991) and Fox and Tversky (1995) observe that the Ellsberg (1961) paradox disappears when people are familiar with the ambiguity they face. Their findings suggest that a DM's perception of ambiguity may depend on the context from which uncertainty is viewed. Roca et al. (2006) confirm this by showing that if individuals are endowed with a status-quo alternative in an Ellsberg (1961) setting, they make choices that are inconsistent with models of ambiguity aversion in which the reference-point is given by a constant act. They interpret this as evidence that status-quo bias leads to ambiguity loving behavior. Instead, the model of reference-dependent ambiguity aversion provides a more natural explanation, namely that the DM's ambiguity attitude does not change with the context of the decision problem, but that the perspective from which they view ambiguity may be context-dependent.

To illustrate, consider at first only bets on urn B and assume that the DM considers a state space $S = \{s_1, s_2\}$, where s_1 is the state in which the ball drawn from urn B is black and s_2 is the state in which the ball drawn from urn B is white. To focus on the response to ambiguity, assume that the DM is risk neutral. The DM has a status-quo alternative $r = (6, 3)$. To think about this status-quo as a reference-point, suppose that the DM must first decide whether to forgo r to realize any other alternative. Now consider two alternatives $f = (8, 2)$ and $g = (4, 4)$. Giving up r for f is profitable in s_1, and detrimental in s_2. Since the realization of states in urn B is subject to ambiguity, the DM may therefore express a strict preference to keep r over f. Analogously, giving up r for g is profitable in s_2 and detrimental in s_1, and the DM may therefore also express a strict preference for r over g. When comparing f and g directly, $f(s_1) = g(s_1) + 4$, and $f(s_2) = g(s_2) - 2$, but g is constant across states. Since the decision maker is (by assumption) risk neutral, indifference between f and g therefore seems plausible.

Now consider the following bet, h. A ball is drawn first from urn A. If the ball drawn is black, the DM receives the bet f on urn B. If the ball drawn is white, the decision maker receives bet g on urn B. A risk neutral decision maker is indifferent between h and r.[3] If the DM's preferences are transitive, the decision maker therefore prefers h to f and g. This preference ordering is a violation of the independence axiom used by Anscombe and Aumann (1963) to characterize SEU. It is also a violation of the certainty independence axiom used by Gilboa and Schmeidler (1989) to characterize MEU, since the alternative g is constant across states. The reason for this violation is intuitive from a reference-point perspective: While g is unambiguous in an absolute sense, g exhibits *relative* ambiguity when viewed from the perspective of the status-quo r. It represents an improvement in s_1 and would make the DM worse off in s_2. Since the state of the world is determined by the unknown composition of balls in urn B the relation between r and g is therefore ambiguous. However, viewed from the perspective of r all ambiguity in bet h has been resolved and "replaced" with the objective uncertainty from urn A. Roca et al. (2006) demonstrate that endowing individuals with a status-quo alternative in an Ellsberg (1961) setting does indeed lead to the kind of violations of certainty independence outlined in this thought experiment. The reference-dependent ambiguity model rationalizes such violations, not by suggesting that DMs become ambiguity loving when endowed with a status-quo, but by pointing out that such violations arise naturally when decision makers are ambiguity averse and view ambiguity *relative* to the status-quo.

The key axiom behind the RMEU representation therefore characterizes behaviorally what it means for a DM to view ambiguity from the perspective of a particular reference-point. I call this axiom "Reference-Independence". It is a weakening of the

[3]Formally this requires that the DM's preferences satisfy a monotonicity assumption, and that the DM evaluates objective lotteries in terms of their expected utility.

independence axiom that relates subjective uncertainty over states to the objective uncertainty over outcomes in the SEU axiomatization of Anscombe and Aumann (1963).[4] In particular, Reference-Independence generalizes the certainty independence axiom in the MEU model. In the MEU model, independence is replaced with two (jointly weaker) axioms. First, an uncertainty aversion axiom (first introduced by Schmeidler, 1989) captures a basic aversion to ambiguity (vs. risk).[5] The RMEU model satisfies uncertainty aversion, and is therefore also a model of ambiguity-aversion. Secondly, Gilboa and Schmeidler (1989) introduce a certainty independence axiom that requires independence to hold only with respect to constant acts.[6] Gilboa and Schmeidler (1989, pp. 144–145) justify certainty independence by arguing that "a DM who prefers f to g can more easily visualize the mixture of f and g with a constant [act] h than with an arbitrary one, hence he is less likely to reverse his preferences." The motivation behind Reference-Independence is that the alternatives a DM "can more easily visualize" mixtures with may be context-depend. In the Reference-Independence axiom it is therefore the reference-point, r, of the DM which occupies the role otherwise occupied by a constant act in the certainty independence axiom. If r is constant across states, the decision model is equivalent to MEU, but the model also allows for a more general perspective on ambiguity aversion in which ambiguity is perceived relative to a non-constant reference-point.

[4] A preference relation $\succeq$ satisfies independence if, for all acts f, g and h, and for all $\alpha \in (0,1)$, $f \succeq g$ if and only if $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$.

[5] A preference relation $\succeq$ satisfies uncertainty aversion if, for all acts $f \sim g$ and for all $\alpha \in (0,1)$, $\alpha f + (1-\alpha)g \succeq f$. This is weaker than independence which requires indifference.

[6] Formally, a preference relation $\succeq$ satisfies certainty independence if, for all acts f, g and all constant acts h, and for all $\alpha \in (0,1)$, $f \succeq g$ if and only if $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$. Certainty independence is weaker than independence which requires the same condition to hold but for all acts, not only acts that are constant across states.

2.2 A Model of Reference-Dependent Maxmin Expected Utility

In this Section, I give a behavioral characterization of the Reference-Dependent Maxmin Expected Utility (RMEU) model in Eq. (2.2). The approach is to identify where the implicit reference-point assumption is made in the axioms that characterize Maxmin Expected Utility (MEU), and then generalize by imposing Reference-Independence instead.

2.2.1 Decision environment

In the traditional setting of Anscombe and Aumann (1963), consider a (non-empty) *state space* S, a (non-empty) set of *outcomes* or *prizes* X, and a set of *lotteries* $\mathcal{P}$ consisting of all probabilities on X with finite support.[7] The state space S is endowed with an algebra, Σ, of *events*. Denote by $\Delta(\Sigma)$ the set of finitely-additive probabilities on Σ. This set is endowed with the event-wise convergence topology. The objects of choice are the set of *acts*, $\mathcal{F}$, consisting of all (simple), finite-valued Σ-measurable functions $f : S \to \mathcal{P}$.[8] The set of acts is endowed with a point-wise mixture operation, such that $\alpha f + (1 - \alpha)g$ is the act given by $\alpha f(s) + (1 - \alpha)g(s)$ for each $s \in S$.

2.2.2 Preferences

The DM's preferences depend on a reference-point (or reference-act) $r \in \mathcal{F}$. A reference-point is an alternative in the choice space that is especially familiar to the

[7]More accurately, the decision environment is based on the extension of the Anscombe and Aumann (1963) framework in Fishburn (1970).

[8]Where it should not be confusing, I abuse notation and identify $\mathcal{P}$ with the subset of $\mathcal{F}$ consisting of constant functions.

DM, or is otherwise salient in the decision making process. I present axioms on preferences given the reference-point that characterize a DM who acts *as if* (s)he perceives ambiguity from the perspective of the reference-point, and has a negative attitude towards this ambiguity. The decision model is therefore a theory about how individuals view and respond to ambiguity from the perspective of a given reference-point, and does not embody any particular assumption about what act will serve as a reference-point for a DM in a given context. In applications the model needs to be augmented by a theory of what constitutes the reference-point. MEU implicitly assumes that the reference-point is a constant act, but other examples might include a status-quo or default option; a particular contract or investment project; or a personal recommendation or social convention.

For any reference-point $r \in \mathcal{F}$, let $\succeq_r \subset \mathcal{F} \times \mathcal{F}$ denote the preferences of the DM given reference-point r, and denote the asymmetric and symmetric parts of $\succeq_r$ by $\succ_r$ and $\sim_r$, respectively. With the conventional abuse of notation, denote the restriction of preferences to $\mathcal{P}$ and X analogously.

2.2.3 Axioms

In the following I introduce three types of axioms:

1. The first set of axioms describe behavior given a particular reference-point, but describe behavior that does not directly depend on the reference-point. In particular, for any given reference-point, the RMEU decision model is an example of a variational preference and so I recall (for completeness) the axioms that characterize variational preferences in Maccheroni et al. (2006). I also provide a somewhat simpler representation result for the case when preferences belong to the class of

unbounded variational preferences.[9]

2. The second set of axioms describe behavior that is specific to preferences given a particular reference-point. In fact, there is only one axiom that fully distinguishes behavior given different reference-points, namely the Reference-Independence axiom which characterizes behaviorally what it means for a DM to view ambiguity from the perspective of a particular reference-point.

3. A last set of axioms describe the relation between behavior given different reference-points. To this end, I assume that the individual has a class of preference relations, $(\succeq_r)_{r\in\mathcal{F}}$, one preference relation for each possible reference-point. Two axioms are introduced that establish connections between these different preference relations. The first is a Reference Translation (RT) condition that imposes a rescaling invariance condition between behavior given different reference-points. The second imposes an equivalence between unambiguous preferences (EUP) given different reference-points. The behavioral content of these axioms is more delicate because they are falsifiable only on the basis of choice data for a decision maker under different reference-points.[10] I view RT and EUP as axioms that provide a natural structure for comparative statics. For example, EUP is the assumption that the unambiguous preference relation (as defined in Ghirardato et al., 2004) given two different reference-points coincides. Hence, EUP captures the idea that the reference-point determines the perspective from which ambiguity is perceived, not the degree of ambiguity perceived *per se*. This seems a natural assumption to make for comparative statics purposes, but it is verifiable only if behavior is observed under different reference-points.

[9] In Maccheroni et al. (2006) unbounded variational preferences are a particularly important example of variational preferences for which they are able to attain tighter uniqueness conditions in the representation.

[10] The extension of the Ellsberg (1961) experiment in Roca et al. (2006) can be viewed as an attempt to study behavior under different ambiguous reference-points. It is clear that experimental solicitation of preferences under different reference-points always hinges on counterfactual assumptions about the invariance of observations across reference-points.

I first present a number of axioms that describe behavior regardless of the reference-point. The following four axioms are standard in the literature on decision making under uncertainty (see, e.g., Fishburn, 1970), and hold for a class of RMEU preference for every reference-point $r \in \mathcal{F}$.

Axiom 1 (Preorder) *For all $f, g \in \mathcal{F}$, not $f \succeq_r g$ implies $g \succeq_r f$; for all $f, g, h \in \mathcal{F}$, $f \succeq_r g$ and $g \succeq_r h$ implies $f \succeq_r h$.*

Axiom 2 (Non-trivial) *There exist $x, y \in X$ such that $x \succ_r y$.*

Axiom 3 (Monotonicity) *For all $f, g \in \mathcal{F}$, $f(s) \succeq_r g(s)$ for all $s \in S$ implies $f \succeq_r g$.*

Axiom 4 ((Mixture) Continuity) *For all $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] | \alpha f + (1-\alpha) g \succeq_r h\}$ and $\{\alpha \in [0,1] | h \succeq_r \alpha f + (1-\alpha) g\}$ are closed.*

The preceding axioms and independence together characterize Subjective Expected Utility (SEU) (Fishburn, 1970). Schmeidler (1989) and Gilboa and Schmeidler (1989) observe that independence excludes the phenomenon of hedging, i.e., a preference for uncertainty that is objective rather than ambiguous in nature. They therefore introduce two axioms that are jointly weaker than independence and that can be combined with Axioms 1-4 to characterize behaviorally a Maxmin Expected Utility (MEU) decision making criterion. The first axiom is uncertainty aversion (UA). It captures the idea that hedging is beneficial, and therefore represents a basic negative attitude towards ambiguity (see also the discussion in Schmeidler, 1989; Ghirardato et al., 2004). A DM whose preferences satisfy UA at least always weakly prefers alternatives in which some subjective uncertainty has been directly replaced with risk, but axiom is permissive of preference reversals due to the ambiguity inherent in the source of uncertainty.

Axiom 5 (Uncertainty Aversion (UA)) *For all* $f, g \in \mathcal{F}$, $f \sim_r g$ *implies*

$$\left(\frac{1}{2}\right) f + \left(\frac{1}{2}\right) g \succeq_r f \, .$$

The second axiom introduced in Gilboa and Schmeidler (1989) is certainty independence (CI).

Axiom 6 (Certainty Independence (CI)) *For all* $f, g \in \mathcal{F}$, *all* $x \in \mathcal{P}$ *and all* $\alpha \in (0, 1)$,

$$f \succeq_r g \;\Leftrightarrow\; \alpha f + (1-\alpha)x \succeq_r \alpha g + (1-\alpha)x \, . \tag{2.5}$$

CI is weaker than the Independence axiom implied by SEU because it rules out preference reversals only when acts are mixed with a constant act. Independence rules out preference reversals due to mixing with any arbitrary act. Gilboa and Schmeidler's (1989) objection to independence is that it rules out all possibility of hedging. They justify CI on the basis that – even when a preference for hedging is permitted – constant acts should be exempt because they are ambiguity neutral. However, whether a constant act is unambiguous or not depends on the perspective from which it is viewed. For example, *relative* to an alternative $f = (4, 2)$ the constant act $x = (3, 3)$ may appear quite ambiguous: It represents an improvement on f in state 1 and a deterioration in state 2. If the realization of states is determined by drawing a ball from an urn with an unspecified proportion of black and white balls, then the benefits of x *relative* to f are clearly ambiguous. Just as the likelihood of states is subjective when the proportion of black and white balls is unspecified, the perspective on what constitutes an ambiguity neutral act is also subjective and depends on the view of the DM. Hence, MEU is based on the assumption that DMs regard constant acts as ambiguity neutral. In certainty contexts it may be a justified assumption, but it is nevertheless an assumption and one that is not well recognized as such in the literature. To make clearer the content of the assumption, it is insightful to decompose CI into two axioms which jointly imply CI but are individually weaker.

Axiom 7 (Weak Certainty Independence (WCI)) *For all $f, g \in \mathcal{F}$, for all $c_1, c_2 \in \mathcal{P}$ and for all $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)c_1 \succeq_r \alpha g + (1-\alpha)c_1 \;\Leftrightarrow\; \alpha f + (1-\alpha)c_2 \succeq_r \alpha g + (1-\alpha)c_2 \,. \tag{2.6}$$

Axiom 8 (Pivotal Independence (PI)) *There exists a $c \in \mathcal{P}$ such that for all $f, g \in \mathcal{F}$ and for all $\alpha \in (0,1)$,*

$$f \succeq_r g \;\Leftrightarrow\; \alpha f + (1-\alpha)c \succeq_r \alpha g + (1-\alpha)c \,. \tag{2.7}$$

A non-trivial preference preorder that is non-trivial, monotone and continuous satisfies WCI and PI if and only if it satisfies the CI axiom in Gilboa and Schmeidler (1989). Axiom 7 states that there should be no preference reversals if acts f and g are mixed in a given proportion with two different constant acts. As the name suggests, it imposes a translation invariance condition. The motivation given for this axiom in Maccheroni et al. (2006) is that, fixing the mixture and varying only in terms of constant acts c does not alter the absolute ambiguity exhibited by the act $\alpha f + (1-\alpha)c$, and hence does not introduce hedging opportunities in comparisons to $\alpha g + (1-\alpha)c$. Varying the mixture can introduce hedging opportunities because for α close to 0, the act $\alpha f + (1-\alpha)c$ is close to constant, while for α close to 1 the act potentially pays off very differently across states. Hence, in comparisons to $\alpha g + (1-\alpha)c$, the source of uncertainty becomes irrelevant as α converges to 0, but may become increasingly important as α converges to 1. Likewise, holding the mixture fixed but varying in terms of non-constant acts also introduces hedging opportunities. For example, if $f = (f_1, f_2) \in \mathbb{R}^2$, and $\alpha \in (0,1)$, the act

$$f' = \alpha f + (1-\alpha)\left(\frac{\alpha}{(1-\alpha)} f_2, \frac{\alpha}{(1-\alpha)} f_1\right) = \alpha(f_1 + f_2, f_1 + f_2)\,,$$

is constant across states. As a result, comparing $f = \alpha f + (1-\alpha) f$ with $\alpha g + (1-\alpha) f$, or f' with $\alpha g + (1-\alpha) f$, may be subject to a very different degree of ambiguity. However, if the mixture is held constant, whatever hedging opportunities can be realized by mixing acts f and g with a constant act c_1 are already realized in the mixture with c_1, and there are no additional hedging opportunities in mixing in the same proportion with an alternative and also constant act c_2. A DM whose preferences satisfy UA and WCI can therefore be interpreted as displaying a type of constant absolute ambiguity aversion.[11]

Given Axioms 1-5 and Axiom 7, Axiom 8 is a reference-point assumption implicitly embedded in the MEU decision model. In general, the mixture of the constant act c in PI can affect the ambiguity exhibited by two acts f and g very differently. For example, suppose that $g = c$. Then mixing g with c provides no hedging opportunity against ambiguity in g, but may well provide hedging opportunities against the ambiguity in an arbitrary act f. One way to justify precluding preference reversals due to mixtures with c is therefore to argue that the DM views ambiguity from the perspective of the constant act c, and can therefore "more easily visualize the mixture of f and g with the constant act [c], than with an arbitrary one" (Gilboa and Schmeidler, 1989). Hence, PI is the assumption that a constant act is ambiguity neutral. Of course, if one constant act is ambiguity neutral all constant acts are ambiguity neutral. Indeed, in the MEU decision model, which constant act serves as a reference-point can not be revealed by behavioral data. PI simply implies that a MEU DM does "reveal" that he uses *some* constant act as a reference-point.

It is not clear, however, that a DM will always view ambiguity from the perspective of a constant act. A large literature in psychology, experimental and behavioral economics demonstrates that decision making under uncertainty is often influenced by fea-

[11]Behavioral differences between WCI and CI are discussed in greater detail in Maccheroni et al. (2006).

tures of the context of the decision problem, or details of the decision making environment (Samuelson and Zeckhauser, 1988, see, e.g,), and it is plausible therefore that the perspective from which ambiguity is viewed by a DM may also be context-dependent. For example, the findings of Roca et al. (2006) suggest that DMs may view ambiguity from the perspective of a status-quo even in simple Ellsberg (1961) settings. The key behavioral axiom for a RMEU preference is therefore the Reference-Independence axiom discussed in the Introduction to this Chapter. Reference-Independence generalizes PI to allow for ambiguity to be perceived from the perspective of a non-constant reference-point. The reference-point can be interpreted as an act that is particularly familiar to a decision maker and therefore plays a focal role in their decision making process, so that the decision maker can "more easily visualize mixtures" with the reference-point than with an arbitrary act. If the DM views ambiguity from the perspective of the reference-point r, then r provides no hedging opportunities against ambiguity because all ambiguity is defined *relative* to r. Hence, mixtures with r should not lead to preference reversals. This is exactly the content of Reference-Independence axiom.

Axiom 9 (Reference-Independence (r-Independence)) *For all $f, g \in \mathcal{F}$ and for all $\alpha \in [0, 1]$, $f \succeq_r g$ if and only if $\alpha f + (1-\alpha)r \succeq_r \alpha g + (1-\alpha)r$.*

It is straightforward to verify that Axioms 1-7 and r-Independence are implied by the utility representation in Eq. 2.1 for a given reference-point r. However, they are sufficient only with some additional structure relating preferences with different reference-points. Moreover, for comparative statics it is also of interest to know when the set of priors Π and the von Neumann/Morgenstern utility payoff u are independent of r in a class of preference relations $(\succeq_r)_{r\in\mathcal{F}}$. I therefore introduce two more axioms that establish a relationship between the preferences of a DM given different reference-points. As the theory stands, the DM could have entirely different preference relations given two different reference-points. For example, the preference order given reference-point r' could be the exact opposite of the preference order given reference-point $r \neq r'$. This complete reversal of the preference order is consistent with the preceding axioms, but it seems neither natural as a model of behavior nor useful as an assumption for comparative statics. It therefore makes sense to impose some additional structure on the class of preference relations to model how a decision maker behaves when the reference-point changes.

The first axiom I introduce on the class of preference relations is motivated by the idea that reference-points affect the perspective from which the decision maker views ambiguity, but do not affect the degree of ambiguity that the decision maker perceives *per se*. To formalize this idea, I follow Ghirardato et al. (2004) in defining an unambiguous preference relation given any reference-point, $\succeq_r^*$, and then assume that this relation does not depend on the reference-point. Hence, the reference-point determines the perspective from which ambiguity is viewed, not the degree of ambiguity that the DM perceives.

Definition 1 (Unambiguous preferences) *Let $f, g \in \mathcal{F}$, then f is unambiguously preferred to g given reference point r, denoted $f \succeq^* g$, if for all $h \in \mathcal{F}$ and for all $\alpha \in (0, 1)$,*

$$\alpha f + (1-\alpha)h \succeq_r \alpha g + (1-\alpha)h\,. \tag{2.8}$$

The unambiguous preference relation $\succeq^*$ is introduced and motivated in Ghirardato et al. (2004). The basic idea is that if $\alpha f + (1-\alpha)h$ is preferred to $\alpha g + (1-\alpha)h$ for all $h \in \mathcal{F}$ and for all $\alpha \in (0, 1)$, there is no hedging opportunity that can lead to a preference reversal between f and g. "Hence", f is unambiguously preferred to g. The unambiguous preference relation is revealed by the DM, but the interpretation is attributed. For a more detailed discussion of the sense in which $\succeq_r^*$ captures unambiguous preferences, the reader is referred to Ghirardato et al. (2004). For the representation of a class of RMEU preferences, I use the unambiguous preference relation to establish a connection between the preference relations of a DM given different reference-points.

Axiom 10 (Equivalent Unambiguous Preferences (EUP)) *For all $r_1, r_2 \in \mathcal{F}$, and for all $f, g \in \mathcal{F}$,*

$$f \succeq^*_{r_1} g \;\Leftrightarrow\; f \succeq^*_{r_2} g\,. \tag{2.9}$$

If we take the perspective that the preferences of a DM are revealed given a particular reference-point, then EUP has no behavioral content. EUP is therefore simply a comparative statics assumption: We may not observe the choices a DM would make given a different reference-point, but EUP formalize the counterfactual assumption that if we were to observe these choices, the primitives of the RMEU model (the set of priors Π and the von Neumann/Morgenstern payoff function u) would not change. Even a reference-free model of preferences must make such counterfactual assumptions about behavior in contexts in which the DM is never observed. For example, if we observe the choices of a DM in a particular context and conclude that the DM is a SEU maximizer, there is nothing to guarantee that in a different (hypothetical) context the DM would not exhibit reference-dependent behavior. Hence, SEU and (all decision models) involve comparative static assumptions. This limitation is overcome only by assuming that the DM's preferences are observed in all possible contexts, and this is exactly the condition under which EUP would also be revealed by preferences (i.e., when EUP has behavioral content). EUP also rules out a number of behaviors that are normatively unappealing, like the complete reversal of preferences or the type of preference cycles discussed in Sagi (2006). Although its behavioral content is therefore delicate, it seems like a natural condition to impose on a class of reference-dependent ambiguity aversion preferences.

The second axiom I introduce on the class of preference relations can be viewed as a translation invariance condition for comparisons across reference-point.

Axiom 11 (Reference Translation (RT)) *For all $r \in \mathcal{F}$ and for all $f, g, h \in \mathcal{F}$ and all $\alpha \in (0, 1)$,*

$$f \succsim_r g \;\Leftrightarrow\; \alpha f + (1-\alpha)h \succsim_{\alpha r+(1-\alpha)h} \alpha g + (1-\alpha)h\ . \tag{2.10}$$

Note that, if f is unambiguously preferred to g, EUP already implies the equivalence (2.10). The motivation behind RT is that, in comparing two alternatives f and g among which a decision maker does not have an unambiguous preference, the decision maker with reference-point r cares in particular about how the alternatives compare relative to the reference-point. Applying a similar transformation to f, g and r simultaneously "should therefore" leave preference orders unaltered. Again, the motivation for this is that changes in the reference-point only change the perspective from which ambiguity is viewed by the decision maker and do not affect the preferences of the decision maker otherwise. Hence, RT formalizes the idea that if the same transformation is applied simultaneously to f, g and r, the decision maker performs essentially the same comparison between the transformed alternatives given the transformed reference-point as between the initial alternatives given the initial reference-point.

Note also that RT represents a generalization of a structure that is already inherent in MEU to a setting where preferences may be reference-dependent. Suppose r and r' are constant acts. Then if $\succsim_r$ satisfies preorder, non-triviality, continuity, monotonicity, UA, WCI and r-Independence it is an MEU preference. Since $\alpha r + (1-\alpha)r'$ is also a constant act, $\succsim_{\alpha r+(1-\alpha)r'}$ is therefore also an MEU preference. Moreover, r-Independence and WCI imply CI when r is a constant act and RT is therefore trivially satisfied when $r, r' \in \mathcal{P}$. It follows that in the subclass of MEU preferences, RT is implied by CI and has no additional behavioral content. RT therefore imposes a structure which is already implicitly embedded in MEU to comparisons across preference relations with non-constant reference-points.

2.2.4 Representation

I call a preference relation satisfying Axioms 1-7 and Axiom 9 a Reference-Dependent Maxmin Expected Utility (RMEU) preference. The following theorem provides a representation for a class of RMEU preferences also satisfying RT and EUP.

Theorem 1 (RMEU Representation) *The following statements are equivalent.*

1. *For all $r \in \mathcal{F}$, $\succeq_r$ satisfies the preorder, non-trivial, monotonicity, continuity, UA, WCI and r-Independence axioms, and the class of preference relations $(\succeq_r)_{r\in\mathcal{F}}$ satisfies RT and EUP.*
2. *There exists a non-constant, mixture-linear von Neumann/Morgenstern utility index $u : \mathcal{P} \to \mathbb{R}$ and a weak*-closed, convex set of priors $\Pi \subset \Delta(S)$, such that for all $r \in \mathcal{F}$ and for all $f, g \in \mathcal{F}$,*

$$f \succeq_r g \;\Leftrightarrow\; \min_{\pi\in\Pi} \int_S [u(f(s)) - u(r(s))]\, \pi(ds) \geq \min_{\pi\in\Pi} \int_S [u(g(s)) - u(r(s))]\, \pi(ds)\ . \tag{2.11}$$

Moreover, the set of priors Π is unique and the von Neumann/Morgenstern utility index u is unique up to positive affine transformations.

Proof. The proof is given in the Appendix. ■

For a particular reference-point, each RMEU preference in a class of RMEU preferences is an example of a variational preference (Maccheroni et al., 2006). I discuss the relation to variational preference in more detail after first providing a number of observations that relate RMEU preferences to other preferences that have been studied in the literature.

Remark 1 *Suppose that each preference relation, $\succeq_r$, in the class $(\succeq_r)_{r\in\mathcal{F}}$ is an RMEU preference, with a representation by way of* (Π_r, u_r).

1. *The set of priors Π_r is a singleton if and only $\succeq_r$ satisfies independence. In this case, if $(\succeq_r)_{r\in\mathcal{F}}$ satisfies EUP, each $\succeq_r$ has a SEU representation.*
2. *$\succeq_r$ has an MEU representation if only if r has constant von Neumann-Morgenstern utility $u(r(s))$ on all events in Σ for which the prior probability in Π_r is not unique.*
3. *$\succeq_r$ is a variational preference (Maccheroni et al., 2006) with ambiguity index given by the product of the the expected utility of the reference-point, $\int_S u(r(s))\pi(ds)$, and the indicator function*

$$\delta_r(\pi) = \begin{cases} 1 & \text{if } \pi \in \Pi_r \\ -\infty & \text{if } \pi \notin \Pi_r \,. \end{cases} \tag{2.12}$$

4. *If $(\succeq_r)_{r\in\mathcal{F}}$ satisfies EUP, the unambiguous preference relation $\succeq^*$ is reference-free and has a unanimity representation in the sense of Bewley (2002). Moreover, if r is the status-quo, $\succeq_r$ satisfies the Inertia assumption in Bewley (2002).*

The relation to variational preferences is of particular interest. Variational preferences are characterized by Axioms 1-7 (Maccheroni et al., 2006) and are therefore a general class of preferences that do not specify a particular perspective from which ambiguity is viewed. For example, they include MEU preferences (where ambiguity is viewed from the perspective of a particular constant act) as well as the multiplier preferences used in the work of Hansen and Sargent (2001) on robust control (in which ambiguity is viewed from the perspective of a particular reference probability distribution). RMEU preferences are the generalization of MEU preferences in which ambiguity is viewed from the perspective of a particular (but not necessarily constant) act in the choice space. As a result, the response to ambiguity in the choice space is very sharp around the reference-point, as suggested by the connection to the Inertia assumption of Bewley (2002). The following example illustrates some implications of the sharp response to ambiguity captured by RMEU preference. The example extends on Dow and Werlang (1992) and a similar example is given in Cao et al. (2009). In the context of Bewley's (2002) decision model, Easley and O'Hara (2010) relate the status quo bias illustrated in this example to the dramatic collapse in trading volumes during the 2008 financial crisis.

A portfolio inertia example

There is one risk free asset, b, with constant value 1, and one risky asset, x, with value $v \sim N(\hat{v}, \sigma^2)$. Suppose that, in addition to knowing the distribution, the variance σ^2 is known. However, the decision maker is uncertain about the mean value of the risky asset, $\hat{v}$, so that $S = \{\hat{v} \in \Re\}$. Acts in this setting are portfolios, (x, b), that map states in S into distributions over final wealth. Consider a decision maker with constant absolute risk aversion preferences and coefficient of risk aversion α, endowment $(\bar{x}, \bar{b}) >> 0$ and priors

$$\Pi = \left\{ \pi | \operatorname{supp} \pi \subset \left[\underline{v}, \bar{v} \right] \text{ , and } \operatorname{supp} \pi \text{ finite} \right\} . \tag{2.13}$$

The CARA assumption is for simplicity. The set of priors reflects the idea that the decision maker knows that $\hat{v} \in [\underline{v}, \bar{v}]$ but is not confident of in any particular distribution over this set. The utility maximization problem of the trader is

$$\max_{(x,b)} \min_{\pi\in\Pi} \int_{\underline{v}}^{\bar{v}} \left[Eu(x,b|\hat{v}) - Eu(\bar{x},\bar{b}|\hat{v})\right]\pi(d\hat{v}) \quad \text{subject to} \quad px + b \leq p\bar{x} + \bar{b}\ , \tag{2.14}$$

where the price of the risk-free asset is normalized to 1, p is the price of the risky asset, and $Eu(.|\hat{v})$ is the (objective) expected utility of a portfolio given a mean of $\hat{v}$ for the risky asset and Bernoulli utility $u(w) = -\exp^{-\alpha w}$ on wealth. With a change of variables, the solution to this problem is equivalent to the solution to the following problem.

$$\max_{t\in\Re} \min_{\pi\in\Pi} \int_{\underline{v}}^{\bar{v}} \left[\exp^{\alpha(\hat{v}\bar{x}-\alpha^2(\sigma^2/2)\bar{x}+\alpha\bar{b})} - \exp^{-\alpha(\hat{v}(\bar{x}-t)+\alpha^2(\sigma^2/2)(\bar{x}-t)^2+\alpha pt+\alpha\bar{b})}\right]\pi(d\hat{v})\ , \tag{2.15}$$

where $t = \bar{x} - x$ represents the trade of the decision maker in the risky asset.

Let δ_v denote the probability measure with point mass on v. For any $t > 0$ the arg-min in Eq. (2.15) is $\delta_{\bar{v}}$, while for $t < 0$ the arg-min is $\delta_{\underline{v}}$. That is, if the trader decreases his holdings in the risky asset, RMEU is determined by the highest mean value of the risky asset, while if the trader increases holdings of the asset the RMEU is determined by the lowest mean value. It follows that the solution to the problem in Eq. (2.15) is of the following form:

$$t^* = \begin{cases} \frac{\alpha\sigma^2\bar{x}+p-\bar{v}}{\alpha\sigma^2} & \text{if } p > \frac{\bar{v}}{\sigma^2\bar{x}} \\ 0 & \text{if } \frac{\underline{v}}{\sigma^2\bar{x}} < p < \frac{\bar{v}}{\sigma^2\bar{x}} \\ \frac{\alpha\sigma^2\bar{x}+p-\underline{v}}{\alpha\sigma^2} & \text{if } p < \frac{\underline{v}}{\sigma^2\bar{x}} \end{cases} \tag{2.16}$$

The trader is a subjective expected utility maximizer if and only if $\underline{v} = \bar{v}$, and in this case there is a unique price at which he does not trade (i.e., a unique price at which the the solution is $t^* = 0$). However, for an ambiguity averse trader, with $\underline{v} < \bar{v}$, there exists a closed interval of prices (with non-empty interior), in which no-trade is optimal. This comes from the status quo bias when the decision maker experiences ambiguity about the mean value of the risky asset. Notice also that the no-trade price interval depends not only on the ambiguity, captured by $\underline{v}$ and $\bar{v}$, but also on the initial endowment of the risky asset, $\bar{x}$, and the variance, σ^2. In particular, notice that the size of the interval is decreasing both in the initial endowment and the variance. This diminishing effect of ambiguity aversion is due to a trade-off between ambiguity-aversion and risk-aversion. The more risky the asset (larger σ^2) or the riskier the initial position (larger $\bar{x}$) the less important ambiguity aversion becomes relative to risk aversion.

Crisp acts

The status-quo bias illustrated in the previous example is due to a "kink" in the indifference curve of a RMEU preference, $\succeq_r$, at the reference-points, r. Such kinks exist as long as Π is not a singleton. However, the set of acts where the family of indifference curves exhibit kinks is not limited to the reference-point. The set of points where kinks occur is most transparent in the special case of MEU preferences: Weak Certainty Independence implies that if a constant act c is a reference-point of $\succeq_c$, then *all* constant acts c' are also reference-points of this preference relation. As a result, indifference curves exhibit kinks at all constant acts. The following definition (suitably adapted from Ghirardato et al., 2004) can be used to formalize behaviorally what is meant by "kinks", and to establish formally the counterpart for general RMEU preferences.

Definition 2 (Crisp acts) *Let $\succeq_r$ be a RMEU preference. An act $r' \in \mathcal{F}$ is crisp with respect to reference-point r if, for all $f, g \in \mathcal{F}$ and for all $\alpha \in (0, 1)$,*

$$f \succeq_r g \;\Rightarrow \alpha f + (1-\alpha) r' \succeq_r \alpha g + (1-\alpha) r' \,. \tag{2.17}$$

An act is therefore crisp if it cannot be used for hedging the ambiguity in other acts. Crisp acts can be interpreted as all those acts which are ambiguity neutral from the perspective of the decision maker. In SEU all acts are assumed to be ambiguity neutral. In MEU constant acts are assumed to be ambiguity neutral. For an RMEU preference the reference-point is by definition assumed to be ambiguity neutral, but WCI and RT in particular also imply that other alternatives in the choice space are viewed as ambiguity neutral. The following Proposition establishes that the set of crisp acts corresponds to the set of points from where the indifference map of a RMEU preference exhibits kinks.

Proposition 1 *Suppose that* $(\succeq_r)_{r\in\mathcal{F}}$ *is a class of RMEU preferences satisfying EUP and RT, and represented by way of* (Π, u)*. Let* $r, r' \in \mathcal{F}$*, then the following statements are equivalent.*

1. *r' is crisp with respect to r.*
2. $\succeq_{r'} = \succeq_r$.
3. *For all $g \in \mathcal{F}$,*

$$g \succeq_r r' \;\Leftrightarrow\; \int_S u(g(s))\pi(ds) \geq \int_S u(r'(s))\pi(ds)\ \forall\, \pi \in \Pi\,. \tag{2.18}$$

Moreover, if there exists $b \in \mathbb{R}$ such that $u(r'(s)) = u(r(s)) + b$ for all $s \in S$, then r' is crisp with respect to r.

Proof. The proof is given in the Appendix. ■

Proposition 1 shows that the reference-point in a RMEU preference is generally not unique. If r' is crisp with respect to r, then r' is also a reference-point of the preference relation $\succeq_r$. This property of RMEU preferences is crucial to ensure that such preferences contain MEU preferences as a special case, because all constant acts are reference-points in MEU. Under RMEU, the position of crisp acts is determined by the interaction of WCI and the reference-point. To illustrate this graphically, suppose that there are two states of the world, the DM has von-Neumann/Morgenstern utility index u on $\mathcal{P}$ and that payoffs are given directly in terms of expected utiles. The set of crisp acts for the reference-point r is illustrated in Figure 2.3. The set of crisp acts is on the translation of the full-insurance line that passes through r. Note that along any other translation of the full-insurance line indifference curves are straight lines (since payoffs are in utiles) and parallel (because the minimization over the set of priors is not affected by translations). The latter is a direct implication of WCI. The reference-point determines on which translation of the full-insurance line the crisp acts will lie. Figure 2.3 illustrates this for the reference-point r and an alternative reference-point r'. Note that for either reference-point indifference curves are straight and parallel on translations of the full-insurance line, and the reference-points then determine along which translation the crisp acts lie (i.e., from where the DM views ambiguity). The substantive implications of the set of crisp acts are illustrated in the following example.

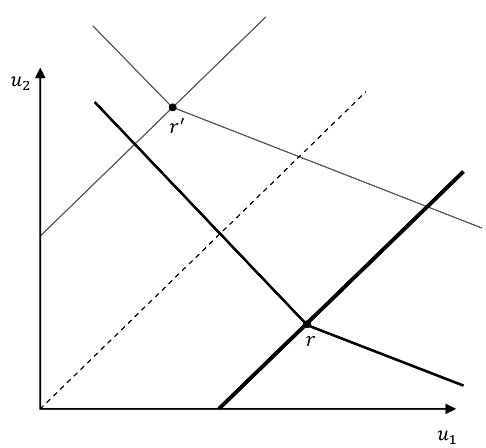


Figure 2.3: *The set of crisp acts for two different reference-points r and r′ (payoffs in expected utiles).*

A diversification-bias example

There are two periods, $t = 0, 1$, and one consumption good to be consumed in $t = 1$. In $t = 0$ the decision maker can trade in contingent claims on the consumption good. There are four states of the world, $S = \{s_1, s_2, s_3, s_4\}$. Assume that the DM is confident about the likelihood of states s_1 and s_2, but completely ignorant about the relative likelihood of states s_3 and s_4: The DM has priors $\Pi = \{\pi | \pi_1 = \pi_2 = 0.25, \pi_3 \in [0, 0.5], \pi_4 = 0.5 - \pi_3\}$. Let the Bernoulli utility of consumption be $v(x) = \log(x)$. The log utility assumption is special because it implies that preferences are homothetic. Hence, if $r : S \to X$ is a reference-point, λr is also a reference-point for any λ such that $\lambda r \in \mathcal{F}$. Suppose for this example that the decision maker has an endowment of contingent claims $e = (e_1, e_2, e_3, e_4) >> 0$, and that ambiguity is viewed from the perspective of this endowment. Denote the price of contingent claims be $p = (p_1, p_2, p_3, p_4)$. Then the DM solves the following utility maximization problem:

$$\max_{(x_1,x_2,x_3,x_4)\geq 0} \min_{\pi\in\Pi} \sum_{s=1}^{4} \pi_s \left[\log(x_s) - \log(e_s)\right] \quad \text{subject to} \quad \sum_{s=1}^{4} x_s \leq \sum_{s=1}^{4} e_s\,. \tag{2.19}$$

With the assumption on Π and v, the maximization problem is equivalent to the following problem:

$$\max_{(x_1,x_2,x_3,x_4)\geq 0} \frac{1}{4}\log(x_1) + \frac{1}{4}\log(x_2) + \frac{1}{2}\min\left\{\log\left(\frac{x_3}{e_3}\right), \log\left(\frac{x_4}{e_4}\right)\right\} \tag{2.20}$$

$$\text{subject to} \quad \sum_{s=1}^{4}(1-\beta_s)e_s = 0\,. \tag{2.21}$$

The demand for contingent consumption is therefore:

$$x_1(p,e) = \frac{p\cdot e}{4p_1}\,,\ x_2(p,e) = \frac{p\cdot e}{4p_2}\,, \tag{2.22}$$

$$x_3(p,e) = \frac{e_3 p\cdot e}{2(p_3e_3 + p_4e_4)}\,,\ x_4(p,e) = \frac{e_4 p\cdot e}{2(p_3e_3 + p_4e_4)} \tag{2.23}$$

The events $E_1 := \{s_1, s_2\}$ and $E_2 := \{s_3, s_4\}$ are not ambiguous, and therefore the allocation across these events is determined by risk preferences alone. As a result, the DM choose the status-quo if and only if $e_1 = \frac{p \cdot e}{4p_1}$ and $e_2 = \frac{p \cdot e}{4p_1}$, because risk preferences alone imply that the decision maker generally demands an allocation that is different from the endowment. However, ambiguity always affects the allocation of contingent-consumption across states s_3 and s_4. In particular, $x_3(p,e)/x_4(p,e) = e_3/e_4$, so even if the DM re-allocates consumption between E_1 and E_2, the allocation within E_2 remains fixed. Such demand functions are inconsistent with any model of context-free preferences. For example, a SEU (or MEU) DM's demand will depend on the endowment only through the wealth (i.e., through the aggregate $(e_1 + e_2 + e_3 + e_4)$). The dependence of demand on e_3 and e_4 directly is a consequence of reference-dependent ambiguity aversion and the assumption that the endowment serves as a point of reference.

To interpret this reference-effect consider, for example, that the endowment of contingent-claims in states 3 and 4 represents ownership of a particular asset with the payoff structure $(0, 0, z, \lambda z)$. The allocation of wealth between the unambiguous events E_1 and E_2 is determined entirely by risk preferences and by the relative value of the endowment in these events, $\kappa := (p_1e_1 + p_2e_2)/(p_3e_3 + p_4e_4)$. For example, if $\kappa > 1$, risk aversion alone makes it optimal for the decision maker to transfer consumption from states 1 and 2 to states 3 and 4. However, the DM is completely ignorant about the relative likelihood of states 3 and 4. The DM does, however, already own an asset that would achieve the desired re-allocation across events, and therefore simply buys more of this asset. Such behavior is at odds with the diversification predicted by reference-free models, but it corresponds well with observations on household investment behavior. For example, amongst households that hold stocks directly in the US "the median number of stocks held was two until 2001, when it rose to three" (Campbell, 2006). That is, even in the subset of households that do trade in stocks, trade is typically restricted to only a very small number of assets. Such "under-diversification" behavior is consistent with RMEU preferences under the assumption that the given asset holdings act as a reference-point.

Similar under-diversification puzzles have been observed in 401(k) investment decisions (see, e.g., Beshears et al., 2009, and the references therein). There are two surprising findings here. The first is that the decision on whether to invest in a 401(k) plan at all seems to depend on whether non-enrollment or enrollment is the default option. This default-effect is suggestive of a reference-effect, where the reference-point is given by the default option. The second surprising finding is that even among households that do enroll, the portfolio held seems to depend on the default portfolio on enrollment. That is, many households that adjust their contributions to 401(k) plans do not adjust the portfolio structure (and are often heavily under-diversified). Moreover, the portfolio structure chosen is sensitive to default manipulation. Given that the stakes are high, this type of investment behavior is consistent with the idea that the default portfolio structure has a reference-effect and DM's are ambiguous about trading options. As a result, even when individuals do change the allocation between consumption and investment, their allocation within the investment plan is affected by the default option.

Comparative Ambiguity Aversion

A RMEU preference order can be represented by a triple of primitives (Π, u, r), where Π is the set of priors, u is the von Neumann/Morgenstern utility index, and r is the reference-point. I have argued above that r represents the perspective from which the decision maker views ambiguity. The decision maker's attitude towards risk is captured by u and can be elicited on the set $\mathcal{P}$ of constant acts as in the seminal paper by Yaari (1969). Intuitively, the the size of the set of priors Π reflects the degree of ambiguity perceived by decision maker. Ghirardato and Marinacci (2002) provide a way to formalize the intuition that the size of the set Π reflects the ambiguity perceived by a decision maker for MEU preferences. Their procedure is motivated by the comparative notion of risk-aversion in Yaari (1969), extended to consider preferences that reflect different attitudes towards ambiguity. In particular, they argue that a preference relation $\succeq_1$ should be considered "more ambiguity averse" than a preference relation $\succeq_2$ if for every act $f \in \mathcal{F}$ and every constant act $x \in \mathcal{F}$,

$$f \succeq_1 (\succ_1) x \;\Rightarrow\; f \succeq_2 (\succ_2) x\,. \tag{2.24}$$

Accordingly, a preference relation is ambiguity neutral if there does not exist another preference relation that is "less ambiguity averse". For the special case of MEU, Ghirardato et al. (2004) show that a preference relation $\succeq_1$ represented by (Π_1, u_1) is more ambiguity averse (in the sense of Ghirardato and Marinacci, 2002) than a preference relation $\succeq_2$ represented by (Π_2, u_2) if and only if u_1 and u_2 are in a positive affine relation (denoted $u_1 \cong u_2$)[12] and $\Pi_1 \supset \Pi_2$. Hence, an MEU preference is ambiguity neutral if and only if it has a SEU representation. I next extend their result on comparative ambiguity to a class of RMEU preferences. Ghirardato and Marinacci (2002) provide the following motivation for their comparative notion of ambiguity.

"If a decision maker prefers an ambiguous act to an unambiguous one, a more ambiguity averse one will do the same. This is natural, but it raises the obvious question of which acts should be used as the "unambiguous" acts for this ranking. Depending on the decision problem the DM is facing and on her information, there might be different sets of "obviously" unambiguous acts; that is, acts that we are confident that any DM perceives as unambiguous. It seems intuitive to us that in any well-formulated problem, the constant acts will be in this set. Hence, we make our first primitive assumption: Constant acts are the only acts that are "obviously" unambiguous in any problem, since other

[12]Formally, $u_1 \cong u_2$ if there exist $(a, b) \in \mathbb{R}_{++} \times \mathbb{R}$ such that $u_1 = au_2 + b$.

acts may not be perceived as unambiguous by some DM in some state of information."

I concur with the first sentence and therefore follow their basic procedure to define a comparative notion of ambiguity. I also concur that while the comparative notion of ambiguity they define is natural, it raises the obvious question of what should constitute the "unambiguous" alternatives that are used for making comparisons. Needless to say, however, I disagree that the set of constants acts should always be regarded as unambiguous. In fact, the overriding premise of this Dissertation is that the alternatives a decision maker regards as unambiguous should be treated as subjective. In fact, as Ghirardato and Marinacci (2002) argue, what constitutes an unambiguous act may well depend on the decision problem that the decision maker is facing. While constant acts may be regarded as unambiguous by a decision maker in particular context, there seems to be no compelling reason why, for example, a decision maker might not view their status-quo as an unambiguous alternative. Moreover, if the status-quo is regarded as unambiguous and the status-quo is itself not constant, then constant acts may well appear ambiguous from the perspective of the decision maker. As a result, for a decision maker who views ambiguity from the perspective of an alternative $r \in \mathcal{F}$, the only alternative which we can confidently say that the decision maker regards as unambiguous is r itself. Of course, this is almost by definition true for the RMEU decision model, and I therefore formulate a notion of reference-dependent comparative ambiguity based on this simple primitive assumption.

Definition 3 (More ambiguity averse) *The class of preference relations $(\succsim_r^1)_{r\in\mathcal{F}}$ is more ambiguity averse than the class of preference relations $(\succsim_r^2)_{r\in\mathcal{F}}$ if for all $r \in \mathcal{F}$ and for all $f \in \mathcal{F}$*

$$f \succsim_r^1 (\succ_r^1) r \;\Rightarrow\; f \succsim_r^2 (\succ_r^2) r\,. \tag{2.25}$$

A class of preference relations is ambiguity neutral if there does not exist another class *of preference relations that is less ambiguity averse.*

The following proposition formalizes the intuition that the set of priors Π captures the degree of ambiguity perceived by a decision maker with Reference-Dependent Maxmin Expected Utility.

Proposition 2 *Suppose that* $(\succeq_r^1)_{r\in\mathcal{F}}$ *is a class of RMEU preference relations satisfying RT and EUP and represented by* (Π_1, u_1)*. Suppose that* $(\succeq_r^2)_{r\in\mathcal{F}}$ *is another class of RMEU preference relations satisfying RT and EUP and represented by* (Π_2, u_2)*.*

- *Then* $(\succeq_r^1)_{r\in\mathcal{F}}$ *is more ambiguity averse than* $(\succeq_r^2)_{r\in\mathcal{F}}$ *if and only if* $u_1 \cong u_2$ *and* $\Pi_1 \supset \Pi_2$.
- $(\succeq_r^1)_{r\in\mathcal{F}}$ *is ambiguity neutral if and only if* Π_1 *is a singleton. Moreover, if* $(\succeq_r^1)_{r\in\mathcal{F}}$ *is ambiguity neutral, then* $\succeq_r^1 = \succeq_{r'}^1$ *for all* $r, r' \in \mathcal{F}$*, and all preference relations in* $(\succeq_r^1)_{r\in\mathcal{F}}$ *have an SEU representation with the same utility index* u_1 *and the same prior* π_1 *(where* $\{\pi_1\} = \Pi_1$*).*

Proof. The proof is given in the Appendix. ■

Proposition 2 provides formal support for comparative static analysis of RMEU preferences in terms of the size of Π. In particular, it highlights that the ambiguity aversion of the decision maker can be cleanly separated from the reference-point from which the decision maker views ambiguity, and the the von Neumann/Morgenstern utility index u which describes the risk-preferences of the decision maker. Equivalence of unambiguous preferences is crucial for this, but I show below that the behavioral content of EUP is not as important as it may appear: When the decision maker's preferences on $\mathcal{P}$ are unbounded either EUP or RT are redundant. Before providing this result, I first provide an axiomatization of a reference-dependent version of Choquet Expected Utility. The axiomatization is also provided for unbounded preferences, but is of interest primarily because CEU is – after MEU – perhaps the most common decision model used in practice to capture ambiguity sensitive behavior and because – unlike MEU – it is also able to accommodate different attitudes to ambiguity (from ambiguity averse to ambiguity loving behavior).

2.3 A Model of Reference-Dependent Choquet Expected Utility

I conclude this Chapter by studying an alternative model of reference-dependent ambiguity sensitive preferences, namely a reference-dependent version of Choquet Expected Utility. When the reference-point is constant, the Choquet Expected Utility (CEU) model of Schmeidler (1989) represents an alternative to MEU for representing a perception of ambiguity. Unlike MEU, CEU accommodates different attitudes towards ambiguity (from ambiguity aversion to ambiguity loving behavior), and a characterization of reference-dependent CEU is therefore of interest because it provides a reference-dependent decision model with differencing ambiguity attitudes. When preferences are ambiguity averse, CEU is a special case of MEU and the reference-dependent CEU model is likewise a special case of the reference-dependent MEU model of the previous section. However, even when the focus is on ambiguity averse preferences (in the sense of Schmeidler, 1989) the added structure of the CEU model is often used in applications because it provides for greater tractability than the more general MEU model. Hence, it is also of interest to know what the behavioral content of this added structure is when ambiguity averse preferences are reference-dependent.

I provide an axiomatization of a class of Reference-Dependent Choquet Expected

Utility (RCEU) for the special case when preferences are unbounded. To formalize this assumption, I recall the following axiom introduced in Maccheroni et al. (2006) for their characterization of unbounded variational preferences.

Axiom 12 (Unbounded) *There exists $y \succ_r x$, $y, x \in \mathcal{P}$, such that for all $\alpha \in (0, 1)$, (1) there exists $\bar{z} \in \mathcal{P}$ such that $\alpha x + (1 - \alpha)\bar{z} \succ_r y$ and (2) there exists $\underline{z} \in X$ such that $x \succ_r \alpha y + (1 - \alpha)\underline{z}$.*[13]

Given that preferences on $\mathcal{P}$ are complete, transitive and continuous, (1) is equivalent to the assumption that these can be represented by a utility function which is unbounded above, while (2) is equivalent to the assumption that the utility representation is unbounded below. Most Bernoulli utility functions defined on monetary outcomes satisfy one or both of these conditions. Moreover, since the axiom restricts preferences only on the set of constant alternatives it does not, in itself, restrict ambiguity attitudes in any way. Nevertheless, the axiom clearly has some important implications. For example, it immediately precludes a finite set of outcomes X. It is also problematic when preferences are extended outside the class of simple functions or to infinite horizon dynamic settings. Despite this, the axiom plays an important role in the analysis of variational preferences. In particular, together with Axioms 1-5 and Axiom 7, Axiom 12 characterize the unbounded variational preferences in Maccheroni et al. (2006) for which they provide a representation result (with stronger uniqueness properties).

The novel axiom used to characterize RCEU is reference dependent version of the comonotonic independence axiom in Schmeidler (1989). To state the axiom, a definition of reference-comonotone acts is required which generalizes the definition of comonotone acts in Schmeidler (1989).

[13] Maccheroni et al. (2006) only impose (1) or (2) in their axiom. It is clear from the proof of Proposition 3 that the same could be done here, but for simplicity I provide a characterization under the stronger axiom where both (1) and (2) are required.

Definition 4 (*r*-comonotonic) *Given an $r \in \mathcal{F}$, two acts $f, g \in \mathcal{F}$ are r-comonotone if there do not exist states $s, s' \in S$ such that*

$$\frac{1}{2}f(s) + \frac{1}{2}r(s') \succ_r \frac{1}{2}f(s') + \frac{1}{2}r(s) \text{ and } \frac{1}{2}g(s') + \frac{1}{2}r(s) \succ_r \frac{1}{2}g(s) + \frac{1}{2}r(s') \tag{2.26}$$

Note that if r is a constant act and $\succeq_r$ satisfies certainty independence (which is implied by the comonotonic independence axiom in Schmeidler (1989)), then (2.26) is equivalent to the condition that there never exist states $s, s' \in S$ such that

$$f(s) \succ_r f(s') \text{ and } g(s') \succ_r g(s) \tag{2.27}$$

which coincides with the definition of comonotonicity used by Schmeidler (1989). r-comonotonicity represents a generalization of the notion of comonotonicity to preferences that do not necessarily have a constant reference-point and therefore do not necessarily satisfy certainty independence. To explicate how r-comonotonicity generalizes comonotonicity, suppose that $r(s) \succ_r r(s')$ (of course, this is possible only if r is non-constant). Then the left side of 2.26 is satisfied only if f in state s is sufficiently preferred to f in state s' to "compensate" for the fact that r is also preferred in state s to s'. Hence, f must be preferred in state s over state s' *relative* to how much r is preferred in state s over s'. Acts are r-comonotonic if they both cross this "sufficiently"-preferred-to hurdle relative to r in exactly the same states. Hence, acts are r-comonotonic if for comparisons across states they have a similar relation to each other *relative* to the reference point r. The key axiom that characterizes a reference-dependent version of CEU is an independence condition for r-comonotone acts.

Axiom 13 (Reference-Comonotonic Independence (RCI)) *For all pairwise r-comonotonic acts $f, g, h \in \mathcal{F}$ and for all $\alpha \in (0, 1)$,*

$$f \succ_r g \Rightarrow \alpha f + (1-\alpha)h \succ_r \alpha g + (1-\alpha)h\,. \tag{2.28}$$

The motivation given for comonotonic independence by Schmeidler (1989) is that, even if a decision maker finds it difficult to make the comparisons needed to justify independence for all acts, comonotone acts are sufficiently alike so that such comparisons are less problematic. Hence, Schmeidler argues that comonotonic independence represents a reasonable weakening of the full independence condition for decision makers that may perceive ambiguity. The generalization to r-comonotonic independence is best viewed in a similar spirit. It states that, even when full independence may be too strong, pairwise r-comonotone acts are sufficiently alike *relative* to the reference-point r so that mixtures between such acts are more easily visualized by the decision maker (see also the discussion on pp. 576–577 of Schmeidler, 1989).

The following proposition provides a representation for (1) a preference relation that satisfies r-comonotonic independence, and (2) a class of preference relations that satisfies r-comonotonic independence for every reference point $r \in \mathcal{F}$ and satisfies Reference Translation (RT). To state the Proposition, say that a set function $v : \Sigma \to [0, 1]$ is a *capacity* if it is (1) normalized ($v(\emptyset) = 0$ and $v(S) = 1$) and (2) monotone ($A, B \in \Sigma$ and $A \subset B$ implies $v(A) \leq v(B)$). A capacity is *convex* if for all $A, B \in \Sigma$, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$. Note that a capacity is a probability measure if and only if the latter condition is satisfied with equality for all $A, B \in \Sigma$. Finally, denote for any $a \in B_0(\Sigma)$ (where $B_0(\Sigma)$ denotes the set of simple, Σ-measurable functions on S[14]) the Choquet integral of a with respect to the capacity v by $\oint a dv$. If v is a probability measure then the Choquet integral coincides with the regular expectations operator, but for non-additive capacities the Choquet integral is a generalized expectations operator suitable also for non-additive capacities (refer to Schmeidler, 1989, for a definition).

Theorem 2 (Reference-Dependent CEU) *The following statements are equivalent:*

[14] See the Appendix for more details.

1. *A preference relation $\succeq_r$ (for some $r \in \mathcal{F}$) satisfies the preorder, monotonicity, continuity, WCI and unboundedness axioms, and satisfies r-comonotonic independence.*
2. *There exists an onto, mixture-linear von Neumann/Morgenstern utility index u : $\mathcal{P} \to \mathbb{R}$ and a unique capacity v, such that for all $f, g \in \mathcal{F}$,*

$$f \succeq_r g \iff \oint_S [u(f(s)) - u(r(s))] \, v(ds) \geq \oint_S [u(g(s)) - u(r(s))] \, v(ds) \, . \quad (2.29)$$

Moreover, u is unique up to positive affine transformations and the capacity v is convex if and only if $\succeq_r$ also satisfies UA.

Further, suppose that $(\succeq_r)_{r \in \mathcal{F}}$ is a class of preference relations such that for each $r \in \mathcal{F}$, $\succeq_r$ satisfies the preorder, monotonicity, continuity, WCI, unboundedness and r-comonotonic independence axioms and is therefore represented by a pair (u_r, v_r) as in Eq. 2.29. Then the class of preference relations $(\succeq_r)_{r \in \mathcal{F}}$ satisfies Reference Translation if and only if for all $r, r' \in \mathcal{F}$

$$u_r \cong u_{r'} \quad \text{and} \quad v_r = v_{r'} \, . \quad (2.30)$$

Proof. The proof is given in the Appendix. ■

The RCEU model in (2.29) is a special case of an unbounded RMEU preference (with the same reference-point) if and only if preferences satisfy uncertainty aversion, i.e., if the capacity in the RCEU representation is convex. When the capacity is convex it is well known (see, e.g., Schmeidler, 1989) that it has a non-empty core (defined as the set of probability measures that set-wise dominate the capacity), and the Choquet integral of a with respect to a convex capacity is then exactly the minimum expectation of a over priors in the core of the capacity. More generally, a reference-dependent CEU preference need not satisfy UA and RCEU can therefore capture different ambiguity attitudes in a decision model in which ambiguity is viewed from the perspective of a non-constant reference-point. Finally, it is worth noting that r-comonotonic independence is the only axiom used in the characterization of RCEU that relates preferences directly to the the reference-point. r-independence is clearly implied by RCEU, but is redundant when preferences satisfy r-comonotonic independence.

2.4 Unbounded Reference-Dependent Maxmin Expected Utility

It is worth noting an important difference in the revealed preference foundations of Theorem 1 and Theorem 2. Theorem 1 provides a representation result only for a class of preference relations. While I refer to preferences satisfying Axioms 1-WCI and r-independence, and preferences represented by the utility functional in Eq. 2.1 interchangeably as RMEU preferences, Theorem 1 does not in fact prove that Axioms 1-WCI and r-independence are sufficient for the utility representation in Eq. 2.1. Sufficient conditions are provided only for each preference relation in a class of preference relations satisfying Axioms 1-WCI and r-independence separately and EUP and RT jointly to be represented by the utility functional in Eq. 2.1. On the other hand, Theorem 2 provides a characterization of a single reference-dependent preference relation in terms of the RCEU functional. However, it would be misleading to conclude from this that EUP and RT are central to the characterization of RMEU preferences (at least, that they are more central to the characterization of the RMEU functional than for characterization of the ambiguity averse RCEU functional). The reason is that, unlike in Theorem 1, the axiomatic characterization of RCEU in Theorem 2 requires that preferences be unbounded. The following Proposition illustrates that when unboundedness is assumed an analog of Theorem 2 is also also true for RMEU.

Proposition 3 (Unbounded RMEU representation) *The following statements are equivalent:*

1. *A preference relation $\succeq_r$ (for some $r \in \mathcal{F}$) satisfies the preorder, monotonicity, continuity, WCI, UA and unboundedness axioms, and satisfies r-Independence.*
2. *There exists an onto, mixture-linear von Neumann/Morgenstern utility index $u : \mathcal{P} \to \mathbb{R}$ and a weak*-closed, convex set of priors $\Pi \subset \Delta(S)$, such that for all $f, g \in \mathcal{F}$,*

$$f \succeq_r g \;\Leftrightarrow\; \min_{\pi \in \Pi} \int_S \left[u(f(s)) - u(r(s))\right] \pi(ds) \geq \min_{\pi \in \Pi} \int_S \left[u(g(s)) - u(r(s))\right] \pi(ds)\ . \tag{2.31}$$

Moreover, the set of priors Π is unique and the von Neumann/Morgenstern utility index u is unique up to positive affine transformations.

Further, suppose that $(\succeq_r)_{r\in\mathcal{F}}$ is a class of preference relations such that for each $r \in \mathcal{F}$, $\succeq_r$ satisfies the preorder, monotonicity, continuity, WCI, UA, unboundedness and r-Independence axioms and is therefore represented by a pair (u_r, Π_r) as in Eq. 2.31. Then the class of preference relations $(\succeq_r)_{r\in\mathcal{F}}$ satisfies Reference Translation if and only if for all $r, r' \in \mathcal{F}$

$$u_r \cong u_{r'} \quad \textit{and} \quad \Pi_r = \Pi_{r'}\ . \tag{2.32}$$

Proof. The proof is given in the Appendix. ■

Subject to the restriction to unbounded preferences, Proposition 3 illustrates that the only behavioral difference between the RMEU functional of Eq. (2.1) and the MEU functional is that in the latter Independence is satisfied with respect to a constant act, while in the former it holds with respect to the reference-point r. Put differently, unbounded MEU is obtained from an unbounded variational preference by assuming independence with respect to a single constant-act, while an unbounded RMEU preference with a general reference-point r is obtained by assuming independence with respect to that reference-point. Clearly, unbounded MEU is the special case of unbounded RMEU when the reference-point is constant. Finally, it is worth noting that for a class of unbounded RMEU preferences, EUP and RT are, in fact, equivalent. Hence, the last part of Proposition 3 could be restated assuming that the class of preferences satisfies EUP instead of RT.[15]

2.5 Conclusion

Ambiguity is pervasive in economic decision problems, which often involve allocation decisions under subjective uncertainty. In this Chapter, I have presented a decision model of an ambiguity averse decision maker who views ambiguity from the perspective of an *ex-ante* reference-point. The decision model is motivated by identifying an implicit constant reference-point assumption in the axiomatic characterization of Maxmin Expected Utility (Gilboa and Schmeidler, 1989), and generalizing this assumption to allow for (possibly) context-dependent preferences in which ambiguity is viewed from the perspective of a reference-point (such as the status quo, a particular contract or a social convention). A representation theorem for Reference-Dependent Maxmin Expected Utility preferences is given that generalizes the MEU decision model of Gilboa and Schmeidler (1989). For the special case when preferences are unbounded a reference-dependent version of CEU is also characterized to provide a representation result for a

[15]There is an interesting relation to the axiomatic representation of Vector Expected Utility (VEU) in Siniscalchi (2009b). There is a clearly apparent relation between Reference Translation and the Complementary Translation Invariance (CTI) condition (Axiom 8 in Siniscalchi, 2009b) that is used to give a behavioral characterization of VEU. Moreover, CTI is redundant for the characterization of VEU in two important special cases: (1) If preferences are unbounded and (2) if they satisfy Certainty Independence. These are also the two important special cases when an RT is redundant for an RMEU preference. The proof that RT is redundant under (1) is provided by Proposition 3, the proof that RT is redundant under (2) follows from Proposition 1 which demonstrates that when CI is satisfied the preferences have an MEU representation.

decision maker who views ambiguity from the perspective of a non-constant reference-point but may not necessarily be ambiguity averse.

The RMEU model helps to formalize an intuition that already exists in the literature on ambiguity aversion. For example, a much cited result in Dow and Werlang (1992) illustrates the possibility that an investor with Choquet expected utility will not trade away from a risk-free portfolio position for a non-singleton, convex set of prices. This result is regarded as an intuitively appealing improvement on the knife-edged predictions regarding no-trade under the subjective expected utility hypothesis. However, the condition that no-trade should depend on the trader holding an *ex-ante* risk-free position is restrictive because it is, in itself, a knife-edged condition. If we consider no-trade to be an intuitive aspect of behavior that a decision model of preferences under ambiguity might capture, it seems unsatisfactory that the result should depend on the specific properties of the status quo. Allowing for a generalization in which no-trade is robust around any status quo, requires that we first account for the fact that preference will somehow be related to features of the environment; and, secondly, requires that we identify what underlying motive of the decision maker restricts trade. The RMEU model addresses these concerns by (1) providing a formal decision theoretic framework within which to model a dependence between the ambiguity preferences of a DM and features of the environment, and (2) identifying in the preferences of the decision maker a trade-off between an insurance motive for trade (due to risk-aversion), and a hedging motive for trade (due to ambiguity aversion). This trade-off is not emphasized by existing models of ambiguity averse preferences in the literature. In Chapter 3 I show that this trade-off between insurance and hedging has substantive implications for the study of markets, by highlighting the possibility of underinsurance and market collapse when consumers view ambiguity from the perspective of their endowments in an Arrow-Debreu exchange economy.

CHAPTER 3

RISK AND AMBIGUITY AVERSION IN MARKETS

3.1 Introduction

Economic models of decision making are primarily judged by their empirical implications. There are two ways to gain insights regarding the empirical implications of a decision model. The first is via an axiomatic characterization of the decision model. An axiomatic characterization is useful because it makes transparent the content of a decision model that is falsifiable in terms of behavioral data. Subjective expected utility (SEU) provides a good illustration. Savage (1954) and Anscombe and Aumann (1963) provide axiomatic characterizations of subjective expected utility. The axioms they provide are simple and intuitive, and are also appealing from a normative viewpoint. However, a number of experimental findings, such as the Ellsberg (1961) paradox, challenge key axioms that characterize subjective expected utility when there is a stark contrast between different sources of uncertainty. The behavior observed in the Ellsberg (1961) paradox is not consistent with preferences that are both complete and satisfy the independence axiom of Anscombe and Aumann (1963) (or the sure thing principle in the context of Savage (1954)). It is this challenge to SEU that motivated the growing literature on ambiguity aversion, and suggested generalizations that would be required to accommodate Ellsberg behavior. Bewley (2002) represents one direction of accommodating Ellsberg behavior by relaxing the completeness axiom. The Maxmin Expected Utility (MEU) and Choquet expected utility (CEU) theories of Schmeidler (1989) and Gilboa and Schmeidler (1989) represent an alternative direction in which the independence axiom is weakened. Hence, the axiomatic characterizations of SEU pointed the way to experimental tests and generalizations of the decision theory that could accommodate concerns over ambiguity. Likewise, the axioms in Bewley (2002) and Gilboa and Schmeidler (1989) pointed the way to include reference-points in a model of ambiguity aversion. Motivated by experimental findings that reference-effects are important for decision making under ambiguity. Chapter 2 of this Dissertation therefore proposes a generalization of MEU theory to accommodate reference-effects, provides an axiomatic characterization of a reference-dependent ambiguity aversion model (Reference-Dependent Maxmin Expected Utility, or RMEU), and relates the axioms to the axioms of MEU theory and the decision theory in Bewley (2002).

The second way to gain insights about the empirical implications of a decision model is to use the decision model to put structure on behavior in markets or other institutions economists care about. To this end, it is generally the utility functional that is of interest, both because it is more tractable to work with and because it suggests alternative interpretations that can be used to impose structure on a model. Again, SEU provides a good example. Consider a simple Arrow-Debreu economy.[1] Without any additional structure, any allocation in such an economy can be rationalized as an equilibrium for SEU investors. Hence, the axioms themselves are not restrictive. However, a key feature of the SEU functional is a separation between beliefs (represented by the prior) and tastes (represented by the Bernoulli payoff function), and these separate components of the SEU representation can be used to impose more structure on equilibrium analysis. Of course, the separation in terms of beliefs and tastes is an interpretation that can not be revealed by behavioral data (the axioms underlying SEU theory characterize all that behavioral data can reveal), but it is an interpretation that can be used to gain further insights about the interaction between market participants. For example, suppose that there is no aggregate uncertainty in the economy and all investors are risk-averse. Then an equilibrium allocation is a full insurance allocation if and only if all investors have

[1]By simple I mean an economy with a finite number of states and dates, one consumption good in each state/date pair and a complete set of contingent claims.

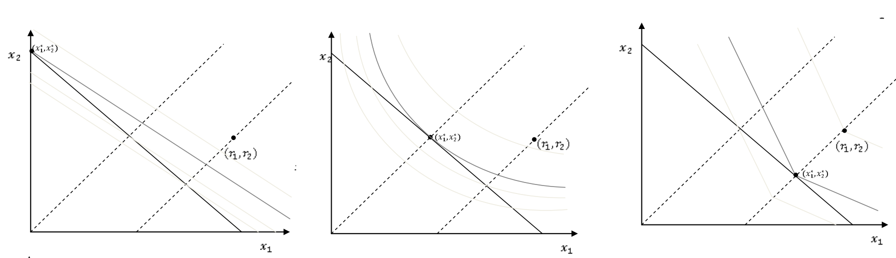
identical beliefs. This is a remarkably stark implication of SEU theory. It says that if all investors agree on the likelihood of states, an equilibrium will feature full insurance or, equivalently, any underinsurance must be due to disagreement on the likelihood of states. The result is independent of the endowments of the investors, the particulars of their risk preferences (i.e. their tastes for consumption in states), and the exact beliefs that investors hold regarding the likelihood of states. Hence, application of the key features of the SEU functional provide another way to study the empirical content of SEU; now looking at market outcomes rather than experimental data.

A second motivation for models of decision making that account for ambiguity has come from empirical market data that seems at odds with the implications of SEU when cross-sectional restrictions are put on the utility representations of multiple decision makers (DMs). For example, underinsurance, non-participation and market collapses are well-documented features of modern asset markets that seem at odds with subjective expected utility theory. The underinsurance observed in markets would require substantial disagreements in beliefs, non-participation is a knife-edged condition for a SEU maximizer and market collapses are non-generic in complete market economies with SEU DMs. The most common way to address these anomalies in the economics literature has been to weaken the assumption of complete markets, but in recent years there has also been growing interest in ambiguity aversion as a possible alternative explanation. The uncertainty faced by investors in asset markets is clearly ambiguous and so the question is whether ambiguity aversion is consistent with the market anomalies observed in a way that SEU is not. Results – mostly from the application of MEU theory – have been mixed. Under no aggregate uncertainty, an equilibrium allocation is a full insurance allocation in an economy with MEU investors if and only if the set of priors of the investors intersect (Billot et al., 2000). Hence, underinsurance requires substantial disagreement (more, one could argue, than with SEU DMs). Non-participation due to ambiguity aversion was first illustrated by Dow and Werlang (1992) in a partial equilibrium setting, but requires that the initial allocation have full insurance. Extensions exist (see, e.g., Epstein and Schneider, 2007) but it is not clear if they are really decidedly more general. Moreover, when the interaction between multiple market participants is considered, it turns out that market collapse can occur due to ambiguity aversion (see,e.g., Billot et al., 2000), but Rigotti and Shannon (2008) demonstrate that market collapse is still non-generic when investors have MEU preferences. In particular, it generally requires no aggregate uncertainty (which is a non-generic restriction on endowments). Overall, it is tempting to conclude from the literature that ambiguity aversion does little to address the empirical short-comings of SEU theory (at least in the context of asset markets).

However, it would be wrong to conclude that ambiguity aversion is not important in markets from the existing research. As the analysis in Chapter 1 has illustrated, MEU does not only model a particular perception of ambiguity (via multiple priors) and a particular attitude to ambiguity (via the minimization over the set of priors), but also fixes a particular perspective from which the DM views ambiguity (i.e., constant acts). In a simple investor problem, constant acts are full insurance allocations. Hence, MEU implies that both risk aversion *and* ambiguity aversion provide a motive for the DM to purchase insurance. On the other hand, the Reference-Dependent Maxmin Expected Utility (RMEU) model introduced in Chapter 2 identifies a fundamental trade-off between an insurance motive for trade (due to risk aversion) and a hedging motive to trade (due ambiguity aversion). This trade-off occurs whenever the set of crisp acts of an RMEU preference does not contain all constant acts. If the set of crisp acts contains all constant acts, the DM's preferences have an MEU representation and risk-aversion and ambiguity-aversion both imply a motive to trade towards full insurance. But if the reference-point is not a constant act – e.g., if it is a non-constant endowment, default option or status-quo – risk-aversion implies a motive to trade towards full insurance while ambiguity-aversion implies a motive to trade towards the non-full-insurance reference-point. In this Chapter, I therefore use the RMEU representation to illustrate that ambiguity can affect market outcomes much more generally than work using the MEU decision model suggests. In Chapter 2 the empirical implications of RMEU preferences were analyzed by studying the axioms underlying the utility representation, and the analysis in this Chapter therefore complements the analysis of Chapter 2 by looking at the empirical implications of RMEU in the context of exchange under uncertainty. The trade-off between insurance and hedging is essential for the analysis, and before introducing equilibrium, I therefore provide a simple example for a single investor to show how the separation between beliefs and tastes in the RMEU decision model can be used to highlight a novel trade-off between hedging and insurance.

3.1.1 Examples from a simple investor choice problem

Consider a DM choosing contingent consumption, $(x_1, x_2) \in \mathbb{R}^2_+$, in two states of the world, s_1 and s_2, given wealth $w \in \mathbb{R}_{++}$ and prices $(p_1, p_2) \in \mathbb{R}^2_{++}$. Figure 3.1 illustrates the trade-off between a speculative motive for trade (due to differences in beliefs), an insurance motive (due to risk aversion), and hedging motive (due to RMEU).



3.1.1: Speculation 3.1.2: Insurance 3.1.3: Hedging

Figure 3.1: *Motives for trade under different decision models: (3.1.1) Speculative motive; (3.1.2) Insurance motive; (3.1.3) Hedging motive.*

In Figure 3.1.1 the DM maximizes SEU and is risk-neutral. Indifference curves are therefore straight lines with slope $\pi/(1-\pi)$, where π is the subjective prior belief on state s_1. The DM has a speculative motive for trade if $\pi/(1-\pi) \neq p_1/p_2$. For example, we can interpret the condition $\pi/(1-\pi) < p_1/p_2$ heuristically as saying that the DM believes state 2 to be more likely than the market does. As a result, the DM uses all wealth for consumption in the state 2 ($x_1^* = 0$, $x_2^* = w/p_2$). There is no speculative motive for trade exactly when $\pi/(1-\pi) = p_1/p_2$ and in this case the quantity demanded can be anywhere on the budget frontier ($x_1^* \in [0, w/p_1]$, $x_2^* = w/p_2 - x_1^*$), otherwise the quantity demanded is always at corner.

In Figure 3.1.2 the DM maximizes SEU but is risk averse. There is no speculative motive for trade when $\pi/(1-\pi) = p_1/p_2$. Risk-aversion implies that better than sets are convex and the condition $\pi/(1-\pi) = p_1/p_2$ implies that the marginal rate of substitution is equal to the price ratio at full insurance allocations. As a result, the DM demands the same quantity of contingent-consumption in both states, and the quantity demanded is therefore a full insurance allocation ($x_1^* = x_2^* = w/(p_1 + p_2)$). In the special case that the DM is risk neutral the quantity demanded can be anywhere on the budget frontier ($x_1(p, w) \in [0, w/p_1]$, $x_2^* = w/p_2 - x_1^*$), otherwise (for a DM who is strictly risk averse) the quantity demanded is always at the full-insurance allocation.

In Figure 3.1.3 the DM perceives ambiguity about the likelihood of states s_1 and s_2. The DM is risk neutral (so there is no insurance motive for trade due to risk aversion). Also, to abstract from a speculative motive for trade, the DM entertains a set of priors for state s_1, $[\underline{\pi}, \bar{\pi}] \subset (0, 1)$ with $p_1/p_2 \in \left(\underline{\pi}/(1-\underline{\pi}), \bar{\pi}/(1-\bar{\pi})\right)$. The DM has RMEU preferences, is ambiguity averse and perceives ambiguity from the perspective of a (possibly) non-constant reference-point (r_1, r_2). The quantity demanded by the DM is the contingent-consumption allocation on the budget set that is on the straight line passing through (r_1, r_2) with slope 1 ($x_1^* = \frac{w+p_2(r_1-r_2)}{p_1+p_2}$ and $x_2^* = \frac{w+p_1(r_2-r_1)}{p_1+p_2}$). Clearly, if the reference-point is a full insurance allocation, $r_1 = r_2$, the DM demands a full insurance allocation. Hence, at the given prices, the quantity demanded by a MEU DM corresponds to the quantity demanded by a risk-averse SEU maximizer.[2] The reason is that, for the MEU DM hedging also implies a motive to trade towards full insurance. However, if $r_2 \neq r_1$, hedging implies a motive to trade towards allocations that are a translation of the reference-point. As a result, the DM demands a quantity that is neither at the corner nor at full insurance. Only in the special case when the DM perceives no ambiguity (and hence, $\underline{\pi} = \bar{\pi}$), the quantity demanded can be anywhere on the budget frontier ($x_1(p, w) \in [0, w/p_1]$, $x_2^* = w/p_2 - x_1^*$). The example therefore illustrates that the affect of ambiguity aversion on the quantity demanded by the investor is distinct from the affect of risk aversion if and only if $r_1 \neq r_2$ (and utility therefore has an RMEU representation that is not in the class of MEU representations).

3.1.2 Outline of the Chapter

In the remainder of this Chapter, I illustrate some consequences of the trade-off between insurance and hedging in an exchange economy where multiple market participants interact and prices and wealth are also determined in equilibrium. I assume throughout that DMs have RMEU preferences with the reference-point given by the endowment. Cao et al. (2009) also study some implications of reference-dependent ambiguity aversion in financial markets using a representation of preferences very closely related to Eq. 2.1. They show that RMEU is consistent with an endowment effect, limited market participation and a home equity bias, but they do not provide axiomatic foundations for their

[2]Note that the demand functions are different. In particular, DM with MEU preferences will demand full insurance for a range of prices, while full insurance is demanded by a strictly risk-averse SEU maximizer only when there is no speculative motive for trade.

representation of preferences and their analysis of exchange is restricted to a CARA-normal framework.[3] I study a general (finite dimensional) Arrow-Debreu exchange economy and show that market collapse is a robust implication of ambiguity aversion. I also give a number of examples to illustrate the possibility of non-participation and underinsurance in asset markets due to reference-dependent ambiguity aversion.

3.2 The Economy

Consider a standard Arrow-Debreu economy. There are two dates 0 and 1. At date 1 one of a finite number of states of nature, $s = 1, ..., S$, is realized. There is a single consumption good available at date 1; for simplicity assume there is no consumption at date 0. At date 0 investors can trade in a complete set of Arrow securities. Denote the price of an Arrow security paying off one unit of consumption in state s by $p(s)$, and the vector of prices by $p \in \Delta(S)$, where Δ denotes the standard simplex in $\mathbb{R}^S_+$. There are finitely many investors indexed $i = 1, ..., I$. Each investor's consumption set is $\mathbb{R}^S_+$, and each investor has an endowment $e^i \in \mathbb{R}^S_{++}$. Each investor has a binary relation $\succeq^i$ that describes the choices the investor will make over $\mathbb{R}^S_+$ and that satisfies the following assumption.

Assumption 1 (Endowment-Dependent Ambiguity Aversion) *For all $i = 1, ..., I$, there exists a compact, convex set $\Pi_i \subset \Delta_{++}$ and a C^2, concave, strictly increasing function $u_i : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $u_i'(c) > 0$ for all $c > 0$ and $u_i'(c) \to \infty$ as $c \to 0$; such that for any $x, y \in \mathbb{R}^S_+$, $x \succeq^i y$ if and only if $V_i(x) \geq V_i(y)$, where*

$$V_i(x) := \min_{\pi \in \Pi_i} \sum_{s=1}^{S} \pi(s) \left(u_i(x_s) - u_i(e_s)\right). \tag{3.1}$$

[3] In the economy they study, assets have normal distributions with unknown mean and all investors have constant absolute risk aversion (CARRA).

Unless otherwise stated, Assumption 1 will be maintained throughout. It is not standard for preferences to depend directly on the endowment. However, it is possible to define equilibrium and a notion of welfare in a standard way.

Definition 5 (Feasible Allocation) *An allocation* $(x^1, ..., x^I)$ *is feasible if* $\sum_i x^i \leq \sum_i e^i$.

Definition 6 (Competitive Equilibrium) *A feasible allocation* $(x^1, ..., x^I)$ *and a price vector* $p \in \Delta(S)$ *are a competitive equilibrium if* $x \succ^i x^i$ *implies* $p \cdot x > p \cdot e^i$ *for all* $x \in \mathbb{R}^S_+$ *and for all* $i = 1, ..., I$.

An economy is parameterized by $(e^i, \Pi_i, u_i)_{i \in I}$. Note that for any given endowment distribution, preferences of each individual are determined, continuous and strictly convex. Hence, existence of a competitive equilibrium follows directly from Debreu (1959).

Proposition 4 (Existence) *Under Assumption 1, a competitive equilibrium of the economy* $(e^i, \Pi_i, u_i)_{i \in I}$ *exists.*

The Pareto optimality criterion usually used to analyze welfare properties of a competitive equilibrium is a context-free criterion; it is defined with respect to preferences and the total endowment of an economy and without reference to any particular distribution of the endowments. However, in the RMEU models with endowments as reference points, preferences depend on the distribution of the endowment. It is nevertheless possible to define the core of the economy in the usual way, by first defining the notion of a blocking coalition and then defining the core as the set of allocations not blocked by any coalition.

Definition 7 (Blocking coalition) *A coalition $C \subset \{1, ..., I\}$ blocks allocation $(x^1, ..., x^I)$ if there exists another allocation $(x^{1\prime}, ..., x^{I\prime})$ satisfying the following conditions:*

$$\sum_{i \in C} (x^{i\prime} - e^i) \leq 0 \tag{3.2}$$

$$x^{i\prime} \succeq^i x^i \text{ for all } i \in C \text{, and } x^{i\prime} \succ^i x^i \text{ for some } i \in C \,. \tag{3.3}$$

Definition 8 (Core) *A feasible allocation $(x^1, ..., x^I)$ is in the core of the economy if there is no coalition $C \subset \{1, ..., I\}$ which blocks $(x^1, ..., x^I)$.*

The core represents a cooperative solution to the allocation problem facing the investors $\mathcal{I}$. Suppose all investors had complete information about the endowments and preferences of all other investors, then the core represents the set of allocations that could be attained as the result of cooperative bargaining between groups of investors. Of course, investors do not have complete information about each other's tastes or endowments and so a cooperative procedure for reaching the core is not feasible. Competitive equilibrium, on the other hand, is interpreted as the outcome of a decentralized process in which each investor makes optimal choices given knowledge of their own preferences and endowments, as well as a market price. The equilibrium problem is exactly to determine at what prices the plans of investors are mutually consistent. It follows from standard arguments that this can occur only for allocations that also happen to lie in the core, i.e., equilibrium allocations are efficient in the sense that with perfect information and a procedure for cooperative bargaining no subset of consumers could reach a Pareto improving outcome relative to any competitive equilibrium in the economy.

Proposition 5 ("1st Welfare Theorem") *Under assumption 1, if $(x^1, ..., x^I)$ is a competitive equilibrium of the economy $(e^i, \Pi_i, u_i)_{i \in I}$ then it is in the core of the economy.*

3.3 Market Collapse

The first result in this Chapter demonstrates that market collapse is a robust implication of ambiguity aversion. This is not true under the MEU decision model, but is true if DMs view ambiguity from the perspective of their initial endowments. Sporadic market collapses are a well-documented feature of modern asset markets, and usually follow periods of extreme uncertainty. For example, some salient features of the 2008 financial crisis include the collapse of trade in markets for many classes of assets, abnormally large ask-bid spreads, and widespread uncertainty about the value of assets (Easley and O'Hara, 2010). Markets that usually witness millions of transactions and billions of dollars of trade each week, froze almost completely overnight and in some cases it took several months for trade to return to usual levels. Motivated by this feature of modern asset markets, I define a market collapse in this Section as an equilibrium allocation in which there is no trade and no unique way to value the set of Arrow securities. Theorem 3 gives necessary and sufficient conditions for this type of market collapse. A no-trade equilibrium can occur also in an economy in which investors are not ambiguity averse, but for SEU investors (with differentiable Bernoulli utility indexes) the equilibrium price is always unique. A market collapse due to ambiguity aversion is therefore distinguished by indeterminacy of the equilibrium price. With MEU investors market collapse can occur, but is non-generic in endowments. Theorem 3 therefore also illustrates that a market collapse is robust if ambiguity is viewed from the perspective of endowments.

Definition 9 (No-trade equilibrium) *The economy* $(e^i, \Pi_i, u_i)_{i \in I}$ *has a no-trade equilibrium if* $(e^1, ..., e^I)$ *is the unique competitive equilibrium allocation.*

Definition 10 (Market Collapse) *The economy* $(e^i, \Pi_i, u_i)_{i \in I}$ *has a market collapse if it has a no-trade equilibrium and the equilibrium price is indeterminate.*

Theorem 3 gives necessary and sufficient conditions for a market collapse. It relies on a characterization of the superdifferential of an RMEU preference relation that is of interest in itself. To this end, define the following set for each investor $i = 1, ..., I$:

$$\Pi_i(e^i) = \left\{ \left(\frac{\pi_1 u_i'(e_1^i)}{\sum_{s\in S} \pi_s u_i'(e_s^i)}, ..., \frac{\pi_S u_i'(e_S^i)}{\sum_{s\in S} \pi_s u_i'(e_s^i)} \right) | \pi \in \Pi_i \right\} . \tag{3.4}$$

The set $\Pi_i(e^i)$ contains the probability distributions in the set of priors of investor i re-weighted by the marginal utilities of the endowment. An important feature of the set $\Pi_i(e^i)$ is therefore that it depends on the ambiguity perceived by the investor (i.e., the size of Π), as well as the risk perceived by the investor at their initial endowment (via the weights on the marginal utilities of the endowment). Hence, the "size, shape and location" of Π_i are determined by an interaction between ambiguity attitudes, risk attitudes and the risk inherent in the distribution of the endowment. Note also that for the special case of SEU the set $\Pi_i(e^i)$ is a singleton, but with multiple priors it is generally a convex set with a non-empty interior. Rigotti and Shannon (2008) refer to each element of the set $\Pi_i(e^i)$ as a subjective belief at e^i, highlighting that each element in this set is a probability distribution. The following result characterizes when a market collapse occurs, and provides sufficient conditions for a market collapse to be a robust feature of the economy.

Theorem 3 (Market Collapse) *Consider the economy* $(e^i, \Pi_i, u_i)_{i\in I}$.

1. *The economy has a no-trade equilibrium if and only if* $\bigcap_i \Pi_i(e^i) \neq \emptyset$.
2. *The economy has a market collapse if and only if* $|\bigcap_i \Pi_i(e^i)| > 1$.
3. *There exists an open ball* $B(e) \subset \mathbb{R}_{++}^{I\times S}$ *such that there is a market collapse for all* $e' \in B(e)$ *if* $int \bigcap_i \Pi_i(e^i) \neq \emptyset$.

Proof. The proof is given in the Appendix. ■

Theorem 3 highlights the trade-off between insurance and hedging discussed in the introduction. For the special case when there is no aggregate or idiosyncratic uncertainty, Theorem 3 demonstrates that the existence of a common prior is necessary and sufficient for a market collapse to occur. Hence, if there is no idiosyncratic uncertainty, markets collapse if there is sufficient agreement about the likelihood of states. Since preferences under no idiosyncratic uncertainty are equivalent to MEU preferences, this is a special case of the result in Billot et al. (2000). However, part (3) also demonstrates that if there is sufficient ambiguity about states, the market collapse is robust to the introduction of idiosyncratic and aggregate uncertainty. Rigotti and Shannon (2008) show that with MEU preferences market collapse is not robust to the introduction of aggregate uncertainty, and is therefore non-generic.

However, market collapse is not ubiquitous. If the distribution of endowments is asymmetric, risk aversion acts against ambiguity aversion and market collapse occurs only if investors are sufficiently ambiguity averse. For example, under the Inada condition assumed on payoff functions, $e_s^i \to 0$ implies $u_i'(e_s^i) \to \infty$. As a result, unless there is complete ignorance ($\Pi = \Delta$) the condition $\bigcap_{i \in I} \Pi_i(e^i) \neq \emptyset$ is necessarily violated as the endowment for any investor in any state is sufficiently small. The economic interpretation for this is that for a investor with an endowment approaching zero in one state, the insurance motive for trade becomes sufficiently strong so that no-trade can not be an equilibrium outcome. On the other hand, part (3) of Theorem 3 demonstrates that market collapse is robust in the sense that (given any aggregate endowment) if investors perceive sufficient ambiguity, the market collapses on a positive measure of endowment distributions. Two corollaries follow immediately from Proposition 3 and further illustrate the point.

The first Corollary demonstrates that for any distribution of endowments, market collapse can occur if there is sufficient ambiguity. This requires a notion of "more ambiguity averse" as formalized in Chapter 2: A preference relation $\succeq_1$ represented by (u_1, Π_1) is more ambiguity averse than a preference relation $\succeq_2$ represented by (u_2, Π_2) if u_1 is a positive affine transformation of u_2 and $\Pi_2 \subset \Pi_1$.[4] It is then possible to study the effects of increased ambiguity by holding reference-points (i.e. endowments) constant, payoff functions (u) constant, and increasing the ambiguity by increasing the size of the sets of priors. This leads to the following Corollary to Theorem 3.

Corollary 1 *Fix any distribution of endowments $e >> 0$. If investors are sufficiently ambiguity averse there is a market collapse.*

Proof. The result follows directly by observing that in the extreme case of complete ignorance, investors have Leontief preferences and the endowment is therefore the unique equilibrium allocation at any price. A formal proof is omitted since the result follows directly from Theorem 3. ■

The second corollary gives conditions on beliefs alone for the existence of a market collapse. Again, the contrast to the MEU model is of interest. Billot et al. (2000) show that there is a no-trade equilibrium in an economy with MEU investors if and only if $\bigcap_{i \in I} \Pi_i$ is non-empty and there is no idiosyncratic uncertainty. However, as Rigotti and Shannon (2008) show, market collapse is non-generic in an MEU economy regardless of the degree of ambiguity, because ambiguity aversion reinforces the insurance motive for trade, and market collapse therefore occurs only if there is no aggregate uncertainty. Intuitively, equilibrium indeterminacy requires that the set of crisp acts of RMEU investors line up in a particular way. For MEU preferences, this occurs when there is no aggre-

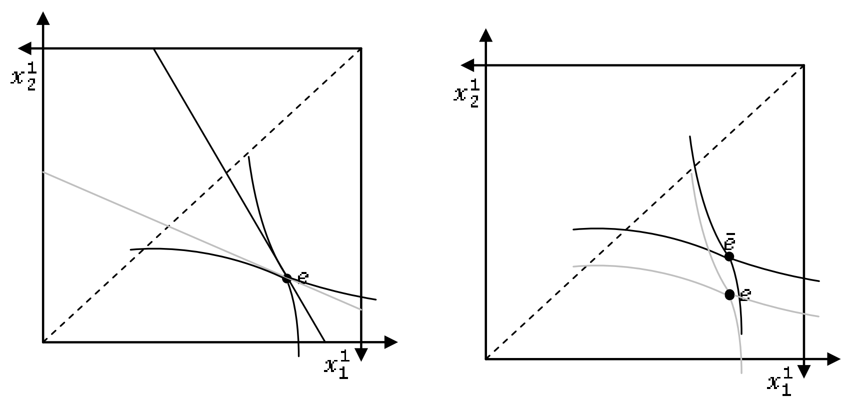
[4]Note that this also corresponds to the the notion of "more ambiguity averse" defined in Ghirardato et al. (2004).

gate uncertainty, and the set of crisp acts is therefore perfectly aligned along the full insurance allocations for all investors. No aggregate uncertainty is clearly not a robust condition, because a perturbation of endowments almost surely leads to idiosyncratic uncertainty and an economy with MEU investors then looks and behaves like an economy with risk-averse SEU investors. However, the condition that the relative interior of $\bigcap_{i \in I} \Pi_i$ is non-empty is sufficient when ambiguity is viewed from the endowment to ensure that market collapse is robust. The reason is that if investors view ambiguity from the perspective of the endowment, comparative statics on the endowment can no longer be performed independently of the preferences of the individuals. Perturbations of the endowment now *imply* perturbations of the preferences of the investors, because they alter the perspective from which ambiguity is viewed. As a result, the set of crisp acts "re-aligns" itself with the new endowment profile and the possibility of market collapse persists (see Figure 3.2.2). Corollary 2 follows exactly from this intuition.

Corollary 2 *Fix the aggregate endowment of an economy and suppose that there exists $\pi^* \in int \bigcap_{i \in I} \Pi_i$. Then there exists an open, convex set of endowment distributions such that there is a market collapse.*

Proof. A formal proof is omitted. The result follows directly by starting with the equilibrium allocation of the SEU economy with common prior π^*, assigning this equilibrium allocation as the endowment distribution of the economy and then observing that the conditions of part (3) of Theorem 3 hold. ■

Corollary 2 demonstrates that market collapse is a robust implication of ambiguity aversion even in an economy with aggregate uncertainty. A simple intuition for market collapse can be derived by looking at the Edgeworth box economy in Figure 3.2. Figure 3.2.1 illustrates a no-trade equilibrium when investors have RMEU preferences and view ambiguity from the endowment. Figure 3.2.2 illustrates a market collapse. Note that there exists an interval of prices such that no-trade is the unique equilibrium allocation. Also, consider a perturbation of the endowment. With non-reference-dependent preferences this would (almost surely) lead to a situation in which the endowment is no longer on the "kink" for both investor; hence, even if no-trade is still an equilibrium the price would be determined uniquely by any investor for whom the endowment is not on the kink of an indifference curve. However, if preferences are reference-dependent a perturbation of the endowment also implies a perturbation of the perspective from which the investors view ambiguity. As a result, the kinks shift with the endowment and for all endowments sufficiently close to the initial endowment there is still a market collapse.



3.2.1: No-trade 3.2.2: Market collapse

Figure 3.2: *Implications of RMEU in an Edgeworth box economy: (3.2.1) No-trade equilibrium; (3.2.2) (Robust) market collapse.*

The complete collapse of markets highlighted by Theorem 3 seems a plausible implication of ambiguity aversion in the idealized Edgeworth box economy. If there are only two investors and two states, no-trade by one investor implies no-trade for both. Likewise, no trade in one asset implies no-trade in both assets. Hence, it is sufficient for one investor to consider one state as ambiguous in order for no-trade to occur. However, in a more general setting with multiple investors and multiple states a complete collapse of all markets is clearly an idealized phenomenon. The following examples therefore illustrate implications of RMEU when there is some, but limited trade. The first example looks at market collapse for a subset of assets, the second example looks at non-participation by a subset of investors. After these examples, I return to provide a more careful analysis of underinsurance in an economy without aggregate uncertainty.

3.3.1 An example of market collapse with risk and ambiguity

The type of market collapse studied in Theorem 3 is reminiscent of the financial crisis of 2008. Theorem 3 shows that market collapse can be a consequence of RMEU when investors view ambiguity from the perspective of their status-quo, and face considerable uncertainty. However, even in the 2008 financial crisis, there was trade in state-contingent consumption. The collapse of markets was restricted to assets over which there was considerable uncertainty regarding value. The diversification bias example of Chapter 2 is suggestive of this type of more restricted market collapse, where speculative and insurance motives lead to trade in some assets but the hedging motive prevents trade in assets with sufficient ambiguity over states.

To illustrate, consider an example of an economy in which the set of states S can be partitioned into events R and A. The likelihood of states in R is subject only to risk, i.e., the probability of all states in s is known (or, alternatively, there is a unique and common subjective probability on each $s \in S$). Event A contains ambiguous states with unknown probabilities. For each investor i denote the linear projection of Π_i on to A by Π_i^A. For simplicity, assume that there is no aggregate uncertainty and investors have log utility. Hence, the utility maximization problem of a investor $i = 1, ..., I$ for any price p

is,

$$\max_{x^i \geq 0} \min_{\pi^i \in \Pi_i} \sum_{s \in R} \pi_s \left(\log x^i_s - \log e^i_s\right) + \sum_{t \in A} \pi^i_t \left(\log x^i_t - \log e^i_t\right) \tag{3.5}$$

$$\text{s.t. } px^i \leq pe^i \tag{3.6}$$

$$\Leftrightarrow \quad \max_{x^i \geq 0} \pi(R) \sum_{s \in R} \pi_s \left(\log x^i_s\right) + \min_{\pi^i \in \Pi^A_i} \sum_{t \in A} \pi^i_t \log\left(\frac{x^i_t}{e^i_t}\right) \tag{3.7}$$

$$\text{s.t. } px^i \leq pe^i\,. \tag{3.8}$$

The maximization problem decomposes and consumption decisions on R are not affected by ambiguity, and hence are not affected by the reference-point in states R. As a result, demand for good $s \in R$ by investor i is $x^i_s(p, e^i) = \frac{\pi_s pe^i}{p_s}$ for all $i = 1, ..., S$. Market clearing then implies that $p_s = \pi_s$ for all $s \in R$, and, hence no aggregate uncertainty implies that, $x^i_s = \pi(R)e$ for all $i = 1, ..., I$. Hence, investors are fully-insured in the risky states, where the insurance motive alone determines allocations. A no-trade equilibrium is therefore knife-edge in this economy: Generally, the insurance motive alone determines the allocation of resources between events R and A, and amongst states $s \in R$. However, a more restrictive form of market collapse is possible in the economy.

Definition 11 (No-ambiguous-trade equilibrium) *The economy $(e^i, \Pi_i)_{i=1,...,I}$ has a no-ambiguous-trade equilibrium if an allocation x with $x^i_s = \pi(R)e$ for all $s \in R$ and $i = 1, ..., I$, and $x^i_t = \pi(A)e^i$ for all $t \in A$ and $i = 1, ..., I$, is the unique competitive equilibrium allocation.*

In a no-ambiguous-trade equilibrium $\frac{x^i_s}{x^i_t} = \frac{e^i_s}{e^i_t}$ for all $s, t \in S$ and for all $i = 1, ..., I$. Hence, there is trade because investors trade contingent-consumption between the risky event R and the ambiguous event A, but there is no trade across states within A. In this sense of no-trade, there is also the possibility of a market collapse.

Definition 12 (Ambiguous Asset Market Collapse) *The economy $(e^i, \Pi_i)_{i=1,...,I}$ has an* *ambiguous-assets market collapse if it has a unique no-ambiguous trade equilibrium and the equilibrium price* $(p_s)_{s\in A}$ *is indeterminate.*

To interpret the definition of a market collapse suppose that investors have access to a complete set of Arrow securities, as well as a single asset, b, that can allocate consumption between events R and A. There is clearly a redundant asset, but in a no-ambiguous-trade equilibrium it is as if investors trade only in risky-assets (Arrow securities that payoff in states R, as well as the security b). Moreover, if the market for ambiguous-assets collapses there is indeterminacy regarding their values. To characterize conditions under which there is an ambiguous-assets market collapse, define the analog for 3.4 in the present case:

$$\Pi_i^A(e^i) = \left\{ \left(\frac{\frac{\pi_t}{e_t^i}}{\sum_{s\in A} \frac{\pi_s}{e_s^i}} \right)_{s\in A} | \pi \in \Pi_i^A \right\} . \tag{3.9}$$

Now observe that, given the allocations amongst risky states and events, remaining allocations are determined independently of outcomes on R. The relevant utility maximization problem for investor i is the following.

$$\max_{(x_s^i)_{s\in A}\geq 0} \quad \min_{\pi^i\in\Pi_i^A} \sum_{s\in A} \pi_s^i \log\left(\frac{x_s^i}{\pi(A)e_t^i}\right) \tag{3.10}$$

$$\text{s.t.} \sum_{s\in A} \frac{p_s}{\pi(R)}\left(x_s^i\right) \leq \sum_{s\in A} \frac{p_s}{\pi(R)}\left(\pi(A)e_s^i\right) . \tag{3.11}$$

The following characterization of conditions under which there will be an ambiguous-assets market collapse therefore follows immediately from Theorem 3.

Corollary 3 *Consider the economy* $(e^i, \Pi_i)_{i=1,\dots,I}$ *with risky event R and ambiguous event A.*

1. *The economy has a no-ambiguous-trade equilibrium if and only if* $\bigcap_i \Pi_i^A\left(e^i\right) \neq \emptyset$.
2. *The economy has a risky assets market collapse if and only if* $|\bigcap_i \Pi_i^A\left(e^i\right)| > 1$.

3. *There exists an open ball $B(e) \subset \mathbb{R}^{I}_{++} \times \mathbb{R}^{I\times S}_{++}$ such that there is an ambiguous-assets market collapse for all $e' \in B(e)$ if $int \bigcap_i \Pi_i^A \left(e^i\right) \neq \emptyset$.*

Proof. The proof follows immediately from Theorem 3 and is therefore omitted. ■

As in the market collapse identified in Theorem 3, an ambiguous-asset market collapse occurs only if the motive to hedge against reference-dependent ambiguity overcomes the motive to insure. Corollary 3 gives sufficient conditions for this to occur that depend on the set of priors and the endowment of the investors. However, unlike a complete market collapse, an ambiguous-assets market collapse is not due to a status-quo bias. The status-quo of a investor i is the endowment, e^i, and risk preferences alone determine that this is generally not an equilibrium allocation. Investors do trade in equilibrium; they just do not re-allocate contingent-consumption across the ambiguous states in event A. As a result, the sufficient conditions given for a robust asset market collapse in part (3) of Proposition 3 are not sufficient for a market collapse if investors had the preferences given in Bewley (2002). If investors had incomplete preferences and chose the endowment if and only if there is no alternative allocation that dominates the endowment, the insurance motive for trade on events that are purely risky would lead them to trade away from the endowment. And if investors trade away from the endowment, Bewley's decision model is indeterminate about what choices they will then make among incomparable alternatives. In particular, an ambiguous-assets market collapse in an economy in which investors follow Bewley's decision rule would be non-generic because no-trade in ambiguous assets might well be part of the equilibrium correspondence but would generally not be unique.

3.3.2 An example of non-participation

It is also interesting to study examples in which only a subset of individuals does not participate in the market. Non-participation in financial markets is common and puzzling under the SEU hypothesis (see, e.g., Campbell, 2006). To illustrate the possibility of non-participation on the part of a subset of investors, suppose now that the set of investors $\mathcal{I}$ can be divided into two subsets, $\mathcal{I} = \mathcal{I}' \cup \mathcal{I}^*$ with $\mathcal{I}' \cap \mathcal{I}^* = \emptyset$. Investors in $\mathcal{I}'$ are SEU maximizers (i.e. $|\Pi_i| = 1$ for all $i \in \mathcal{I}'$). Investors in $\mathcal{I}^*$ are ambiguity averse and have RMEU preferences as in Assumption 1 with a non-singleton set of priors $|\Pi| > 1$. For simplicity, assume again that all investors have log preferences over consumption in each state. Assume also that there is no disagreement over the likelihood of states, in the sense that investors in $\mathcal{I}'$ have a common prior $\hat{\pi}$ and $\hat{\pi} \in \Pi_i$ for all $i \in \mathcal{I}^*$. Define a non-participation equilibrium as a competitive equilibrium price and allocation in which all ambiguity averse investors do not participate in the market.

Definition 13 *The economy* $((e^i, \hat{\pi})_{i\in\mathcal{I}'}, (e^i, \Pi_i)_{i\in\mathcal{I}^*})$ *has a non-participation equilibrium if there exists an equilibrium price* p *and a corresponding equilibrium allocation* x *such that* $x^i = e^i$ *for all* $i \in \mathcal{I}^*$.

The following characterization of conditions under which there is a non-participation equilibrium follows immediately from Theorem 3. To simplify the exposition, we use the following notation $e^{\mathcal{I}'} := \sum_{i\in\mathcal{I}'} e^i$.

Corollary 4 *The economy* $((e^i, \hat{\pi})_{i\in\mathcal{I}'}, (e^i, \Pi_i)_{i\in\mathcal{I}^*})$ *has a non-participation equilibrium if and only if there exists* $\pi^* \in \bigcap_{i\in\mathcal{I}^*} \Pi_i(e^i)$ *such that*

$$\frac{\hat{\pi}_s e_t^{\mathcal{I}'}}{\hat{\pi}_t e_s^{\mathcal{I}'}} = \frac{\pi_s^*}{\pi_t^*} \quad \forall\ s, t \in S \tag{3.12}$$

Proof. The proof is given in the Appendix. ■

Corollary 4 demonstrates that non-participation can be a robust implication of ambiguity aversion, even in markets where trade occurs amongst ambiguity neutral investors. Suppose, first, that there is no idiosyncratic risk in the endowments. Then the common prior assumption is sufficient to ensure that there is a unique no-trade equilibrium.[5] This is true also if investors in $\mathcal{I}^*$ have MEU preferences. However, with RMEU preferences, non-participation is robust to the introduction of idiosyncratic and aggregate risk. As long as there is sufficient agreement over the likelihood of states between different market participants, RMEU investors may not participate in the market because the expected returns from investments must be calculated with too much precision in order to justify trade.

Corollary 4 also illustrates the trade-off between hedging and insurance. For a given sets of priors (other than complete ignorance), as the endowment of any one ambiguity averse investor in one state converges to zero, the condition in Eq. 3.12 is clearly violated and that investor will participate in the market. Note also that there is an interesting interaction between the ambiguity neutral and ambiguity averse investors. With sufficient ambiguity, there is always a non-participation equilibrium. But if the distribution of the aggregate endowment across ambiguity neutral and ambiguity investors is highly asymmetric, then it requires greater ambiguity in order for ambiguity averse investors not to participate in the market. Economic intuition for this comes from realizing that it is the SEU investors who determine prices in a non-participation equilibrium, and the prices they determine are affected by their share of aggregate consumption in each state. If their share of the aggregate consumption differ drastically from those of the ambiguity averse investors, the prices they determine are more likely to induce ambiguity averse

[5]Note that there is, however, no market collapse since the equilibrium price is uniquely determined by the set of ambiguity neutral investors

investors to enter the market.

3.4 Underinsurance

The no-trade equilibrium allocations studied in the preceding Section appear like examples of underinsurance. The direction of underinsurance is clearly correlated (perfectly in fact) with the endowments, and so the equilibrium allocations look much like the allocation in an economy without ambiguity aversion in which markets are (exogenously) incomplete. Ambiguity aversion can therefore provide an alternative explanation for market observations that are often attributed to incompleteness of markets. There is, however, an important distinction. No-trade allocations due to ambiguity aversion are (in an appropriate sense) efficient: There does not exist a Pareto improving allocation. The reason is that the commodity which would make ambiguity averse DMs better off is information that reduces the ambiguity about states; and this information is not in the aggregate endowment. On the other hand, when there is no-trade due to incomplete markets, equilibrium allocations are generally inefficient. The reason is that Pareto improving trades are possible within the given aggregate endowment, but not feasible given the existing market structure.

As a result, underinsurance due to ambiguity aversion needs to be interpreted carefully. Underinsurance does not refer to inefficiency, but must instead refer to underinsurance relative to allocations that seem like a reasonable benchmark for what insurance "should" be in the absence of ambiguity aversion. By Proposition 2 in Chapter 2 subjective expected utility is a natural benchmark for what "absence of ambiguity aversion" could mean. In general, defining a suitable benchmark for what insurance means is not straightforward. However, in the special case of an economy without aggregate uncertainty a natural benchmark exists. Hence, in the following, I consider only economies without aggregate uncertainty.

Assumption 2 (No aggregate uncertainty) *There exists $e \in \mathbb{R}_{++}$ such that for all $s \in S$, $\sum_i e^i_s = e$.*

The importance of assumption 2 comes from the following definition that can be used as a benchmark for considering underinsurance: An allocation x is a *full-insurance allocation* if x is feasible and for each $i \in \mathcal{I}$ and for all $s, t \in S$, $x^i_s = x^i_t$. Economies with no aggregate uncertainty provide a natural benchmark for studying underinsurance because a full-insurance allocation exists in an economy if and only if Assumption 2 is satisfied. Moreover, there is a natural condition for an economy with SEU investors under which all equilibrium allocations are full-insurance allocations: If all investors are ambiguity neutral then the equilibrium allocation is a full insurance allocation if and only if the investors have a common prior with full support. Note that it does not matter which prior is common (as long as the prior has full support), what the distribution of endowments is, or what the risk-preferences of the investors are. In terms of the discussion in the Introduction to this Chapter, equilibrium allocations are full insurance allocations whenever there is no disagreement about the likelihood of states or, equivalently, there is no speculative motive for investors to trade.

It is therefore of interest to see if – for a suitable counterpart of the condition that there be no disagreement about the likelihood of states – ambiguity aversion can lead equilibrium allocations to deviate from full-insurance. Two counterparts of the condition that there is no disagreement about the likelihood of states have been suggested in the literature to study this problem. The first is a weak form of agreement about the likelihood of states: Investors have multiple priors but there exists a common prior

$\pi \in \bigcap_i \Pi_i$ (see, e.g., Billot et al., 2000, for motivation). The second is a stronger form of agreement: Investors have multiple priors and all priors coincide in the sense that $\Pi_i = \Pi_j$ for all $i, j \in \mathcal{I}$. An important special case of the latter is when investors have CEU with a common, convex capacity (i.e., a common, non-additive prior) $\nu_i = \nu_j$ for all $i, j \in \mathcal{I}$ (see, e.g., Chateauneuf et al., 2000).

With the conditions on agreement in mind, the most common existing models of ambiguity aversion (MEU and CEU) suggest that ambiguity aversion does not lead to underinsurance. For the strong sense of agreement, Chateauneuf et al. (2000) demonstrate that in a CEU economy, a common, convex capacity is sufficient to ensure that all equilibria of the economy have full-insurance. Billot et al. (2000) demonstrate that the weak notion of agreement about states is both necessary and sufficient for equilibria to be in the set of full-insurance allocations for the case of MEU. The economic intuition is that, within the MEU decision model, ambiguity aversion and risk aversion both imply a motive to purchase insurance. The risk-aversion motive come from the assumption that marginal *ex-post* utility in each state (captured by the curvature of the Bernoulli utility index u) is decreasing in consumption. As a result, a risk-averse investor prefers the average of consumption in two states over different consumption in two states.[6] If, in addition, the DM has MEU with multiple priors, then full-insurance allocations serve as reference-points and the DM will purchase insurance as long as it is optimal according to *any* of the priors in the set of priors. It follows that there is not a unique price at which the DM will purchase insurance, but rather full-insurance is utility optimizing for a range of prices. This is the intuition for the well-known non-participation result of Dow and Werlang (1992), and for the finding in Billot et al. (2000) that there generally

[6]The reason is that, starting from average consumption, an increase in consumption in one state with a corresponding decrease in the other state (that holds average consumption equal) leads to increased utility in the first state and decreased utility in the second. However, with decreasing marginal utility, the increase in utility in the first state is smaller than the decrease in utility in the second state. Hence, the motive to purchase insurance.

exist multiple full-insurance equilibria when investors have MEU preferences.

However, when ambiguity is viewed from the perspective of endowments, ambiguity aversion can (and, in fact, generally does) lead to underinsurance. Theorem 3 and the discussion surrounding it already illustrates an important example: Endowment-dependent ambiguity aversion can lead to robust market collapse in an economy without aggregate uncertainty with both the strong and weak form of agreement. Moreover, robust market collapse with weak agreement is “most likely” when there is no aggregate uncertainty and, of course, if there is no aggregate uncertainty but there is idiosyncratic uncertainty then market collapse leads to underinsurance (in the sense that equilibrium allocations are not full insurance allocations). The intuition for this is straightforward: Even if there is no disagreement about the likelihood of states, when there is idiosyncratic risks investors do have asymmetric perspectives on ambiguity. Hence, ambiguity aversion leads investors to be more cautious about trade (since they are not confident about the likelihoods of states), and the resulting hedging motive is a source of disagreement (not captured by differences in beliefs) about the benefits of trading. This intuition is formalized in the following Theorem. To state the Theorem succinctly, denote for each $i \in \mathcal{I}$

$$\hat{\Pi}_i(e^i) := \arg\max_{\pi \in \Pi_i} \int_S u\left(e^i_s\right) d\pi \,, \tag{3.13}$$

which is the set of probabilities in Π_i that maximize the expected utility of the endowment of investor i. Note that as for the set $\Pi_i(e^i)$ defined in the previous Section, $\hat{\Pi}_i(e^i)$ depends on “size, shape and location” of the set of multiple priors Π_i, as well as on the risk-preferences u, and the distribution of endowments e^i. However, while $\Pi_i(e^i)$ generally always has a non-empty interior (except for the trivial case of SEU), the set $\hat{\Pi}_i(e^i)$ will often be a singleton. For example, most prominent examples of sets of multiple priors are finitely generated (i.e., have a finite number of extremal points)[7] and if Π_i is finitely generated it is straightforward to see that $\hat{\Pi}_i(e^i)$ is generically (in endowments) a singleton. The following Theorem relates full-insurance to agreements in the sets $\hat{\Pi}_i(e^i)$.

Theorem 4 (Underinsurance) *Suppose that the economy $(\Pi_i, u_i, e^i)_{i\in\mathcal{I}}$ satisfies Assumption 2. Then there exists a full-insurance equilibrium allocation if and only if $\bigcap_{i\in\mathcal{I}} \hat{\Pi}_i(e^i) \neq \emptyset$.*

Proof. The proof is given in the Appendix. ■

Theorem 4 provides both necessary and sufficient conditions for the existence of a full-insurance equilibrium allocation in an economy without aggregate uncertainty. Theorem 3 demonstrates that under both weak and strong agreement about the likelihood of states, ambiguity aversion can lead to underinsurance when ambiguity is viewed from the perspective of endowments. The following example provides a further illustration.

3.4.1 An example of underinsurance with weak agreement and two states of the world

Consider an economy with only two states of the world, $S = \{s_1, s_2\}$. There is no aggregate uncertainty, and so weak agreement about the likelihood of states would be necessary and sufficient for all equilibrium allocations to be full-insurance allocations if investors had MEU preferences. However, suppose that (1) all investors perceive ambiguity about the likelihood of states ($|\Pi_i| > 1$ for all $i \in \mathcal{I}$), and (2) investors view

[7] See Siniscalchi (2006) for a behavioral characterization of preferences admitting an MEU representation with a finitely generated set of priors.

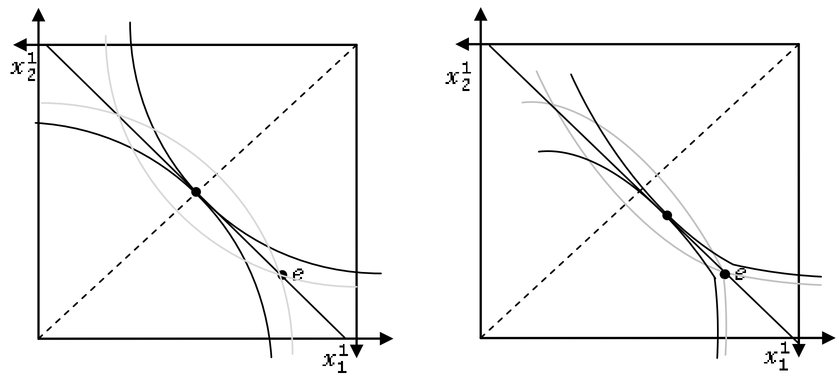
ambiguity from the perspective of their initial endowments. Then the following Corollary to Theorem 4 illustrates that with weak agreement about the likelihood of states (and hence also with strong agreement), the equilibrium allocation is a full-insurance allocation if and only if there is no idiosyncratic risk. Hence, a sunspots economy is the special case of an economy without aggregate uncertainty in which there exists a full-insurance equilibrium allocation.

Corollary 5 *Suppose that there are two states of the world* $S = \{s_1, s_2\}$ *and the economy* $(\Pi_i, u_i, e^i)_{i \in \mathcal{I}}$ *satisfies Assumption 2. If there exists* $\pi^* \in int \bigcap \Pi_i$ *then the following statements are equivalent:*

1. *There exists a full-insurance equilibrium allocation.*
2. *For all* $i \in \mathcal{I}$, $e^i_1 = e^i_2$.

Proof. The proof is given in the Appendix. ■

Since weak agreement implies strong agreement about the likelihood of states, Corollary 5 demonstrates that idiosyncratic risk is a source of underinsurance when investors view ambiguity from the perspective of endowments. Note that a similar result does not hold when ambiguity aversion is modeled with the decision model in Bewley (2002). In particular, if investor preferences have a unanimity representation and choices satisfy inertia (as in Rigotti and Shannon, 2005) it is possible to construct examples in which there is underinsurance as long as the endowment is sufficiently close to full-insurance, but as the idiosyncratic risk in the endowment increases inertia is eventually overcome and the equilibrium correspondence includes full-insurance equilibria. On the other hand, with RMEU preferences, as the idiosyncratic risk in the endowment increases there is also eventually trade (investors purchase insurance), but the hedging motive continues to act against the insurance motive and full-insurance is therefore not an equilibrium outcome. A graphical illustration in an Edgeworth box economy is provided in Figure 3.3.



3.3.1: Without ambiguity

3.3.2: With ambiguity

Figure 3.3: *Underinsurance in an Edgeworth box economy: (3.3.1) illustrates a full-insurance equilibrium in an economy in which investors are ambiguity neutral and have a common prior; (3.3.2) illustrates underinsurance when the investors are ambiguity averse.*

3.4.2 An example of underinsurance with strong agreement and multiple states of the world

Consider another example in which there is an arbitrary, finite number of states (at least two). Again, the economy has no aggregate uncertainty and there is now strong agreement about the likelihood of states. In particular, I assume that each investor $i \in \mathcal{I}$ has the following set of priors: There exists a reference probability $\pi^* \in \Delta(S)_{++}$ and an $\epsilon \in (0,1)$ such that

$$\Pi_i(\pi^*, \epsilon) = \left\{(1-\epsilon)\pi^* + \epsilon\pi \,\middle|\, \pi \in \Delta(S)\right\} \tag{3.14}$$

for all i. This is known as an ϵ-contamination (see, e.g., Epstein and Schneider, 2003) of the baseline probability π^*. The model is popular in the literature that has applied MEU to the study of markets because of its easy interpretation, simple comparative statics (in terms of ϵ) and tractability. Note also that the set $\Pi_i(\pi^*, \epsilon)$ defined in (3.14) is the core of a convex capacity, so an economy in which all investors have an ϵ-contamination with common baseline probability π^* and common ϵ is a special case of a CEU economy with common convex capacities (as studied in Chateauneuf et al., 2000). The following Corollary to Theorem 4 demonstrates that when ambiguity is captured via a common ϵ-contamination, underinsurance is generic in an economy without aggregate uncertainty.

Corollary 6 *Suppose that the economy* $(\Pi_i(\pi^*, \epsilon), u_i, e^i)_{i \in \mathcal{I}}$ *satisfies Assumption 2. Then the following statements are equivalent:*

1. *There exists a full-insurance equilibrium allocation.*
2. *For all* $i \in \mathcal{I}$ *and for all* $s, t \in S$, $e^i_s = e^i_t$.

Proof. The proof is given in the Appendix. ■

Since the condition that e is a full-insurance allocation is non-generic, Corollary 6 demonstrates that even with strong agreement about states, reference-dependent ambiguity aversion generically leads to underinsurance. This underinsurance is not due to any particular assumption about risk-preferences (all investors are risk-averse and they can have identical or very different risk-preferences). It is also not due to disagreement about the likelihood of states: All investors have the same beliefs over states (although the beliefs are non-additive). It is due to asymmetric responses to ambiguity which are determined by the asymmetric endowment allocations. Comparative statics in terms of ϵ are insightful to get a better sense for the nature of underinsurance. It is straightforward to verify that as $\epsilon \to 0$, the equilibrium allocation converges to the full-insurance allocation that would obtain in the SEU economy with common prior π^*. In terms of the definition of comparative ambiguity aversion in Chapter 2, an increase in ϵ is equivalent to an increase in the ambiguity perceived by investors. For any non-full-insurance endowment e, as ϵ increases from 0 there exists a critical $0 < \bar{\epsilon} < 1$ such that there is a market collapse (in the sense of the previous Section) for all $\epsilon \geq \bar{\epsilon}$. Hence, for $\epsilon \geq \bar{\epsilon}$ there is a sufficient ambiguity aversion for no-trade to occur, while for $\epsilon < \bar{\epsilon}$ there is trade for insurance purposes but not enough to lead to a full-insurance allocation. Hence, while no-trade may require substantial ambiguity about the likelihood of states, an arbitrarily small (but strictly positive) amount of ambiguity is sufficient for underinsurance. This illustrates the knife-edged nature of a key implication of SEU in exchange under uncertainty, a point that is missed in the existing literature which has focused on ambiguity aversion only when ambiguity aversion reinforces the insurance motive for trade.

3.4.3 Underinsurance with aggregate uncertainty

In an economy without aggregate uncertainty, full-insurance is a natural benchmark with respect to which absolute notion of underinsurance can be defined. When there is aggregate uncertainty it is less clear what constitutes a reasonable benchmark. One alternative is to consider a SEU economy. It is another stark implication of SEU that if investors have a common prior (with full support), the set of Pareto optimal allocations is independent of the prior (i.e., does not depend on what exactly the common prior is), and depends only on the risk preferences of the investors. For any given endowment and risk-preferences of the investors, one could therefore define the Pareto optimal set of a common prior SEU economy as a set of "maximal" insurance allocations. This set is well-defined since it is independent of the common prior, and the definition coincides with the set of full-insurance allocations when there is no aggregate uncertainty. Formally, the set of maximal insurance allocations is then defined as the set of allocations x that are (1) feasible, and (2) satisfy that for all $i, j \in \mathcal{I}$ and all $s, t \in S$

$$\frac{u_i'\left(x_s^i\right)}{u_i'\left(x_t^i\right)} = \frac{u_j'\left(x_s^i\right)}{u_j'\left(x_t^i\right)}. \tag{3.15}$$

Using the notion of maximal insurance, Corollary 5 and Corollary 6 have natural counterparts in economies with aggregate uncertainty. For example, in an economy with two states of the world and weak agreement about the likelihood of states, there exists a maximal insurance allocation if and only if the endowment is a maximal insurance allocation. This is an immediate analog for Corollary 5 for economies with aggregate uncertainty. Likewise, if priors are described by a common ϵ-contamination of a common baseline prior π^*, then in an economy with multiple states and aggregate uncertainty, there exists a maximal insurance equilibrium allocation if and only if the endowment is a maximal insurance allocation. Compare this to the findings in Chateauneuf et al. (2000), who show that if investors have CEU with a common, convex capacity, equilibrium allocations are always maximal insurance allocations. This further illustrates the point that reference-dependent ambiguity aversion is a source of underinsurance that is not recognized with models of ambiguity aversion in which ambiguity is perceived from the perspective of full-insurance.

The result in Chateauneuf et al. (2000) illustrates that a particular type of strong agreement about beliefs (common, convex capacities) is sufficient for the existence of maximal insurance equilibria in the context of MEU preferences. However, Strazlecki and Werning (2011) illustrate that weak agreement is generally not sufficient for the existence of maximal insurance equilibria when investors have MEU preferences. In fact, Strazlecki and Werning (2011) provide an example that demonstrates something considerably stronger. They consider an economy with with two investors and three states of the world. One investor has MEU preferences and the other has SEU with a prior contained in the set of priors of the MEU investor. In this economy they show that the set of interior Pareto optimal allocations does not intersect with the set of comonotonic allocations. An allocation x is comonotonic (with the aggregate endowment) if for all $i \in \mathcal{I}$ and all $s, t \in S$, $e_s \geq e_t$ implies $x^i_s \geq x^i_t$. Maximal insurance allocations are comonotonic and since the equilibria in the economy they construct are interior and Pareto optimal, they show therefore that there does not exist a maximal insurance equilibrium. They interpret their example as an illustration of underinsurance due to ambiguity aversion. However, there is an important distinction between the underinsurance in their example and the underinsurance demonstrated for RMEU preferences above. In the example in Strazlecki and Werning (2011), there is underinsurance regardless of the endowment. In fact, if underinsurance is interpreted to mean that investors purchase less insurance than some given benchmark (e.g., the set of comonotonic or maximal insurance allocations), then their example can as easily be used to demonstrate overinsurance as underinsurance. On the other hand, when ambiguity is viewed by investors from the perspective of endowments, underinsurance is directly related to lack of trade. Hence, the empirical predictions of this behavioral hypothesis are closer to the prediction of incomplete market models.

3.5 Conclusion

Equilibrium analysis based on the Maxmin Expected Utility model suggests that ambiguity aversion has limited implications for asset markets. In this Chapter, I have shown that the Reference-Dependent Maxmin Expected Utility model can be used to show that affects of ambiguity aversion are substantially more robust. In particular, ambiguity aversion can not only lead to underinsurance, non-participation or market collapse for a knife-edged set of initial conditions, but generally for any distribution of endowments.

In particular, I show that how ambiguity aversion affects markets outcomes depends not only on how much ambiguity investors perceive, but also on the perspective from which they view ambiguity. The analysis of this Chapter therefore complements and extends on the existing literature on ambiguity aversion in asset markets by clarifying that results attained under the assumption of MEU hold much more generally when reference-dependence of ambiguity aversion is permitted.

CHAPTER 4

RECURSIVE REFERENCE-DEPENDENT AMBIGUITY AVERSION

4.1 Introduction

In this Chapter I extend the Reference-Dependent Maxmin Expected Utility (RMEU) decision model introduced in Chapter 2 to a dynamic setting. An extension of RMEU preferences to dynamic choice settings is desirable because many applications to problems of choice under uncertainty – such as in finance and macroeconomics – also have an intertemporal dimension. It is also of particular interest for any model of decision making under uncertainty to address how decision makers learn from information revealed over time. Subjective expected utility (SEU) comes equipped with an essentially in-built theory of Bayesian learning, but the same does not hold for models that allow for a behavioral distinction between risk and ambiguity attitudes. For example, in models of ambiguity aversion with multiple priors it is not obvious what constitutes the appropriate generalization of Bayesian updating. Prior-by-prior Bayesian updating is an obvious candidate, but a number of alternative updating rules are also plausible. For example, when there are multiple priors, new information can lead to updating of existing priors but could also suggest a selection from among the set of priors according to some criterion of how well they "fit" to the information revealed. A study of the axiomatic foundations of dynamic decision models of ambiguity averse preferences is therefore important to shed light on the behavioral content of different updating rules.

The literature on dynamic decision models of ambiguity averse preferences also indicates that variety in terms of updating rules corresponds to a particularly interesting trade-off that arises when ambiguity aversion is accommodated in dynamic choice settings. Epstein and LeBreton (1993) show that a decision model that allows for ambiguity averse behavior (consistent with the Ellsberg (1961) paradox) can not simultaneously also satisfy consequentialism and dynamic consistency. In a decision tree, consequentialism is the requirement that conditional preferences at a particular decision node should not depend on parts of the tree that are no longer payoff relevant (i.e., the past or parts of the tree that can no longer be reached). Dynamic consistency is the requirement that a plan which is optimal *ex-ante* should also be optimal *ex-post* to the realization of any non-null event (i.e., any event which the decision maker initially perceived as possible). Since both consequentialism and dynamic consistency have considerable normative appeal in dynamic settings, this impossibility result highlights a clear trade-off that must be addressed by any dynamic decision model that aims to integrate ambiguity averse behavior. A special interest in an axiomatic analysis of dynamic models of ambiguity aversion is therefore to highlight how different updating rules correspond to different emphases in the trade-off between ambiguity aversion, consequentialism and dynamic consistency.

A number of dynamic decisions models that allow for ambiguity averse behavior have been proposed in the literature (see Siniscalchi, 2009a, for an excellent summary of dynamic choice under ambiguity). Most build on the multiple priors (or Maxmin Expected Utility (MEU)) model of Gilboa and Schmeidler (1989). A crucial feature of the RMEU model introduced in Chapter 2 is that reference-effects are directly related to the perception of and response to sources of uncertainty, and relative to MEU the reference-effect is due solely to a change in the perspective from which the decision maker views ambiguity. As a result, I conjecture that any one of the numerous decision models that extend MEU to dynamic choice problems could be adapted to study the implications of intertemporal reference-dependent ambiguity aversion. In this Chapter I provide one important illustration by generalizing the Recursive Multiple-Priors (RMP) model in Epstein and Schneider (2003) to characterize Recursive Reference-Dependent

Maxmin Expected Utility (RRMEU).

Epstein and Schneider (2003) study the preferences of a decision maker in a given decision-event tree, with conditional preferences at each date-event pair (or after every *history*). The decision maker has Maxmin Expected Utility (MEU) preferences at every decision node, so that both the initial and conditional preferences satisfy the axioms of Gilboa and Schmeidler (1989). To provide a recursive representation, Epstein and Schneider (2003) then impose a connection between the conditional preferences at different histories. In particular, they assume that preferences satisfy consequentialism and that they are dynamically consistent with respect to the *given* decision tree. Hence, they state the following objective of their dynamic extension of the MEU model.

"We view intertemporal utility as a summary of dynamic behavior in settings where complete commitment to a future course of action is not possible. Accordingly, foundations are provided by axioms imposed on the entire utility (or preference) process, rather than merely on initial utility. Importantly, axioms do not simply apply to conditional preference after each history separately. To ensure that dynamic behavior is completely determined by preferences, a connection between conditional preferences is needed. This connection is provided by dynamic consistency."

The axioms provided by Epstein and Schneider (2003) characterize a recursive version of the MEU model. Preferences at each decision node have an MEU representation and the decision maker updates priors in the set of priors by applying Bayes rule prior-by-prior. In addition to the usual conditions on the set of priors (e.g., convexity), priors also satisfy a dynamic condition that is defined with respect to the given decision tree. This condition – called *rectangularity* – implies that preferences have a recursive structure. However, rectangularity is satisfied with respect to a given decision tree only and the same set of priors need not be rectangular with respect to an alternative decision tree. Hence, the same preferences will generally not be dynamically consistent with respect to an alternative structure on the resolution of uncertainty. This is the cost of accommodating ambiguity aversion in their setting. They retain full consequentialism, but study preferences that are dynamically consistent only with respect to a given decision tree. The latter condition is a reflected in a restriction on the ambiguity perceived by the decision maker *ex-ante*, namely the requirement that the set of priors be rectangular. Nevertheless, their framework provides a recursive model of ambiguity averse behavior that is both flexible (in terms of modeling ambiguity attitudes) and tractable (in terms of its dynamic properties).[1]

In applications of a decision model to dynamic choice settings, a recursive structure is often crucial in order to keep the analysis tractable. Moreover, despite the existence of sensible alternatives, prior-by-prior Bayesian updating remains perhaps the most natural starting point for a theory of learning in decision models with multiple priors. The RMP model therefore seems to strike a sensible balance given the trade-offs highlighted by Epstein and Wang (1994). As a result, I follow the framework of Epstein and Schneider (2003) in this Chapter to provide an extension of the RMEU model to dynamic choice settings. I provide an axiomatic characterization of a Recursive Reference-Dependent Maxmin Expected Utility (RRMEU) model which represents the natural generalization of the RMP model to reference-dependent perceptions of ambiguity. I then illustrate the potential for applications of RRMEU by extending the intertemporal asset pricing model in Epstein and Wang (1994) to the case when ambiguity is viewed from the perspective of the endowment process. In the context of the RMP model, Epstein and Wang (1994) demonstrate that ambiguity aversion can lead to excess price volatility in an intertemporal asset pricing model, but it remains unclear from their analysis how

[1]Epstein and Schneider (2003) also provide a number of examples to illustrate the rich potential for modeling ambiguity averse behavior within their approach that are easily extended to the RRMEU model introduce in this Chapter.

frequent and economically relevant the price indeterminacy they identify is. Using the RRMEU model, I provide a more complete characterization of the price process in an intertemporal asset pricing model with an ambiguity averse investor, and illustrate that price indeterminacy is ubiquitous when ambiguity is viewed from the perspective of the endowment process.

4.2 A Dynamic Decision Problem

The decision making environment is as in Epstein and Schneider (2003), but I adapt their notation to make it consistent with the remainder of the Dissertation. I concentrate on the infinite horizon case, but it should be clear that the method of proof is easily adapted to a finite case (requiring somewhat less structure as in Epstein and Schneider (2003)).

Time is discrete and varies over $\mathcal{T} = \{0, 1, ..., \infty\}$. Information about the state space S is represented by a filtration $\{\mathcal{G}_t\}_{t=0}^{\infty}$, where for each finite t, $\mathcal{G}_t$ corresponds to a finite partition and $\mathcal{G}_t(s)$ denotes the the part containing state $s \in S$. Hence, if s is the true state, the decision maker (DM) knows $\mathcal{G}_t(s)$ at time t. Assume that $\mathcal{G}_\infty$ is the σ-algebra generated by $\bigcup_0^\infty \mathcal{G}_t$. The set of outcomes in each time period is denoted $X \subset \mathbb{R}$ and the set of lotteries over X having finite support is denoted $\mathcal{P}$. Hence, the objects of choice are acts $f = (f_t) \in \mathcal{F}$, where $f_t : S \to \mathcal{P}$ is $\mathcal{G}_t$ measurable for all $t = 1, ..., \infty$. Note that $\mathcal{F}$ is a mixture space under the usual point-wise mixture operation. Denote by $\mathcal{P}^\infty$ the set of constant acts (i.e., acts for which the outcome depends on time and the realization of objective randomization, but not on the state of the world s).

The DM has preferences on $\mathcal{F}$ at each time-event pair $(t, s) \in T \times S$, and for each reference-point $r \in \mathcal{F}$. The reference-point affects preferences *ex-ante* and I therefore assume that the reference-point is given at time 0 and does not change. A crucial axiom that leads to the recursive utility is dynamic consistency. Given dynamic consistency, it would be equivalent to assume that the reference-point in $(t+1, s)$ is the continuation of the plan formed at (t, s) for $(t+1, s)$, since dynamic consistency is exactly the requirement that *ex-ante* optimal plans should also be *ex-post* optimal. Hence, I avoid the unnecessary notational complications that would arise from allowing for this type of endogenous evolution of the reference-point because observationally the two models are equivalent. Formally, therefore, given an *ex-ante* reference point $r \in \mathcal{F}$, the DM has preferences $\succeq^r_{t,s}$ at the time event pair (t, s). Axioms are imposed on the collection of preferences $\{\succeq^r_{t,s}\}_{(t,s)\in T\times S}$ for a given reference-point $r \in \mathcal{F}$, and on the collection of these collections for each reference-point: $\left(\{\succeq^r_{t,s}\}_{(t,s)\in T\times S}\right)_{r\in\mathcal{F}}$. As in the static decision model, the restriction on the collection of preferences given different reference-points can be viewed as a comparative statics assumptions. However, in the dynamic model these Axioms also have substantive implications. In particular, the observational equivalence of a model with a fixed reference-point, and one in which the reference-point changes with the optimal plan, does not necessarily hold when the dynamic counterpart of the equivalence of unambiguous preferences (EUP) is not imposed. For simplicity, I concentrate only on preferences for which a suitably adapted version of EUP holds.

4.2.1 Axioms

The following axioms will be satisfied for all reference-points $r \in \mathcal{F}$.

Axiom 14 (Conditional Preference (CP)) *1. $\succeq^r_{t,s}$ coincides with $\succeq^r_{t,s'}$ if $\mathcal{G}_t(s) = \mathcal{G}_t(s')$.*

2. If $f_\tau(s') = f'_\tau(s')$ for all $\tau \geq t$ and $s' \in \mathcal{G}_t(s)$, then $f \sim^r_{t,s} f'$.

Axiom 14 is a form of consequentialism. It states that conditional preferences depend only on uncertainty that has not yet been resolved, and on continuation acts. Consequentialism has considerable normative appeal in dynamic choice settings, and is crucial for a recursive structure. Of course, the impossibility result of Epstein and LeBreton (1993) demonstrates that for decision making under ambiguity consequentialism can be problematic because of a trade-off with dynamic consistency. As in Epstein and Schneider (2003), I only study preferences that satisfy CP (i.e., consequentialism) and restrictions are imposed on the dynamic consistency of preferences. An alternative model of dynamic ambiguity averse behavior (also extending on the atemporal MEU decision model of Gilboa and Schmeidler (1989)) that is fully dynamically consistent but violates consequentialism is presented by Hanany and Klibanoff (2007).[2]

The main result in this Chapter provides an axiomatization of Recursive RMEU preferences. For any reference-point $r \in \mathcal{F}$ the following axiom therefore simply imposes that conditional preference at any given time-event pair are RMEU preferences given the reference-point r. The interpretation of the axiom is analogous to the interpretation in the static case presented in Chapter 2 and I therefore refer to the discussion in that Chapter for more details.

Axiom 15 (Reference-Dependent Maxmin Expected Utility (RMEU)) *The following holds for all* $(t, s) \in T \times S$.

1. $\succeq^r_{t,s}$ *is complete and transitive.*
2. *For all* $f, f' \in \mathcal{F}$ *and* $x, x' \in \mathcal{P}^\infty$, *and for all* $\alpha \in (0, 1)$, *if* $\alpha f + (1-\alpha)x \succeq^r_{t,s} \alpha f' + (1-\alpha)x$ *then* $\alpha f + (1-\alpha)x' \succeq^r_{t,s} \alpha f' + (1-\alpha)x'$.

[2] I conjecture that the set up in Hanany and Klibanoff (2007) can also be extended to study fully dynamically consistent RMEU, but analysis of this conjecture is left for future research.

3. *For all* $f, f', f'' \in \mathcal{F}$*, the sets* $\{\alpha \in [0,1] | \alpha f + (1-\alpha) f' \succeq^r_{t,s} f''\}$ *and* $\{\alpha \in [0,1] | f'' \succeq^r_{t,s} \alpha f + (1-\alpha) f'\}$ *are closed.*

4. *For all* $f, f' \in \mathcal{F}$*, if* $(f_0(s'), ..., f_\infty(s')) \succeq^r_{t,s} (f'_0(s'), ..., h'_\infty(s'))$ *for all* $s' \in S$*, then* $f' \succeq^r_{t,s} f$*.*

5. *For all* $f, f' \in \mathcal{F}$*, if* $f \sim^r_{t,s} f'$*, then* $\alpha f + (1-\alpha) f' \succeq^r_{t,s} f$ *for all* $\alpha \in (0,1)$*.*

6. *For all* $f, f' \in \mathcal{F}$*,* $f \succeq^r_{t,s} f'$ *if and only if* $\alpha f + (1-\alpha) r \succeq^r_{t,s} \alpha f' + (1-\alpha) r$ *for all* $\alpha \in (0,1)$*.*

The remaining axioms are more directly related to the dynamic structure of the decision problem. The following axiom imposes restrictions only on preferences on the subset of constant act, and is therefore not central to the issue of ambiguity that is the main purpose of the axiomatic characterization. The axiom, which is introduced in Epstein and Schneider (2003), is responsible primarily for the additive separation over time that is familiar from dynamic models of SEU.

Axiom 16 (Risk Preference (RP)) *For any* $y \in \mathcal{P}^\infty$ *and all* $x, x', x'', x''' \in \mathcal{P}$*, if*

$$y_{-\{\tau,\tau+1\},x,x'} \succeq^r_{t,s} y_{-\{\tau,\tau+1\},x'',x'''}$$

holds for some $(t,s) \in T \times S$ *and some* $\tau \geq t$*, then it holds for all* $(t,s) \in T \times S$ *and all* $\tau \geq t$*.*

For simplicity, I assume that all events are nonnull at time 0. This axiom is without loss of generality because if some component of the partition defined by some $\mathcal{G}_t$ were null according to time 0 preferences, it could be discarded and all axioms applied to the (suitably adjusted) smaller state space (refer to Epstein and Schneider, 2003). To state the axiom, define nullity in the standard way: For any $\tau > t$, say that the event A is $\mathcal{G}_t$ is $\succeq^r_{t,s}$-null if

$$h'(.) = h(.) \text{ on } A^c \;\Rightarrow\; h' \sim_{t,s} h\,. \tag{4.1}$$

Axiom 17 (Full Support (FS)) *Each non-empty event in $\bigcup_{t=0}^{\infty} \mathcal{G}_t$ is $\succeq^r_{0,s}$-nonnull.*

The crucial axiom that ties together the conditional preferences of a decision maker at different time-event pairs for a given reference point is dynamic consistency. This axiom is discussed at length in Epstein and Schneider (2003), who provide normative and analytic justifications for the axiom. It states (essentially) that a plan that is optimal *ex-ante* should be optimal also from the point of view of conditional preferences *ex-post*. The axiom provides the crucial additional structure on preferences required for a recursive representation, but the motivation for the axiom is analogous to the motivation given for the RMP model and I therefore defer to Epstein and Schneider (2003) for more details.[3]

Axiom 18 (Dynamic Consistency (DC)) *For all $(t,s) \in T \times S$ and for all $f, f' \in \mathcal{F}$, if $f_\tau = f'_\tau$ for all $\tau \leq t$ and $h \succeq^r_{t+1,s'} f'$ for all $s' \in S$, then $f \succeq^r_{t,s} f'$; and the latter ranking is strict if the former ranking is strict at every s' in a $\succeq^r_{t,s}$-nonnull event.*

Finally, two axioms are required given any reference-point $r \in \mathcal{F}$ to deal with technical difficulties that arise because the range of an act $f : T \times S \to \mathcal{P}$ need not be finite. The first imposes the existence of a best and worst lottery, while the second requires a

[3]It is worth noting, however, that in the context of the reference-dependent decision model, dynamic consistency also imposes consistency conditions across reference-points and the axiom therefore has additional bight in the current setting. These consistency conditions across reference points are apparent in the proof of the representation result.

form of impatience. Together these axioms impose that the utility profiles from constant acts are bounded. They would not be required for a model of recursive RMEU preferences with some finite time horizon $1 < T < \infty$.[4]

Axiom 19 (Best and Worst (BW)) *For each* $(t, s) \in T \times S$*, there exist lotteries* $x^*, x^{**} \in \mathcal{P}$ *such that*

$$(x^{*}*)_0^\infty \succeq_{t,s}^r (x)_0^\infty \succeq_{t,s}^r (x^*)_0^\infty \tag{4.2}$$

for all $x \in \mathcal{P}$.

Axiom 20 (Impatience (IMP)) *For all* $(t, s) \in T \times S$*,* $x \in \mathcal{P}$ *and acts* $f, f', f'' \in \mathcal{F}$*, if* $f'' \succ_{t,s}^r f \succ_{t,s}^r f'$ *and* $f^n = (f_0, ..., f_n, x, x, ...)$*, then* $f'' \succ_{t,s}^r f^n \succ_{t,s}^r f'$ *for all sufficiently large n.*

In addition to the axioms given a particular reference point, two axiom are also introduced to tie together the collection of preference relations given different *ex-ante* reference-points. To this end, define again an unambiguous preference relation: If, for all $f'' \in \mathcal{F}$ and for all $\alpha \in (0, 1)$, $\alpha f + (1 - \alpha) f'' \succeq_{t,s}^r \alpha f' + (1 - \alpha) f''$, write $f \succeq_{t,s}^{*r} f'$ and say that f is unambiguously preferred at time-event pair (t, s) to f'. I impose only that unambiguous preferences be equivalent at time 0. This axiom is sufficient to ensure that endogenous evolution of reference-points does not introduce issues of dynamic inconsistency arising from an individual "switching" between the collections of internally dynamically consistent preferences given a particular reference point. It therefore has the same appeal in an axiomatic study of dynamic choice as dynamic consistency within a collection of dynamic preferences (for a given reference point), which is discussed at length in Epstein and Schneider (2003).

[4]Note that a proof of the finite version of Recursive RMEU preferences can also be obtained by following Maccheroni et al. (2006), because of the relation between RMEU preferences and variational preferences. Following their method requires an unboundedness axiom which clearly violates the Best and Worst axiom stated below. Details are left to the interested reader.

Axiom 21 (Equivalent Unambiguous Preferences (EUP)) *For all $r_1, r_2 \in \mathcal{F}$, for all $s \in S$, and for all $f, f' \in \mathcal{F}$, $f \succeq^{*r_1}_{0,s} f'$ if and only if $f \succeq^{*r_2}_{0,s} f'$.*

Finally, the Reference-Translation (RT) axiom discussed already in Chapter 1 is also introduced. The motivation is as for the atemporal model, but in a dynamic setting the axiom is assumed to hold at all date-event pairs.

Axiom 22 (Reference Translation (RT)) *For all $r \in \mathcal{F}$, for all $f, g \in \mathcal{F}$, for all $(t, s) \in T \times S$, and for all $x \in \mathcal{P}^\infty$, and all $\alpha \in (0, 1)$,*

$$f \succeq^{r}_{t,s} g \;\Leftrightarrow\; \alpha f + (1-\alpha)x \succeq^{\alpha r + (1-\alpha)x}_{t,s} \alpha g + (1-\alpha)x \tag{4.3}$$

4.2.2 Representation

To state the representation theorem, let $\Delta(S, \mathcal{G}_t)$ denote the set of probability measures on the Σ-algebra $\mathcal{G}_t$. Measures in $\Delta(S, \mathcal{G}_\infty)$ are restricted to be finitely additive, and $\Delta(S, \mathcal{G}_t)$ is equipped with the weak topology induced by the set of all bounded measurable real-valued functions. A measure $\pi \in \Delta(S, \mathcal{G}_t)$ has full local support if $\pi(A) > 0$ for every $\emptyset \neq A \in \bigcup_{t=0}^{\infty} \mathcal{G}_t$. Let Π be a set of priors on $(S, \mathcal{F}_\infty)$. Then for each s, define the set of Bayesian of updates of Π by

$$\Pi_t(s) = \{\pi(\cdot|\mathcal{G}_t)(s) | \pi \in \Pi\} \ . \tag{4.4}$$

Denote by $\pi^{+1}(\cdot|\mathcal{G}_t)(s)$ the restriction of $\pi(\cdot|\mathcal{G}_t)(s)$ to $\mathcal{G}_{t+1}$, and define the set of one-step-ahead measures by

$$\Pi^{+1}_t = \left\{\pi^{+1}(\cdot|\mathcal{G}_t)(s) | \pi \in \Pi\right\} \ . \tag{4.5}$$

Definition 14 (Rectangularity) *A set of priors* Π *is* $\{\mathcal{G}_t\}$*-rectangular if for all t and s,*

$$\Pi_t(s) = \left\{ \int \pi_{t+1}(s')(\cdot)dm(s') | \pi_{t+1}(s') \in \Pi_t(s') \, \forall s' \; m \in \Pi_t^{+1}(s) \right\} . \qquad (4.6)$$

Rectangularity is a restriction on the set of priors in the RRMEU model characterized by the preceding axioms. As Epstein and Schneider (2003) argue, rectangularity requires that the set of initial priors be "sufficiently large" in the following sense. Begin with *any* collection of one-step-correspondences, defined as correspondences $P_t^+ : S \rightarrow \Delta(S, \mathcal{G}_{t+1})$ where P_t^+ is $\mathcal{G}_t$-measurable for each t. Each measure in $P_t^+(s)$ can be viewed as a one-step-ahead measure describing beliefs about how uncertainty will resolve in the next period. Then there exists a unique set of priors Π whose 1-step-ahead-conditionals exactly correspond to the P_t^+, and that set can be constructed by backward induction. The set Π then satisfies rectangularity and is the smallest rectangular set to imply the one-step ahead conditionals P_t^+. Hence, it is observationally equivalent to model the evolution of beliefs using an arbitrary collection of one-step-ahead-correspondences, or to model the evolution of beliefs in terms of the constructed rectangular set of priors they imply, and then apply Bayes rule prior-by-prior for updating. Hence, rectangularity is not particularly restrictive in terms of modeling ambiguity attitudes *per se*. The restriction comes from the fact that the definition of a rectangular set of priors is given with respect to the particular filtration $\mathcal{G}_t$. The same set of priors need not be rectangular given a different filtration, which corresponds precisely to the fact that dynamic consistency is required only with respect to the given filtration. This is the cost of accommodating ambiguity, consequentialism and dynamic consistency in the RMP model of Epstein and Schneider (2003), and explains how the recursive RMEU model is also able to escape the impossibility result in Epstein and LeBreton (1993).

The following theorem (essentially) shows that if a collection of preferences at each time-event pair is an RMEU preference, then it satisfies dynamic consistency if and only if the collection of sets of priors is rectangular and obtained from the period 0 set of priors by prior-by-prior Bayesian updating. It therefore gives a recursive model of dynamic RMEU preferences, with an intuitive model of learning that preserves maximal ambiguity (in the sense that there is no endogenous selection among priors).

Theorem 5 (Recursive RMEU) *The following statements are equivalent:*

1. $\left(\left(\succeq^r_{t,s}\right)_{(t,s)\in\mathcal{T}\times S}\right)_{r\in\mathcal{F}}$ *satisfies CP, RMEU, RP, FS, DC, BW, IMP, EUP and RT.*
2. *There exists a weak*-closed, convex and* $\mathcal{G}_t$*-rectangular set of priors* Π*, with all measures in* Π *having full local support; a* $0 < \beta < 1$ *and a mixture-linear, non-constant* $u : \mathcal{P} \to \mathbb{R}$*, where* $\max_{\mathcal{P}} u$ *and* $\min_{\mathcal{P}} u$ *exist, such that for all* $r \in \mathcal{F}$ *and for every* t *and* s*,* $\succeq^r_{t,s}$ *is represented on* $\mathcal{F}$ *by* $V^r_t(\cdot, s)$*, where*

$$V^r_t(f, s) = \min_{m\in\Pi_t(s)} \int \sum_{\tau\geq t} \beta^{\tau-t} \left[u(f_t) - u(r_t)\right] dm\,. \tag{4.7}$$

Moreover, β *is unique,* Π *is unique on* $\bigcup_1^\infty \mathcal{G}_t$*, and* u *is unique up to a positive affine transformations.*

Proof. The proof is given in the Appendix. ■

Note that the RMP model is the special case of the RRMEU model when the reference point $r \in \mathcal{P}^\infty$, and a dynamic SEU representation is characterized by extending independence from reference-independence to all acts in $\mathcal{F}$. Perhaps the most important aspect of the RMEU decision model is the sense in which it is dynamically consistent. In fact, dynamic consistency clearly restricts the behavior that can be implied by ambiguity in the recursive representation of dynamic RMEU preferences. These restrictions are embodied in the rectangularity condition, and they are the price that must be paid for the tractability of a recursive structure. However, the advantage of a recursive structure is obvious. To provide an illustration, I study a simple intertemporal asset pricing model in the following Section and show that ambiguity aversion can lead to ubiquitous price indeterminacy. Epstein and Wang (1994) study dynamic asset pricing using the RMP model and illustrate the possibility of price indeterminacy due to ambiguity aversion. However, as they themselves note, it is difficult to judge how frequent or economically relevant the price indeterminacy in their setting is. Moreover, price indeterminacy can result also from a model with a representative consumer who has SEU preferences when the felicity function of the SEU functional is non-differentiable. Although Epstein and Wang (1994) discuss conceptional differences between the latter source of price indeterminacy and the price indeterminacy implied by ambiguity aversion, the differences are not empirical in nature. Both models imply price indeterminacy when the consumption process happens to coincide with kinks in the utility functional describing preferences. Since these kinks are fixed independent of the consumption process, price indeterminacy is somewhat coincidental. Hence, it is easy to conclude that the excess price volatility due to ambiguity aversion is not empirically relevant. However, as I show in the following, if ambiguity is viewed by the representative agent from the perspective of the endowment process, price indeterminacy is ubiquitous. The following application of RRMEU preferences therefore illustrates that ambiguity aversion can provide a robust explanation for excess price volatility and complements the static analysis of the implications of reference-dependent ambiguity aversion in Chapter 3.

4.3 An Intertemporal Asset Pricing Model

The framework and results in this Section build closely on Epstein and Wang (1994), which is itself a natural extension of the classic Lucas (1978) representative agent pure exchange economy to accommodate for ambiguity averse behavior. The example illustrates that RRMEU preferences are sufficiently tractable to lead to new economic insights regarding the affects of ambiguity aversion on decision making under uncertainty when there is also an intertemporal dimension to the decision problem. Where possible I follow the notation in Epstein and Wang (1994), and make adjustments only to make the exposition consistent with the remainder of the Dissertation.

4.3.1 The economy

There is one representative agent. The agent has RRMEU preferences described in further detail below. Time is infinite and discrete and varies over $t = 1, ..., \infty$. In each period there is a set of states of the world S, assumed to be a compact metric space with Borel σ-algebra $\mathcal{B}(S)$. At each date t a state of the world $s_t \in S$ is realized and, hence, the evolution of states in the economy lies in the underlying measurable space denoted S^∞ with the product Borel σ-algebra $\mathcal{B}(S^\infty)$. For a given state of the economy $s \in S^\infty$, the history of states up to (and including) some time period $t \geq 1$ is denoted $s^t = (s_1, ..., s_t)$, and the collection of such partial histories up to (and including) time t is denoted S^t. A real valued process $\{X_t\}$ defined on S^∞, $X_t : S^\infty \to \mathbb{R}$ for each t, is *adapted* if X_t is $\mathcal{B}(S^t)$-measurable for all t, and continuous if X_t is continuous for all t. The adapted process $X = \{X_t\}$ is Markovian if for each t and $s_t \in S$, $X_t(\cdot, s_t)$ is constant on S^{t-1} and time-homogeneous if, in addition, there exists $\bar{X} : S \to \mathbb{R}$ such that $X_t = \bar{X}$ for all t. I refer to a real valued process that is adapted, continuous, Markovian and time-homogeneous as a *basic* process.

In the economy there is a single perishable consumption good at each date and in each state. Denote by $\mathcal{D}$ the set of all adapted consumption processes $c = \{c_t\}$ such that (1) c is continues, adapted and positive valued for all t, and (2) $||c|| =$

$\sup_t \sup_{s^t} |c_t(s^t)|/b^t < \infty$, for some $b \geq 1$. The constant real value b in condition (2) provides a bound on the average rate of consumption growth given the consumption process c. Also, for a consumption process c, denote by $c^{t+}(s^{t-1})$ the continuation consumption process given the history s^{t-1} at time t, i.e., $c^{t+}(s^{t-1}) := \{c_\tau(s^{t-1}, \cdot)\}_{\tau=t}^{\infty}$.

Preferences are defined on the adapted consumption processes in $\mathcal{D}$. Utility has four primitives, the first three are as in Epstein and Wang (1994) and the last is the reference-point which is novel. The first primitive used to define utility is a probability kernel correspondence that describes the evolution of beliefs of the agent in terms of one-step-ahead correspondences. Beliefs evolve according to a time-homogeneous Markov structure in which the beliefs over S at $t+1$ are described at date t by the correspondence $\Pi : S \to \Delta(S)$, where $\Delta(S)$ denotes the set of Borel probability measures on S.[5] The probability kernel correspondence Π is assumed to be non-empty, continuous, compact-valued, convex-valued and to have full support (in the sense that each measure in Π has full support on S). Recall from the discussion preceding Theorem 5 that such a probability kernel correspondence implies a unique rectangular set of priors that generates Π via prior-by-prior Bayesian updating. Hence, the probability kernel correspondence Π represents a convenient and appropriate way to capture ambiguous beliefs corresponding to the recursive RMEU decision model. In particular, any class of preference relations satisfying the Axioms 14-22 in the previous Section can be represented by way of a discount factor $0 < \beta < 1$ a felicity function u and either (1) a rectangular set of priors Π_0 or (2) a set of one-step-ahead correspondences Π_t. For the following, the representation of beliefs in terms of one-step-ahead correspondences is both natural and convenient. Ambiguity neutrality corresponds to the case when Π is a (unique valued) probability kernel. This description of beliefs of the agent also corresponds exactly to the one given in Epstein and Wang (1994), and they provide a number of examples of

[5]Note that under the weak convergence topology, $\Delta(S)$ is a compact metric space.

probability kernels correspondences that correspond to interesting specifications of sets of priors (see Section 2.3 of Epstein and Wang, 1994).

The second primitive is a discount factor $\beta \in (0,1)$, and assumed to satisfy the condition that $\beta b < 1$ (recall that b is a bound on the rate of growth of consumption process in $\mathcal{D}$). This condition ensures that utility exists given that a counterpart to the Best/Worst act axiom will not be invoked (since this rules out a number of felicity functions that are commonly applied in the literature). The third primitive is a felicity function $u : \mathbb{R}_+ \to \mathbb{R}_+$ that describes instantaneous utility in each time period and each state, and that is assumed to be continuous, twice continuously differentiable on $\mathbb{R}_{++}$, increasing, concave, and to satisfy the normalization $u(0) = 0$, as well as the Inada condition $\lim_{c\to 0} u'(c) = \infty$, and the following growth condition: There exist constant real values $k_1, k_2 > 0$ such that $u(x) \leq k_1 + k_2 x$ for all $x \in \mathbb{R}_+$. As with the Assumption $b\beta < 1$, the latter condition is required to ensure that utility exists. In the characterization of RRMEU this is ensured by the existence of best and worst alternatives, but this assumption seems overly restrictive in an equilibrium model. A slight deviation from the decision model characterized in the previous Section seems appropriate here to allow for easier comparisons with results derived in the absence of ambiguity aversion.

The total supply of the consumption good available at each date and in each state is described by a basic endowment process $e = \{e_t\} \in \mathcal{D}$. In particular, assume that there exists an $e^* : S \to \mathbb{R}_{++}$ such that $e_t(s^t) = e^*(s_t)$ for all $t \geq 1$ and all $s^t \in S^t$. The endowment process is the reference-point of the agent. Hence, for each consumption process in $\mathcal{D}$, the agent's preferences are described by a utility process $V_t(c)$ satisfying the following recursive relation: For all $t \geq 1$ and all $s^t \in S^t$,

$$V_t(c; s^t) = [u(c_t(s^t) - u(e_t(s^t))] + \beta \min_{\pi \in \Pi(s_t)} \left\{ \int V_{t+1}(c; s^t, \cdot) d\pi \right\} \tag{4.8}$$

It is straightforward to adapt the contraction mapping argument in Epstein and Wang (1994) to show that there exists a unique $V(c) \in \mathcal{D}$ satisfying Eq. 4.8, and that for all $c \in \mathcal{D}$, $t \geq 1$ and $s^t \in S^t$,

$$V_t(c; s^t) = V_1(c^{t+}(s^{t-1}; s_t)) \tag{4.9}$$

The recursive structure in (4.8) and (4.9) is exactly the appeal of the recursive RMEU model. Moreover, the dynamic structure implies that the time subscript on initial utility is irrelevant and it is therefore appropriate and convenient to denote V_1 simply by V. Epstein and Wang (1994) also show that for each $s \in S$, $V(\cdot; s)$ is strictly increasing and concave on $\mathcal{D}$; and that $V(c; s)$ is jointly continuous on $\mathcal{D} \times S$, and their proof extends directly to the utility specification presented here.

In addition to the endowment process there is a finite number N of securities, available in zero net-supply in each period t and each state of the world s_t. Each security $k \in \{1, ..., N\}$ provides a dividend process $d_k = \{d_{k,t}\} \in \mathcal{D}$, and is traded in each period at a competitive price denoted $p_{k,t}$. Consumption in each period is the numeraire and $p_t := (p_{1,t}, ..., p_{N,t})$ denotes the vector of period t security prices, and $p \in \mathcal{D}^N$ denotes the complete price process of all securities.

4.3.2 Equilibrium analysis

I study a competitive equilibria of the representative agent economy outlined above. An equilibrium is a price process that ensures that the consumers' excess demand at every date-history pair corresponds exactly to the endowment process. Hence, a competitive equilibrium is similar to the rational expectations equilibrium introduced in Lucas (1978), suitably reinterpreted to allow for ambiguous beliefs.

Definition of equilibrium

A plan for the consumer consists of a pair (c, θ), where $c \in \mathcal{D}$ is a consumption process and $\theta = \{\theta_t\}$ is a continuous process with $\theta_t = (\theta_{1,t}, ..., \theta_{N,t})$ representing the portfolio plan for period t. Given a price process p, a plan (c, θ) is feasible for a time-history pair (t, s^t) if for all $\tau \geq t$ (1) the plan satisfies the following budget constraint: $p_\tau \cdot \theta_\tau + c_\tau = \theta_{\tau-1} \cdot [p_\tau + d_\tau] + e_\tau$, (2) prior asset holdings are zero, $\theta_{t-1}(s^{t-1}) \equiv 0$, and (3) the plan satisfies the following short-sale constraint $\inf_{k,\tau,s^\tau} \theta_{k,\tau}(s^\tau) > -\infty$. The plan is optimal at the time history-pair (t, s^t) if it is feasible and $V\left(c^{t+}(s^{t-1}); s^t\right) \geq V\left(\hat{c}^{t+}(s^{t-1}); s^t\right)$ for all other plans $(\hat{c}, \hat{\theta})$ that are feasible at (t, s^t). Modulo the differences in how preferences are defined, the following is the definition of equilibrium in Lucas (1978).

Definition 15 (Equilibrium) *An equilibrium is a price process $p \in \mathcal{D}^N$ such that the plan $\{(e_\tau, 0)\}_{\tau=1}^\infty$ is optimal for all $t \geq 1$ and $s^t \in S^t$.*

Hence, an equilibrium is a price process such that zero excess demand for all securities is optimal at all time periods and in all states, given the agent's (ambiguous) beliefs about the evolution of states. Epstein and Wang (1994) note that, given the assumption that Π has full-support, it would be equivalent to define equilibrium requiring only that $\{(e_\tau, 0)\}_{\tau=1}^\infty$ is optimal at the time-history pair $(1, s_1)$. This is due precisely to dynamic consistency of the utility process.

Characterization of equilibrium

When the agent is ambiguity neutral, i.e., Π is a probability kernel, there exists a unique equilibrium price process of the economy outlined above. This result follows directly from Lucas (1978) who (for the special case of an ambiguity neutral representative agent) studies a slightly more general economy than the one outlined above, demonstrates existence and characterizes equilibrium in terms of Euler equations. However, due to ambiguity aversion, the price process when Π is not a probability kernel is generally not unique. Moreover, whenever the equilibrium price is not unique there is an uncountable infinity of equilibrium prices and therefore, following standard nomenclature, the equilibrium price is said to be indeterminate. The following Proposition provides a characterization of the set of all equilibrium price processes of the economy and indicates that indeterminacy is a common feature of the economy. As a result, RRMEU provides an intuitive explanation for the excess price volatility puzzle (see, e.g., Shiller, 1981) on the basis that ambiguity leads to difficulties in determining unique values for assets.

Proposition 6 (Characterization of Equilibrium)

- *There exists an equilibrium price process* $p \in \mathcal{D}^N$.
- *The price process* $p \in \mathcal{D}^N$ *is an equilibrium if and only if there exists a continuous selection* $\{\pi_t\}$ *from* Π *(i.e., a sequence of probability kernels* $\{\pi_t\}$ *with* $\pi_t(s_t, \cdot) \in \Pi(s_t)$ *for all* t *and all* $s_t \in S$ *) such that*

$$p_{k,t} = \beta \int \left[\frac{u'(e_{t+1})}{u'(e_t)} \left(\bar{p}_{k,t+1} + d_{k,t+1} \right) \right] d\pi_t(s_t, \cdot) \tag{4.10}$$

for all $t \geq 1$, $s_t \in S$ *and* $k = 1, ..., N$.

Proof. The proof is given in the Appendix. ■

Since Π is a continuous correspondence it generally admits multiple continuous selections. Moreover, since Π is convex and compact-valued, there are in fact generally an infinity of continuous selections. Proposition 6 shows that for each continuous selection there is a unique corresponding equilibrium price process. Moreover, if any of these prices process differ, there is a continuum of price process (i.e., the equilibrium price is indeterminate).[6] Since it is quite coincidental for all selections to imply the same price process, Proposition 6 therefore demonstrates that ambiguity can lead to excess price volatility for any basic endowment process.

Epstein and Wang (1994) also provide a characterization of equilibrium prices in the context of the RMP decision model, which corresponds to the decision model used above when the reference-point is constant over time and states. They obtain a similar condition to (4.10) for their characterization of prices, but restricted to continuous selections from

$$Q(s) := \left\{ \pi \in \Pi(s) | \int \hat{V}(e; s) d\pi = \min_{\pi \in \Pi(s)} \hat{V}(e; s) d\pi \right\}, \tag{4.11}$$

where $\hat{V}$ corresponds to the functional V evaluated at a constant reference-point. When e is non-constant, $Q(s)$ may frequently be unique valued and it is therefore not clear how frequent or economically relevant the price indeterminacy they identify is. However, when the reference-point is given by the endowment process e, $\Pi(s)$ replaces for the $Q(s)$ in Epstein and Wang (1994) since for all $s \in S$

$$Q(s) := \left\{ \pi \in \Pi(s) | \int V(e; s) d\pi = \min_{\pi \in \Pi(s)} V(e; s) d\pi \right\} = \Pi(s) , \tag{4.12}$$

by the fact that $V(e; s)$ is constant for all s. Hence, when e is the reference-point and Π is not single-valued (reflecting ambiguity aversion), Π will admit multiple continuous selections each of which implies a unique equilibrium price. Since it can be only coincidental, i.e., for very special dividend processes, that all of these equilibrium price processes coincide, part (b) of Proposition 6 illustrates that price indeterminacy is substantially more prevalent when ambiguity is viewed from the perspective of the endowment than when it is viewed from the perspective of a constant reference-point. Hence,

[6] It is straightforward to adapt the proof of Theorem 3(a) in Epstein and Wang (1994) to show that the set of equilibrium prices is a closed and connected subset of $\mathcal{D}^N$, and hence if there are multiple equilibria the equilibrium is, in fact, indeterminate.

one could summarize the contribution of Proposition 6 as follows. Epstein and Wang (1994) identify sources of price indeterminacy that can be related directly to ambiguity aversion. However, indeterminacy in the context of the RMP decision model arises only when there is a very special (knife-edge) relationship between the endowment process and the dividend processes of the securities. It is therefore difficult to say with much confidence whether ambiguity aversion as captured by the RMP model is really consistent with the excess price volatility observed in asset markets. On the other hand, the equilibrium price is determinate in the context of the RRMEU decision model (with the endowment as a reference-point) only if there is a very special (knife-edge) relationship between the endowment process and the dividend process of the securities. Hence, while price indeterminacy remains an exception with the RMP decision model, it is the norm if the agent views ambiguity from the endowment process. There is considerable experimental and empirical evidence to suggest that individuals have a status-quo bias (see, e.g., Samuelson and Zeckhauser, 1988; Campbell, 2006), and the hypothesis that this is a reflection of ambiguity aversion is at least plausible. The result summarized in Proposition 6 further illustrates that if investors view ambiguity from the perspective of a status-quo consumption process, ambiguity aversion generally leads to price indeterminacy and is therefore consistent with the excess price volatility observed in asset markets.

4.4 Conclusion

The main result of this Chapter gives an extension of the Reference-Dependent Maxmin Expected Utility model to dynamic settings. The Recursive RMEU decision model generalizes the Bayesian learning inherent in subjective expected utility to the case when there are multiple priors (as in Epstein and Schneider, 2003) and has appealing axiomatic foundations. In particular, it satisfies consequentialism and dynamic consistency (at least within a given decision problem). Moreover, an application to a dynamic asset pricing model (extending on Epstein and Wang, 1994) illustrates that the Recursive Reference-Dependent Maxmin Expected Utility model can allow for a more general perspective on the influence of ambiguity aversion in markets than the Recursive Multiple Priors model of Epstein and Schneider (2003) would suggest. I illustrate that price indeterminacy is a robust implication of ambiguity aversion in asset markets when investors view ambiguity from the perspective of their status-quo, an assumption that corresponds well with data on individual investment behavior and leads to implications that are consistent with the excess price volatility observed in market prices. How the trade-off between insurance and hedging inherent in the Reference-Dependent Maxmin Expected Utility model affects individual behavior and market outcomes in other dynamic settings – such as dynamic labor choices, saving and investment decisions, or dynamic social interaction models – should be a topic of considerable interest for future research.

APPENDIX A

PROOFS FOR THE RESULTS IN THE TEXT

A.1 Proofs for results in Chapter 2

A.1.1 Mathematical notation

I use the following notation in some proofs. Let $B(\Sigma)$ be the set of bounded, Σ-measurable functions on S, and $B_0(\Sigma)$ be the set of all real valued, simple, Σ-measurable functions on S. If $f \in \mathcal{F}$ and $u : \mathcal{P} \to \mathbb{R}$ then $u \circ f$ is the element of $B_0(\Sigma)$ defined by $u \circ f = u(f(s))$ for all $s \in S$. Denote by $ba(\Sigma)$ the set of all finitely additive and bounded set-functions on S. A nonnegative element of $ba(\Sigma)$ that assigns value 1 to S is called a *probability*, and it is typically denoted by π. Since $ba(\Sigma)$ is (isometrically isomorphic to) the norm dual of $B_0(\Sigma)$[1] all of its subsets inherit a weak* topology. Finally, given a non-singleton interval $K \subset \mathbb{R}$, denote by $B_0(\Sigma, K)$ the subset of functions in $B_0(\Sigma)$ taking values in K. Note that for any mixture-linear $u : \mathcal{P} \to \mathbb{R}$, $u(\mathcal{F}) := \{u \circ f | f \in \mathcal{F}\} = B_0(\Sigma, u(\mathcal{P}))$, where $u(\mathcal{P})$ is the convex hull of $u(X) = \{y \in \mathbb{R} | y = u(x) \text{ for some } x \in \mathcal{P}\}$. Suppose that $u, v : \mathcal{P} \to \mathbb{R}$ and there exist $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $u = av + b$, then v is a postive affine transformation of u and I write $u \cong v$. If for some $f, g \in \mathcal{F}$ $u(f(s)) = u(g(s))$ for all $s \in S$, then I write $u \circ f \equiv u \circ g$.

[1]Provided $ba(\Sigma)$ is endowed with the total variation norm and $B_0(\Sigma)$ is endowed with the sup-norm.

A.1.2 Proofs

Proof of Theorem 1. First show that (2) implies (1). If $\succsim_r$ has an RMEU representation for some $r \in \mathcal{F}$, it is trivial to see that the relation is a non-trivial, continuous and monotone preorder. To verify UA let $f, g \in \mathcal{F}$, $f \sim_r g$ and $\alpha \in (0,1)$. Then

$$\begin{aligned}
& \min_{\pi\in\Pi} \int_S \left[(\alpha u(f(s)) + (1-\alpha)u(g(s))) - u(r(s))\right] \, d\pi \\
= & \min_{\pi\in\Pi} \left[\alpha \int_S [u(f(s)) - u(r(s))] d\pi + (1-\alpha) \int_S [u(g(s)) - u(r(s))] \, d\pi\right] && \text{(A.1)} \\
\geq & \left[\min_{\pi\in\Pi} \alpha \int_S [u(f(s)) - u(r(s))] d\pi\right] + \left[\min_{\pi\in\Pi} (1-\alpha) \int_S [u(g(s)) - u(r(s))] \, d\pi\right] && \text{(A.2)} \\
= & \min_{\pi\in\Pi} \int_S [u(f(s)) - u(r(s))] d\pi \, . && \text{(A.3)}
\end{aligned}$$

To verify RI with respect to r, let $\alpha \in (0,1)$ and $f, g \in \mathcal{F}$. Then,

$$\begin{aligned}
& \min_{\pi\in\Pi} \int_S (u(f(s)) - u(r(s))) \, d\pi \\
\geq \; & \min_{\pi\in\Pi} \int_S (u(g(s)) - u(r(s))) \, d\pi && \text{(A.4)} \\
\Leftrightarrow \quad & \alpha \min_{\pi\in\Pi} \int_S (u(f(s)) - u(r(s))) \, d\pi \\
\geq \; & \alpha \min_{\pi\in\Pi} \int_S (u(g(s)) - u(r(s))) \, d\pi && \text{(A.5)} \\
\Leftrightarrow \quad & \min_{\pi\in\Pi} \int_S [(\alpha u(f(s)) - (1-\alpha)u(r(s))) - u(r(s))] \, d\pi \\
\geq \; & \min_{\pi\in\Pi} \int_S [(\alpha u(g(s)) - (1-\alpha)u(r(s))) - u(r(s))] \, d\pi \, . && \text{(A.6)}
\end{aligned}$$

To verify WCI, let $f, g \in F$, $c_1, c_2 \in \mathcal{P}$ and $\alpha \in (0,1)$. Then,

$$\begin{aligned}
& \min_{\pi\in\Pi} \int_S [(\alpha u(f(s)) + (1-\alpha)u(c_1)) - u(r(s))]\, d\pi \\
\geq\ & \min_{\pi\in\Pi} \int_S [(\alpha u(g(s)) + (1-\alpha)u(c_1)) - u(r(s))]\, d\pi \qquad \text{(A.7)} \\
\Leftrightarrow\ & \left[\min_{\pi\in\Pi} \int_S [\alpha u(f(s)) - u(r(s))]\, d\pi\right] + (1-\alpha)u(c_2) \\
\geq\ & \left[\min_{\pi\in\Pi} \int_S [\alpha u(g(s)) - u(r(s))]\, d\pi\right] + (1-\alpha)u(c_2) \qquad \text{(A.8)} \\
\Leftrightarrow\ & \min_{\pi\in\Pi} \int_S [(\alpha u(f(s)) + (1-\alpha)u(c_2)) - u(r(s))]\, d\pi \\
\geq\ & \min_{\pi\in\Pi} \int_S [(\alpha u(g(s)) + (1-\alpha)u(c_2)) - u(r(s))]\, d\pi\,. \qquad \text{(A.9)}
\end{aligned}$$

To verify EUP for a class of RMEU preference relations, let $f, g \in \mathcal{F}$ and define $f \succeq' g$ if and only if $\int_S u(f(s))d\pi \geq \int_S u(g(s))d\pi$ for all $\pi \in \Pi$. For any $r \in \mathcal{F}$, for all $f, h \in \mathcal{F}$ and all $\alpha \in (0,1)$, define the non-empty set

$$\pi_r^f(h,\alpha) = \arg\min_{\pi\in\Pi} \int_S ([\alpha u(f(s)) + (1-\alpha)u(h(s))] - u(r(s)))\, d\pi \qquad \text{(A.10)}$$

Now observe that for all $f, g \in \mathcal{F}$, $f \succeq' g$ implies that for all $h \in \mathcal{F}$ and $\alpha \in (0,1)$,

$$\begin{aligned}
& \min_{\pi\in\Pi} \int_S ([\alpha u(f(s)) + (1-\alpha)u(h(s))] - u(r(s)))\, d\pi \\
=\ & \int_S ([\alpha u(f(s)) + (1-\alpha)u(h(s))] - u(r(s)))\, d\hat{\pi}\ \forall \hat{\pi} \in \pi_r^f(h,\alpha) \qquad \text{(A.11)} \\
\geq\ & \int_S ([\alpha u(g(s)) + (1-\alpha)u(h(s))] - u(r(s)))\, d\hat{\pi}\ \forall \hat{\pi} \in \pi_r^f(h,\alpha) \qquad \text{(A.12)} \\
\geq\ & \min_{\pi\in\Pi} \int_S ([\alpha u(g(s)) + (1-\alpha)u(h(s))] - u(r(s)))\, d\pi \qquad \text{(A.13)}
\end{aligned}$$

Let $f, g \in \mathcal{F}$ and suppose that there exists $\alpha \in (0,1)$ and $h \in \mathcal{F}$ such that

$$\begin{aligned}
& \min_{\pi\in\Pi} \int_S ([\alpha u(f(s)) + (1-\alpha)u(h(s))] - u(r(s)))\, d\pi \qquad \text{(A.14)} \\
\geq\ & \min_{\pi\in\Pi} \int_S ([\alpha u(g(s)) + (1-\alpha)u(h(s))] - u(r(s)))\, d\pi\,. \qquad \text{(A.15)}
\end{aligned}$$

Then,

$$\begin{aligned} &\int_S ([\alpha u(f(s)) + (1-\alpha)u(h(s))] - u(r(s)))\, d\hat{\pi} \\ \geq \quad &\int_S ([\alpha u(g(s)) + (1-\alpha)u(h(s))] - u(r(s)))\, d\hat{\pi} \quad \forall \hat{\pi} \in \pi_r^g(h,\alpha)\,. \end{aligned} \tag{A.16}$$

Hence,

$$\int_S u(f(s))\, d\hat{\pi} \geq \int_S u(g(s))\, d\hat{\pi}\ \forall \hat{\pi} \in \pi_r^g(h,\alpha)\,. \tag{A.17}$$

So not $g \succeq' f$. Consequently, for any $r \in \mathcal{F}$, $f \succeq' g$ if and only if for all $h \in \mathcal{F}$ and all $\alpha \in (0,1)$, $\alpha f + (1-\alpha)h \succeq_r \alpha g + (1-\alpha)h$. Conclude that $\succeq'$ is the unambiguous preference relation given r and observe that $\succeq'$ depends only on Π and u (hence not r). Hence, the class of preference relations satisfies EUP. Finally, to verify RT for a class of RMEU preferences simply observe that for $\alpha \in (0,1)$ and $x \in \mathcal{P}$

$$\min_{\pi\in\Pi} \int [u(f(s)) - u(r(s))]\, d\pi \quad \geq \quad \min_{\pi\in\Pi} \int [u(g(s)) - u(r(s))]\, d\pi \tag{A.18}$$

$$\Leftrightarrow \alpha \min_{\pi\in\Pi} \int [u(f(s)) - u(r(s))]\, d\pi \quad \geq \quad \alpha \min_{\pi\in\Pi} \int [u(f(s)) - u(r(s))]\, d\pi \tag{A.19}$$

$$\begin{aligned} &\Leftrightarrow \min_{\pi\in\Pi} \int [(\alpha u(f(s)) + (1-\alpha)u(x)) - (\alpha u(r(s)) + (1-\alpha)u(x))] \quad d\pi \quad \geq \\ &\min_{\pi\in\Pi} \int [(\alpha u(g(s)) + (1-\alpha)u(x)) - (\alpha u(r(s)) + (1-\alpha)u(x))] \quad d\pi \end{aligned} \tag{A.20}$$

To prove (1) implies (2) assume that $\succeq_r$ satisfies preorder, non-triviality, monotonicity, continuity, UA, WCI and RI with respect to r for all $r \in \mathcal{F}$, and that the class $(\succeq_r)_{r\in\mathcal{F}}$ satisfies EUP and RT. For clarity, the proof is divided into a number of lemmas. The following lemma uses the fact that $\succeq_r$ is a variational preference in the sense of Maccheroni et al. (2006). A function V defined on $B_0(\Sigma)$ is said to be supperadditive if for all $\rho, \phi \in B_0(\Sigma)$ and all $\lambda \in (0,1)$, $V(\lambda\rho + (1-\lambda)\phi) \geq \lambda V(\rho) + (1-\lambda)V(\phi)$ and constant additive if for all $\rho \in B_0(\Sigma)$, for all $\alpha \in [0,1]$ and for all constants $x \in \mathbb{R}$,

$$V[\alpha\rho + (1-\alpha)x] = V(\alpha\rho) + (1-\alpha)x\,. \tag{A.21}$$

Lemma 1 (Variational preference) *For all $r \in \mathcal{F}$ there exists a non-constant, mixture linear von Neumann/Morgenstern utility index $u : \mathcal{P} \to \mathbb{R}$ with $0 \in u(\mathcal{P})$, and a supper-additive, constant additive, real valued, functional V_r defined on $B_0(\Sigma, u(X))$ such that for all $f, g \in \mathcal{F}$, $f \succeq_r g$ if and only if $V_r(u \circ f) \geq V_r(u \circ g)$.*

Proof. The proof follows directly from Maccheroni et al. (2006). The fact that u is independent of r follows from the fact that preferences on $\mathcal{P}$ are unambiguous and therefore u does not depend on r by EUP. In fact, a weaker version of WCI would be sufficient to derive a variational representation. Specifically, it is enough to require that there exist one $\alpha \in (0, 1)$ such that for all $f, g \in F$, and for all $c_1, c_2 \in \mathcal{P}$, $\alpha f + (1-\alpha)c_1 \succeq_r \alpha g + (1-\alpha)c_1$ if and only if $\alpha f + (1 - \alpha)c_2 \succeq_r \alpha g + \alpha g + (1 - \alpha)c_2$. The following proof (given for completeness) establishes that this condition implies WCI when preferences satisfy RI. Hence, suppose only that the weaker version of WCI holds, then

$$\begin{aligned} & \left(\frac{\alpha\gamma}{1-\alpha+\alpha\gamma}\right) f + \left(\frac{1-\alpha}{1-\alpha+\alpha\gamma}\right) c_1 \\ \succeq_r \quad & \left(\frac{\alpha\gamma}{1-\alpha+\alpha\gamma}\right) g + \left(\frac{1-\alpha}{1-\alpha+\alpha\gamma}\right) c_1 \end{aligned} \tag{A.22}$$

$$\begin{aligned} \Leftrightarrow \quad & (1-\alpha+\alpha\gamma)\left[\left(\frac{\alpha\gamma}{1-\alpha+\alpha\gamma}\right) f + \left(\frac{1-\alpha}{1-\alpha+\alpha\gamma}\right) c_1\right] + \alpha(1-\gamma)r \\ \succeq_r \quad & (1-\alpha+\alpha\gamma)\left[\left(\frac{\alpha\gamma}{1-\alpha+\alpha\gamma}\right) g + \left(\frac{1-\alpha}{1-\alpha+\alpha\gamma}\right) c_1\right] + \alpha(1-\gamma)r \end{aligned} \tag{A.23}$$

$$\Leftrightarrow \quad (\alpha\gamma)f + (1-\alpha)c_1 + \alpha(1-\gamma)r \succeq_r (\alpha\gamma)g + (1-\alpha)c_1 + \alpha(1-\gamma)r \tag{A.24}$$

$$\Leftrightarrow \quad \alpha\left[\gamma f + (1-\gamma)r\right] + (1-\alpha)c_1 \succeq_r \alpha\left[\gamma g + (1-\gamma)r\right] + (1-\alpha)c_1 \tag{A.25}$$

$$\Leftrightarrow \quad \alpha\left[\gamma f + (1-\gamma)r\right] + (1-\alpha)c_2 \succeq_r \alpha\left[\gamma g + (1-\gamma)r\right] + (1-\alpha)c_2 \tag{A.26}$$

$$\Leftrightarrow \quad (\alpha\gamma)f + (1-\alpha)c_2 + \alpha(1-\gamma)r \succeq_r (\alpha\gamma)g + (1-\alpha)c_2 + \alpha(1-\gamma)r \tag{A.27}$$

$$\begin{aligned} \Leftrightarrow \quad & (1-\alpha+\alpha\gamma)\left[\left(\frac{\alpha\gamma}{1-\alpha+\alpha\gamma}\right) f + \left(\frac{1-\alpha}{1-\alpha+\alpha\gamma}\right) c_2\right] + \alpha(1-\gamma)r \\ \succeq_r \quad & (1-\alpha+\alpha\gamma)\left[\left(\frac{\alpha\gamma}{1-\alpha+\alpha\gamma}\right) g + \left(\frac{1-\alpha}{1-\alpha+\alpha\gamma}\right) c_2\right] + \alpha(1-\gamma)r \end{aligned} \tag{A.28}$$

$$\Leftrightarrow \quad \left(\frac{\alpha\gamma}{1-\alpha+\alpha\gamma}\right)f+\left(\frac{1-\alpha}{1-\alpha+\alpha\gamma}\right)c_2$$
$$\succeq_r \quad \left(\frac{\alpha\gamma}{1-\alpha+\alpha\gamma}\right)g+\left(\frac{1-\alpha}{1-\alpha+\alpha\gamma}\right)c_2 \tag{A.29}$$

for all $c_1, c_2 \in \mathcal{P}$. Note that for $\alpha = 1/2$, $(1-\alpha)/(1-\alpha+\alpha\gamma) = 1/(1+\gamma)$ is strictly monotone, continuous and onto $[1/2, 1]$ as a function of $\gamma \in [0, 1]$. Hence, for all $\beta \in [0, 1/2]$ and for all $f, g \in \mathcal{F}$, $\beta f + (1-\beta)c_1 \succeq_r \beta g + (1-\beta)c_1$ if and only if $\beta f + (1-\beta)c_2 \succeq_r \beta g + (1-\beta)c_2$ for all $c_1, c_2 \in \mathcal{P}$. Now note that $x, y \in \mathcal{P}$ and $\alpha \in [0, 1]$ implies $\alpha x + (1-\alpha)y \in \mathcal{P}$. It follows that for any $\alpha \in [0, 1]$ and any $f, g \in \mathcal{F}$, $\alpha f + (1-\alpha)c_1 \succeq_r \alpha g + (1-\alpha)c_1$ if and only if $\alpha f + (1-\alpha)c_2 \succeq_r \alpha g + \alpha g + (1-\alpha)c_2$ for all $c_1, c_2 \in \mathcal{P}$. Hence, $\succeq_r$ satisfies WCI. ■

The next lemma is a result in the spirit of Bewley (2002) who provided a unanimity representation for an incomplete preference relation satisfying independence. The lemma stated here is the extension of Bewley's due to Gilboa et al. (2010, see also Ghirardato et al. (2004)).

Lemma 2 *There exists a unique, weak*-closed, convex set of priors $\Pi \subset \Delta(S)$ and a unique (up to positive affine transformations), non-constant, mixture linear von Neumann/Morgenstern index $u : \mathcal{P} \to \mathbb{R}$, such that for all $f, g \in \mathcal{F}$, $f \succeq^* g$ if and only if*

$$\int_S u(f(s))d\pi \geq \int_S u(g(s))d\pi \quad \forall \pi \in \Pi\,. \tag{A.30}$$

Proof. For the unanimity representation Gilboa et al. (2010) require that $\succeq^*$ satisfy independence, certainty-completeness, Archimeadian continuity, non-triviality, transitivity and monotonicity. Certainty-completeness follows from the the variational representation of preferences stated above and monotonicity of $\succeq_r$, independence follows by definition of $\succeq^*$ and the other properties follow directly from the corresponding property of $\succeq_r$. The proof then follows directly from Gilboa et al. (2010) (see, also, the proof in Ghirardato et al., 2004). ■

Since the representation of preferences on $\mathcal{P}$ via u is unique up to positive affine transformations, completeness of $\succeq^*$ on $\mathcal{P}$ implies that (subject to a suitable normalization) the von Neumann/Morgenstern utility index in the variational representation of preferences and in the unanimity representation of Eq. A.30 coincide. The next lemma states that as a completion of $\succeq^*$, the preference relation $\succeq_r$ satisfies Bewley's (2002) inertia condition for all $r \in \mathcal{F}$.

Lemma 3 (Inertia) *For all $r \in \mathcal{F}$ and for all $f \in \mathcal{F}$*

$$f \succeq^* r \;\Leftrightarrow f \succeq_r r\,. \tag{A.31}$$

Proof. First show that the representation V_r satisfies that for all $f \in F$ and all $\alpha \in (0,1)$, $V_r(\alpha u \circ r + (1-\alpha)u \circ f) = \alpha V_r(u \circ r) + (1-\alpha)V_r(u \circ f)$. Note that Axioms 1-3 imply that every $f \in F$ has a certainty equivalent $c_f \in P$ satisfying that $c_f \sim_r f$ (see, e.g., Maccheroni et al., 2006). It follows from RI that $r \sim_r c_r$ implies $r \sim_r \alpha r + (1-\alpha)c_r$. Hence,

$$\begin{aligned} V_r(u \circ r) &= V_r[\alpha u \circ r + (1-\alpha)u \circ c_r] && \text{(A.32)} \\ &= V_r[\alpha u \circ r] + (1-\alpha)u \circ c_r && \text{(A.33)} \\ &= V_r[\alpha u \circ r] + (1-\alpha)V_r(u \circ r)\,. && \text{(A.34)} \end{aligned}$$

So $V_r(\alpha u \circ r) = \alpha V_r(u \circ r)$. For all $f \in \mathcal{F}$ RI implies that $\alpha r + (1-\alpha)f \sim_r \alpha r + (1-\alpha)c_f$. Hence,

$$\begin{aligned} V_r[\alpha u \circ r + (1-\alpha)u \circ f] &= V_r[\alpha u \circ r + (1-\alpha)u \circ c_f] && \text{(A.35)} \\ &= V_r[\alpha u \circ r] + (1-\alpha)u \circ c_f && \text{(A.36)} \\ &= \alpha V_r(u \circ r) + (1-\alpha)V_r(u \circ f)\,, && \text{(A.37)} \end{aligned}$$

where the last equality follows from the construction of V_r in the proof of Lemma 28 in Maccheroni et al. (2006).

Now let $r, f \in \mathcal{F}$ with $f \succeq_r r$ and let $g \in \mathcal{F}$ and $\lambda \in (0,1)$. Then

$$\begin{aligned} V_r[\lambda u \circ f + (1-\lambda) u \circ g] &\geq \lambda V_r(u \circ f) + (1-\lambda) V(u \circ g) && \text{(A.38)} \\ &\geq \lambda V_r[u \circ r] + (1-\lambda) V_r[u \circ g] && \text{(A.39)} \\ &= V_r(\lambda u \circ r + (1-\lambda) u \circ g)\,. && \text{(A.40)} \end{aligned}$$

Hence, since the choice of g and α were arbitrary, $f \succeq_r r$ implies $f \succeq^* r$. Clearly the converse holds, so this establishes the equivalence, and the inertia condition follows. ■

The following observations will be useful in the remainder of the proof (for each observation fix some arbitrary $r \in \mathcal{F}$):

Obs (1): Suppose that for $f, g \in \mathcal{F}$ and for all $\pi \in \Pi$

$$\int u \circ f d\pi > \int u \circ g d\pi\,. \tag{A.41}$$

Let $x, y \in \mathcal{P}$ with $x \succ_r y$. Then there exists an $\alpha \in (0,1)$ such that

$$\alpha \min_{\pi \in \Pi} \int [u \circ f - u \circ g] d\pi = (1-\alpha)[u(x) - u(y)] \tag{A.42}$$

Hence, for all $\pi \in \Pi$

$$\int u \circ (\alpha f + (1-\alpha) y) d\pi \geq \int u \circ (\alpha g + (1-\alpha) x) d\pi \tag{A.43}$$

and therefore $\alpha f + (1-\alpha)y \succeq^* \alpha g + (1-\alpha)x$ and thereby $\alpha f + (1-\alpha)y \succeq_{\alpha r+(1-\alpha)x}$ $\alpha g + (1-\alpha)x$. By monotonicity $\alpha f + (1-\alpha)x \succ_{\alpha r+(1-\alpha)x} \alpha f + (1-\alpha)y$ and hence by transitivity $\alpha f + (1-\alpha)x \succ_{\alpha r+(1-\alpha)x} \alpha g + (1-\alpha)x$. Hence, by RT, $f \succ_r g$. The contrapositive is that if $g \succeq_r f$ then there exists at least one $\pi \in \Pi$ such that the expected utility of g under π is greater than/equal to the expected utility of f.

Obs (2): Suppose that for some $f \in \mathcal{F}$, $f \succ_r r$. Let $x, y \in \mathcal{P}$ with $x \succ_r y$. By RT $\alpha f + (1 - \alpha)x \succ_{\alpha r+(1-\alpha)x} \alpha r + (1-\alpha)x$ for all $\alpha \in (0, 1)$. For any $\alpha \in (0, 1)$, by monotonicity, for all $\beta \in (0, 1)$, $\alpha f + (1-\alpha)x \succ_{\alpha r+(1-\alpha)x} \alpha f + (1-\alpha)[\beta x + (1-\beta)y]$. Moreover, for $\alpha = 0, \beta = 0$, $\alpha r + (1-\alpha)x \succ_{\alpha r+(1-\alpha)x} \alpha f + (1-\alpha)[\beta x + (1-\beta)y]$; while for $\alpha = 1$, $\beta = 1$ the reverse is true. It follows by continuity, that there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1-\alpha)[\beta x + (1-\beta)y] \sim_{\alpha r+(1-\alpha)x} \alpha r + (1-\alpha)x$. Hence, by straightforward algebra manipulations, for all $\pi \in \Pi$

$$\int u \circ f d\pi > \int u \circ r d\pi\,. \tag{A.44}$$

Moreover, there exist $\alpha \in (0, 1)$ and $x, y \in \mathcal{P}$ with $x \succ_r y$ such that $\alpha f + (1 - \alpha)y \sim_{\alpha r+(1-\alpha)x} \alpha r + (1-\alpha)x$.

Obs (3): Suppose that for some $f \in \mathcal{F}$, $f \sim_r r$. Then it follows from RT that for any $\alpha \in (0, 1)$ and $x \in \mathcal{P}$ $\alpha f + (1-\alpha)x \sim_{\alpha r+(1-\alpha)x} \alpha r + (1-\alpha)x$.

Obs (4): Suppose that for some $f \in \mathcal{F}$, $r \succ_r f$. Let $x, y \in \mathcal{P}$ with $x \succ_r y$. Then there exists a unique $\alpha \in (0, 1)$ such that

$$\alpha \min_{\pi\in\Pi} \int [u \circ f - u \circ r] d\pi + (1-\alpha)[u(x) - u(y)] = 0\,. \tag{A.45}$$

Then for all $\pi \in \Pi$,

$$\int u \circ [\alpha f + (1-\alpha)x] d\pi \geq \int u \circ [\alpha r + (1-\alpha)y] d\pi \tag{A.46}$$

and there is equality for at least one $\pi \in \Pi$. The former implies $\alpha f + (1-\alpha)x \succeq_{\alpha r+(1-\alpha)y} \alpha r + (1-\alpha)y$. By Obs (2), the latter then implies that $\alpha f + (1-\alpha)x \sim_{\alpha r+(1-\alpha)y} \alpha r + (1-\alpha)y$.

The next lemma gives a characterization of the preference relation $\succeq_r$ as a completion of $\succeq^*$ for a particular reference-point $r \in \mathcal{F}$.

Lemma 4 *For all $r \in \mathcal{F}$ and for all $f, g \in \mathcal{F}$ the following are equivalent:*

1. $f \succ_r g$
2. *There exist $\alpha \in (0,1)$ and $x, y \in \mathcal{P}$ such that $\alpha f + (1-\alpha)x \succeq^* \alpha r + (1-\alpha)y$ and not $\alpha g + (1-\alpha)x \succeq^* \alpha r + (1-\alpha)y$ (denoted $f \gg_r g$).*

Proof. First suppose that $f \gg_r g$. Clearly, $\alpha f + (1-\alpha)x \succeq_{\alpha r+(1-\alpha)y} \alpha r + (1-\alpha)y$. Also, by inertia, $\alpha r + (1-\alpha)y \succ_{\alpha r+(1-\alpha)y} \alpha g + (1-\alpha)x$. Hence, by transitivity, $\alpha f + (1-\alpha)x \succ_{\alpha r+(1-\alpha)y} \alpha g + (1-\alpha)x$, and by WCI, $\alpha f + (1-\alpha)y \succ_{\alpha r+(1-\alpha)y} \alpha g + (1-\alpha)y$. It then follows from RT that $f \succ_r g$. This proves one implication.

To prove the converse, suppose that $f \succ_r g$ for some $f, g \in \mathcal{F}$. We consider three cases. First, consider the case $f \succ_r r$. By Obs (2) there exist $\alpha \in (0,1)$ and $x, y \in \mathcal{P}$ with $x \succ_r y$ such that $\alpha f + (1-\alpha)y \sim_{\alpha r+(1-\alpha)x} \alpha r + (1-\alpha)x$. Since $f \succ_r g$, RT implies that $\alpha f + (1-\alpha)x \succ_{\alpha r+(1-\alpha)x} \alpha g + (1-\alpha)x$. Hence, by WCI, $\alpha f + (1-\alpha)y \succ_{\alpha r+(1-\alpha)y} \alpha g + (1-\alpha)y$. By inertia we have $\alpha f + (1-\alpha)y \succeq^* \alpha r + (1-\alpha)x$ and by transitivity we have that $\alpha r + (1-\alpha)x \succ_{\alpha r+(1-\alpha)y} \alpha g + (1-\alpha)y$, hence not $\alpha g + (1-\alpha)y \succeq^* \alpha r + (1-\alpha)x$. Therefore $f \gg_r g$.

Now consider the case $f \sim_r r$. Then by RT there exists $\alpha \in (0,1)$ and $x \in \mathcal{P}$ such that $\alpha f + (1-\alpha)x \sim_{\alpha r+(1-\alpha)x} \alpha r + (1-\alpha)x$. The remaining steps to show $f \gg_r g$ then follow essentially as for the first case. Finally, suppose that $r \succ_r f$. Then by Obs (4) there exist $\alpha \in (0,1)$ and $x, y \in \mathcal{P}$ such that $\alpha f + (1-\alpha)x \sim_{\alpha r+(1-\alpha)y} \alpha r + (1-\alpha)y$. Again, the remaining steps then parallel the first case, and we obtain for all cases that $f \gg_r g$. Hence, the equivalence follows. ■

The next Lemma provides a characterization of $\gg_r$ for a given reference point $r \in \mathcal{F}$.

Lemma 5 *Let (u, Π) represent $\succeq^*$. Then for all $r \in \mathcal{F}$ and for all $f, g \in \mathcal{F}$, the following* *statements are equivalent:*

1. $f \gg_r g$
2. $\min_{\pi\in\Pi} \int_S [u \circ f - u \circ r] d\pi > \min_{\pi\in\Pi} \int_S [u \circ f - u \circ r] d\pi$

Proof. Let $f \gg_r g$. Then there exist $x, y \in \mathcal{P}$ and $\alpha \in (0, 1)$ such that $\alpha f + (1-\alpha)x \succeq^* \alpha r + (1-\alpha)y$ and not $\alpha g + (1-\alpha)x \succeq^* \alpha r + (1-\alpha_r)y$. Hence,

$$\alpha \int_S u(f(s))d\pi + (1-\alpha)u(x) \geq \alpha \int_S u(r(s))d\pi + (1-\alpha)u(y) \tag{A.47}$$

for all $\pi \in \Pi$, and there exists $\hat{\pi} \in \Pi$ such that

$$\alpha \int_S u(g(s))d\hat{\pi} + (1-\alpha)u(x) < \alpha \int_S u(r(s))d\hat{\pi} + (1-\alpha)u(y) . \tag{A.48}$$

Consequently, for all $\pi \in \Pi$,

$$\int_S \left[u(f(s)) - u(r(s))\right] d\pi \geq \frac{1-\alpha}{\alpha} \left[u(y) - u(x)\right] , \tag{A.49}$$

and, for some $\hat{\pi} \in \Pi$,

$$\int_S \left[u(g(s)) - u(r(s))\right] d\pi < \frac{1-\alpha}{\alpha} \left[u(y) - u(x)\right] . \tag{A.50}$$

It follows that

$$\min_{\pi\in\Pi} \int_S \left[u(f(s)) - u(r(s))\right] d\pi \geq \frac{1-\alpha}{\alpha} \left[u(y) - u(x)\right] , \tag{A.51}$$

and

$$\min_{\pi\in\Pi} \int_S \left[u(g(s)) - u(r(s))\right] d\pi < \frac{1-\alpha}{\alpha} \left[u(y) - u(x)\right] . \tag{A.52}$$

and, hence,

$$\min_{\pi\in\Pi} \int_S \left[u(f(s)) - u(r(s))\right] d\pi > \min_{\pi\in\Pi} \int_S \left[u(g(s)) - u(r(s))\right] d\pi. \tag{A.53}$$

Now, suppose that

$$\min_{\pi\in\Pi}\int_S [u\circ f(s) - u\circ r(s)]\,d\pi > \min_{\pi\in\Pi}\int_S [u\circ g(s) - u\circ r(s)]\,d\pi. \tag{A.54}$$

Then, there exist $x, y \in \mathcal{P}$ and $\alpha \in (0,1)$ such that

$$\begin{aligned}&\min_{\pi\in\Pi}\int_S [u(f(s)) - u(r(s))]\,d\pi \geq \frac{1-\alpha}{\alpha}[u(y) - u(x)] > \\ &\min_{\pi\in\Pi}\int_S [u(g(s)) - u(r(s))]\,d\pi.\end{aligned} \tag{A.55}$$

Hence, for all $\pi \in \Pi$,

$$\alpha\int_S u(f(s))d\pi + (1-\alpha)u(x) \geq \alpha\int_S u(r(s))d\pi + (1-\alpha)u(y)\,, \tag{A.56}$$

and for some $\hat{\pi} \in \Pi$,

$$\alpha\int_S u(g(s))d\hat{\pi} + (1-\alpha)u(x) < \alpha\int_S u(r(s))d\hat{\pi} + (1-\alpha)u(y)\,. \tag{A.57}$$

Hence, there exist $x, y \in \mathcal{P}$ such that $\alpha f + (1-\alpha)x \succeq^* \alpha r + (1-\alpha)y$ and not $\alpha g + (1-\alpha)x \succeq^* \alpha r + (1-\alpha)y$, so $f \gg_r g$. The required equivalence therefore follows. ∎

Observe that $\succ_r$ is asymmetric and negatively transitive and that $\succeq_r$ is therefore its unique transitive completion. The preceding Lemma establishes that the unique transitive completion of $\gg_r$ is exactly given by the desired representation. This therefore completes the proof of the representation for any given $r \in \mathcal{F}$. Finally, observe that it follows directly from EUP that (u, Π) in Lemma 3 are unique and independent of the reference point, and hence in the RMEU representation derived above (u, Π) are independent of the choice of $r \in \mathcal{F}$. This therefore completes the proof. ∎

Proposition 1. If r' is crisp with respect to r, then r' and r are interchangeable in the RI axiom of $\succeq_r$, hence using either as the reference-point imposes the same structure on the preference relation, so (1) implies (2). (2) implies (1) follows from observing that if $\succeq_r = \succeq_{r'}$ then $\succeq_r$ must satisfy RI with respect to r', and hence r' is by definition crisp with respect to r. (2) implies (3) follows by observing that the value of the RMEU functional is 0 at r. For (3) implies (1) argue as follows. Suppose that $\succeq' \mathcal{F} \times \mathcal{F}$ is a preference relation that is transitive, complete on $\mathcal{P}$, non-trivial, continuous, monotonic and satisfies independence. For any $r \in \mathcal{F}$ define the relation $\triangleright_r$ as in the proof of Theorem 1. Let $\succeq_r$ be the transitive completion of $\triangleright_r$. Observe that from the proof of Theorem 1 $\succeq'$ has a unanimity representation via (Π, u) (Π unique, u unique up to positive affine transformations) and $\succeq_r$ has an RMEU representation with respect to the same (Π, u). Moreover, the unambiguous preference relation defined by $\succeq_r$ is $\succeq'$. If $g \succeq_r r' \Leftrightarrow \int_S u(g(s))\pi(ds) \geq \int_S u(r'(s))\pi(ds)\ \forall \pi \in \Pi$, then $g \succeq_r r'$ if and only if $g \succeq^* r'$, hence $\triangleright$ can be define also in terms of r' and this implies the same RMEU preference. For the last statement of the Proposition observe that if $u(r'(s)) = u(r(s))+b$, the constant $-b$ is an ordinal transformation of the RMEU functional. ■

Proof of Proposition 2. Suppose that $(\succeq_r^1)_{r\in\mathcal{F}}$ is a class of RMEU preference relations satisfying RT and EUP and represented by (Π_1, u_1). Suppose that $(\succeq_r^2)_{r\in\mathcal{F}}$ is another class of RMEU preference relations satisfying RT and EUP and represented by (Π_2, u_2). Denote the unambiguous preference relation of the class $(\succeq_r^1)_{r\in\mathcal{F}}$ by $\succeq_1^*$ and note that this preference relation has a unanimity representation in terms of (Π_1, u_1). Likewise, denote the unambiguous preference relation of the class $(\succeq_r^2)_{r\in\mathcal{F}}$ by $\succeq_2^*$.

First show that the following statements are equivalent:

1. The class $(\succeq_r^1)_{r\in\mathcal{F}}$ is more ambiguity averse than the class $(\succeq_r^2)_{r\in\mathcal{F}}$.
2. For all $f, g \in \mathcal{F}$, $f \succeq_1^* g \Rightarrow f \succeq_2^* g$.

First show that (1) implies (2). Fix any $f, g \in \mathcal{F}$ with $f \succeq_1^* g$. Then it follows that $f \succeq_g^1 g$. By (1) $f \succeq_g^2 g$. Hence by Lemma 3 $f \succeq_2^* g$. Now show that (2) implies (1). Fix any $r \in \mathcal{F}$ and any $f \in \mathcal{F}$ such that $f \succeq_r^1 r$. Then by Lemma 3 it follows that $f \succeq_1^* r$. By

(2) it follows that $f \succeq_2^* r$, so $f \succeq_r^2 r$. Since r and f were arbitrary, it follows that (1) and (2) are equivalent.

By Proposition 6 in Ghirardato et al. (2004) (2) is equivalent to the condition that $u_1 \cong u_2$ and $\Pi_1 \supset \Pi_2$. Hence, this establishes the first part of the Proposition. The second part follows trivially from the first, and the fact that the von Neumann/Morgenstern utility index in the SEU functional is unique only up to positive affine transformations. ■

I prove Proposition 3 before providing the proof for Theorem 2.

Proof of Proposition 3. Start with the first part of the proof which provides a representation theorem for a preference relation $\succeq_r$ for some $r \in \mathcal{F}$. The proof that (2) implies (1) follows as in the proof of Theorem 1. That u is onto implies unboundedness follows from Maccheroni et al. (2006). To prove (1) implies (2), fix some $r \in \mathcal{F}$ as a reference-point and first proceed as in the previous proof to establish a variational representation, (u, V_r), of $\succeq_r$ and a unanimity representation, (u, Π), for $\succeq_r^*$. Observe again that (with an appropriate renormalization) the von-Neumann/Morgenstern index u in both representations coincides. Moreover, Maccheroni et al. (2006) show that u is onto.

Write $u \circ f \equiv u \circ g$ if $u(f(s)) = u(g(s))$ for all $s \in S$. Denote by $\mathcal{R}_r := \{r' \in \mathcal{F} | u(r') \equiv u(r) + k_{r'}$ for some $k_{r'} \in \mathbb{R}\}$. Note that $\mathcal{R}_r$ is a mixture space: Let $r_1, r_2 \in \mathcal{R}_r$, then there exist $k_1, k_2 \in \mathbb{R}$ such that $u \circ r_1 \equiv u \circ r + k_1$ and $u \circ r_2 \equiv u \circ r + k_2$. Let $\alpha \in [0, 1]$, then

$$u \circ (\alpha r_1 + (1-\alpha) r_2) \equiv \alpha u \circ r_1 + (1-\alpha) u \circ r_2 \tag{A.58}$$

$$\equiv \alpha(u \circ r + k_1) + (1-\alpha)(u \circ r + k_2) \tag{A.59}$$

$$\equiv u \circ r + (\alpha k_1 + (1-\alpha) k_2) \tag{A.60}$$

Hence, there exists $k_3 \in \mathbb{R}$ such that for $r_3 := \alpha r_1 + (1-\alpha) r_2$ we have $u \circ r_3 \equiv u \circ r + k_3$.

Now consider any $f \in \mathcal{F}$. Since f is simple, u onto implies that there exist $\bar{r}, \underline{r} \in \mathcal{R}_r$ such that $\bar{r} \succ_r f \succ_r \underline{r}$. By continuity there exists $\alpha \in (0,1)$ such that $f \sim_r \alpha\bar{r} + (1-\alpha)\underline{r}$. Since $\mathcal{R}_r$ is a mixture space, $\alpha\bar{r} + (1-\alpha)\underline{r} \in \mathcal{R}_r$. Hence, for all $f \in \mathcal{F}$ there exists $r_f \in \mathcal{R}_r$ such that $f \sim_r r_f$.

Now suppose that $r' \in \mathcal{R}_r$. Fix any $\alpha \in (0,1)$, then it follows from RI and constant-additivity of V_r (an implication of WCI) that

$$f \succeq_r g \Leftrightarrow \alpha f + (1-\alpha)r \quad \succeq_r \quad \alpha g + (1-\alpha)r \tag{A.61}$$

$$\Leftrightarrow V_r(\alpha u \circ f + (1-\alpha)u \circ r) + (1-\alpha)k_{r'} \quad \geq \quad V_r(\alpha u \circ g + (1-\alpha)u \circ r) + (1-\alpha)k_{r'} \tag{A.62}$$

$$\Leftrightarrow V_r(\alpha u \circ f + (1-\alpha)[u \circ r + k_{r'}]) \quad \geq \quad V_r(\alpha u \circ g + (1-\alpha)[u \circ r + k_{r'}]) \tag{A.63}$$

$$\Leftrightarrow V_r(\alpha u \circ f + (1-\alpha)u \circ r') \quad \geq \quad V_r(\alpha u \circ g + (1-\alpha)u \circ r') \tag{A.64}$$

$$\Leftrightarrow \alpha f + (1-\alpha)r' \quad \succeq_r \quad \alpha g + (1-\alpha)r' \tag{A.65}$$

Since α was arbitrary and $r' \in \mathcal{R}_r$ was arbitrary, it follows that for all $r' \in \mathcal{R}_r$ $\succeq_r$ satisfies r'-Independence with respect to r'. Since RI is the only axiom defined specifically with respect to r, all properties that $\succeq_r$ satisfies relative to r are also satisfied relative to r'. In particular, inertia holds relative to any $r' \in \mathcal{F}$: $f \succeq_r r'$ if and only if $f \succeq_r^* r'$. This leads immediately to the following analog of Lemma 4.

Lemma 6 *For all $r \in \mathcal{F}$ and for all $f, g \in \mathcal{F}$ the following are equivalent:*

1. *$f \succ_r g$*
2. *There exist $r' \in \mathcal{R}_r$ such that $f \succeq_r^* r'$ and not $g \succeq_r^* r'$ (denoted $f \gg_r g$ with the obvious abuse of notation).*

Proof. Suppose $f \succ_r g$. Then $r_f \succ_r r_g$. Since $f \succeq_r r_f$ it follows by inertia that $f \succeq_r^* r_f$. Since $r_f \succ_r g$ it follows immediately that not $g \succeq_r^* r_f$ Hence, (1) implies (2). Now suppose that there exists $r' \in \mathcal{R}_r$ such that $f \succeq_r^* r'$ and not $g \succeq_r^* r'$. By inertia, not $g \succeq_r^* r'$ implies $r' \succ_r g$. Clearly, $f \succeq_r^* r'$ implies $f \succeq_r r'$. Hence, by transitivity, $f \succ_r g$. ■

Finally, provide a characterization of $\gg_r$ as in Theorem 1.

Lemma 7 *Let (u, Π) represent $\succeq_r^*$. Then for all $f, g \in \mathcal{F}$, the following statements are equivalent:*

1. $f \gg_r g$
2. $\min_{\pi\in\Pi} \int_S [u \circ f - u \circ r] d\pi > \min_{\pi\in\Pi} \int_S [u \circ f - u \circ r] d\pi$

Proof. Let $f \gg_r g$. Then there exist $r' \in \mathcal{R}_r$ such that $f \succeq_r^* r'$ and not $g \succeq_r^* r'$. Hence,

$$\int_S u(f(s))d\pi \geq \int_S u(r'(s))d\pi = \int_S u(r(s))d\pi + k_{r'} \tag{A.66}$$

for all $\pi \in \Pi$, and there exists $\hat{\pi} \in \Pi$ such that

$$\int_S u(g(s))d\hat{\pi} < \int_S u(r'(s))d\hat{\pi} + k_{r'}\,. \tag{A.67}$$

Consequently, for all $\pi \in \Pi$,

$$\int_S \left[u(f(s)) - u(r(s))\right] d\pi \geq k_{r'}\,, \tag{A.68}$$

and, for some $\hat{\pi} \in \Pi$,

$$\int_S \left[u(g(s)) - u(r(s))\right] d\pi < k_{r'}\,. \tag{A.69}$$

It follows that

$$\min_{\pi\in\Pi} \int_S \left[u(f(s)) - u(r(s))\right] d\pi \geq k_{r'}\,, \tag{A.70}$$

and

$$\min_{\pi\in\Pi} \int_S \left[u(g(s)) - u(r(s))\right] d\pi < k_{r'}\,. \tag{A.71}$$

and, hence,

$$\min_{\pi\in\Pi}\int_S [u(f(s)) - u(r(s))]\, d\pi > \min_{\pi\in\Pi}\int_S [u(g(s)) - u(r(s))]\, d\pi\,. \tag{A.72}$$

Now, suppose that

$$\min_{\pi\in\Pi}\int_S [u\circ f(s) - u\circ r(s)]\, d\pi > \min_{\pi\in\Pi}\int_S [u\circ g(s) - u\circ r(s)]\, d\pi. \tag{A.73}$$

Then, there exist $r' \in \mathcal{R}_r$ such that

$$\min_{\pi\in\Pi}\int_S [u(f(s)) - u(r(s))]\, d\pi \geq k_{r'} > \min_{\pi\in\Pi}\int_S [u(g(s)) - u(r(s))]\, d\pi. \tag{A.74}$$

Hence, for all $\pi \in \Pi$,

$$\int_S u(f(s))d\pi \geq \int_S u(r(s))d\pi + k_{r'} = \int_S u(r'(s))d\pi\,, \tag{A.75}$$

and for some $\hat{\pi} \in \Pi$,

$$\alpha\int_S u(g(s))d\hat{\pi} < \int_S u(r(s))d\hat{\pi} + k_{r'} = \int_S u(r'(s))d\hat{\pi}\,. \tag{A.76}$$

Hence, there exist $r' \in \mathcal{R}_r$ such that $f \succeq_r^* r'$ and not $g \succeq_r^* r'$, so $f \gg_r g$. The required equivalence therefore follows. ■

By the same observations at the end of the proof of Theorem 1, this therefore completes the proof of the first part of the theorem. It remains to show that every preference relation in the class $(\succeq_r)_{r\in\mathcal{F}}$ has an RMEU representation via (u_r, Π_r), then the class satisfies RT if and only if for all $r, r' \in \mathcal{F}$ (1) $u_r \cong u_{r'}$ and (2) $\Pi_r = \Pi_{r'}$.

First show that if the class satisfies RT, then (1) holds. For any $r \in \mathcal{F}$, $x, y \in \mathcal{P}$ we have that $x \succeq_r y$ if and only if

$$\min_{\pi\in\Pi_r}\int_S u_r(x(s)) - u_r(r(s))d\pi \quad\geq\quad \min_{\pi\in\Pi_r}\int_S u_r(y(s)) - u_r(r(s))d\pi \tag{A.77}$$

$$\Leftrightarrow\; u_r(x) - \max_{\pi\in\Pi_r}\int_S u_r(r(s))d\pi \quad\geq\quad u_r(y) - \max_{\pi\in\Pi_r}\int_S u_r(r(s))d\pi \tag{A.78}$$

$$\Leftrightarrow\; u_r(x) \quad\geq\quad u_r(y) \tag{A.79}$$

where $u_r(x)$ denotes $u_r(x(s))$ for any $s \in S$ (which is well-defined since $x, y \in \mathcal{P}$). Now let $r, r' \in \mathcal{F}$, $x, y \in \mathcal{F}$, and denote $(1/2)r + (1/2)r' =: \bar{r}$. Then

$$\frac{1}{2}x + \frac{1}{2}r' \succeq_{\bar{r}} \frac{1}{2}y + \frac{1}{2}r' \tag{A.80}$$

$$\Leftrightarrow \min_{\pi \in \Pi_{\bar{r}}} \int_S \left[\frac{1}{2}u_{\bar{r}}(x) + \frac{1}{2}u_{\bar{r}}(r'(s)) - \left(\frac{1}{2}u_{\bar{r}}(r(s)) + \frac{1}{2}u_{\bar{r}'}(r(s))\right)\right] d\pi \quad \geq \tag{A.81}$$

$$\min_{\pi \in \Pi_{\bar{r}}} \int_S \left[\frac{1}{2}u_{\bar{r}}(y) + \frac{1}{2}u_{\bar{r}}(r'(s)) - \left(\frac{1}{2}u_{\bar{r}}(r(s)) + \frac{1}{2}u_{\bar{r}'}(r(s))\right)\right] \quad d\pi \tag{A.82}$$

$$\Leftrightarrow u_{\bar{r}}(x) - \max_{\pi \in \Pi_{\bar{r}}} \int_S u_{\bar{r}}(r(s))d\pi \quad \geq \quad u_{\bar{r}}(y) - \max_{\pi \in \Pi_{\bar{r}}} \int_S u_{\bar{r}}(r(s))d\pi \tag{A.83}$$

$$\Leftrightarrow u_{\bar{r}}(x) \quad \geq \quad u_{\bar{r}}(y) \tag{A.84}$$

Hence, for any $r, r' \in \mathcal{F}$ (with $(1/2)r + (1/2)r' =: \bar{r}$) and for any $x, y \in \mathcal{P}$ we have that $u_r(x) \geq u_r(y)$ iff $x \succeq_r y$. By RT this is equivalent to $(1/2)x + (1/2)r' \succeq_{\bar{r}} (1/2)y + (1/2)r'$, which is equivalent to $u_{\bar{r}}(x) \geq u_{\bar{r}}(y)$, which in turn is equivalent to $(1/2)x + (1/2)r \succeq_{\bar{r}} (1/2)y + (1/2)r$. By RT, the latter is equivalent to $x \succeq_{r'} y$, which is equivalent to $u_{r'}(x) \geq u_{r'}(y)$. Since the representation of preferences on $\mathcal{P}$ via u_r is unique up to positive affine transformations given the reference-point r, and the representation of preferences on $\mathcal{P}$ via $u_{r'}$ is unique up to positive affine transformations given the reference-point r', it follows that $u_r \cong u_{r'}$.

Next show that if the class satisfies RT (2) also holds. Let $r, r' \in \mathcal{F}$ and denote by $\bar{r} := (1/2)r + (1/2)r'$. Let $\succeq_r$, $\succeq_{r'}$ and $\succeq_{\bar{r}}$ be represented by (u_r, Π_r), $(u_{r'}, \Pi_{r'})$ and $(u_{\bar{r}}, \Pi_{\bar{r}})$, respectively. By (1) it is without loss of generality to assume $u_r = u_{r'} = u_{\bar{r}} = u$. We have by the first part of the proposition that for all $f, g \in \mathcal{F}$

$$f \succeq_r g \Leftrightarrow \min_{\pi \in \Pi_r} \int_S u(f(s)) - u(r(s))d\pi \geq \min_{\pi \in \Pi_r} \int_S u(g(s)) - u(r(s))d\pi \tag{A.85}$$

By RT $f \succeq_r g$ iff $(1/2)f + (1/2)r' \succeq_{\bar{r}} (1/2)g + (1/2)r'$, and by the first part of the proposition this is equivalent to

$$\min_{\pi\in\Pi_{\bar r}} \int_S \left[\left(\frac{1}{2}u(f(s)) + \frac{1}{2}u(r'(s))\right) - \left(\frac{1}{2}u(r(s)) + \frac{1}{2}u(r'(s))\right)\right] d\pi \quad \geq \tag{A.86}$$

$$\min_{\pi\in\Pi_{\bar r}} \int_S \left[\left(\frac{1}{2}u(g(s)) + \frac{1}{2}u(r'(s))\right) - \left(\frac{1}{2}u(r(s)) + \frac{1}{2}u(r'(s))\right)\right] \quad d\pi \tag{A.87}$$

$$\Leftrightarrow \min_{\pi\in\Pi_{\bar r}} \int_S u(f(s)) - u(r(s)) d\pi \quad \geq \quad \min_{\pi\in\Pi_{\bar r}} \int_S u(g(s)) - u(r(s)) d\pi\,. \tag{A.88}$$

Hence, $f \succeq_r g$ iff (A.88) holds. Hence, $(u, \Pi_{\bar r})$ is an alternative representation of $\succeq_r$. It follows by the uniqueness of Π_r in the first part of the theorem that $\Pi_r = \Pi_{\bar r}$. Likewise, we can show that $\Pi_{r'} = \Pi_{\bar r}$, so $\Pi_r = \Pi_{r'}$.

Finally, show that if (1) and (2) hold, then the class satisfies RT. Suppose (1) and (2) hold, then the class of preference relations $(\succeq_r)_{r\in\mathcal{F}}$ can be represented by (u, Π). Let $r \in \mathcal{F}$, $f, g, h \in \mathcal{F}$ and $\alpha \in (0, 1)$. Then

$$f \quad \succeq_r \quad g \tag{A.89}$$

$$\Leftrightarrow \min_{\pi\in\Pi} \int_S u(f(s)) - u(r(s)) d\pi \quad \geq \quad \min_{\pi\in\Pi} \int_S u(g(s)) - u(r(s)) d\pi \tag{A.90}$$

$$\Leftrightarrow \min_{\pi\in\Pi} \int_S \alpha u(f(s)) - \alpha u(r(s)) d\pi \quad \geq \quad \min_{\pi\in\Pi} \int_S \alpha u(g(s)) - \alpha u(r(s)) d\pi \tag{A.91}$$

$$\Leftrightarrow \min_{\pi\in\Pi} \int_S \left[(\alpha u(f(s)) + (1-\alpha)u(h(s))) - (\alpha u(r(s)) + (1-\alpha)u(h(s)))\right] d\pi \quad \geq \tag{A.92}$$

$$\min_{\pi\in\Pi} \int_S \left[(\alpha u(g(s)) + (1-\alpha)u(h(s))) - (\alpha u(r(s)) + (1-\alpha)u(h(s)))\right] \quad d\pi \tag{A.93}$$

$$\Leftrightarrow \alpha f + (1-\alpha)h \quad \succeq_{\alpha r+(1-\alpha)h} \quad \alpha g + (1-\alpha)h\,. \tag{A.94}$$

Since, r, r', f, g, h, α were arbitrary, (1) and (2) therefore imply that the class of preferences satisfies RT. This completes the proof. ■

Proof of Theorem 2. Start with the first part of the Theorem which provides a representation for a preference relation for a given $r \in \mathcal{F}$. To show that (1) implies (2) assume that $\succeq_r$ satisfies Axioms 1-4, Axiom 7, Axiom 12 and Axiom 13 with respect to

$r \in \mathcal{F}$. Following the first step in the proof of Lemma 28 in Maccheroni et al. (2006) there exists an onto, mixture-linear von Neumann/Morgenstern utility index $u : \mathcal{P} \to \mathbb{R}$ such that preferences on $\mathcal{P}$ are represented by u. Define $\mathcal{R}_r$ as in the proof of Proposition 3 and note that, by unboundedness, for all $f \in \mathcal{F}$ there exists $r_f \in \mathcal{R}_r$ such that $f \sim_r r_f$. Moreover, by unboundedness, there exists $x_f^r \in \mathcal{P}$ such that $u(x_f^r) = k_{r_f}$. Now define a function $U : \mathcal{F} \to \mathbb{R}$ by $U(f) = u(x_f^r)$ for all $f \in \mathcal{F}$. x_r^f is not unique, but $u(x_r^f)$ is unique by monotonicity, hence U is well-defined. Now observe that $f \succeq_r g$ if and only if $r_f \succeq_r r_g$ by transitivity. Since, by definition $u \circ r_f \equiv u \circ r + u(x_f^r)$ and $u \circ r_g \equiv u \circ r + u(x_g^r)$, it follows from monotonicity that $r_f \succeq_r r_g$ if and only if $u(x_f^r) \geq u(x_g^r)$. Hence, U represents $\succeq_r$.

By unboundedness, $\{\phi \in B_0(\Sigma, \mathbb{R}) | \exists f \in \mathcal{F} \text{ s.t. } \phi \equiv u \circ f - u \circ r\} = B_0(\Sigma, \mathbb{R})$. Hence, define a functional $I : B_0(\Sigma, \mathbb{R}) \to \mathbb{R}$ by $I(u(f) - u(r)) = U(f)$ for all $f \in \mathcal{F}$. Observe three properties of this functional.

1. If 1_S denotes the indicator function on S and $k \in \mathbb{R}$, by unboundedness there exists $r' \in \mathcal{R}_r$ such that $u \circ r' - u \circ r \equiv k$. Hence, $I(k1_S) = I(u \circ r' - u \circ r) = U(r') = k$.
2. Suppose $\phi, \gamma \in B_0(\Sigma, \mathbb{R})$, then there exist $f, g \in \mathcal{F}$ such that $\phi \equiv u \circ f - u \circ r$ and $\gamma \equiv u \circ g - u \circ r$. Now suppose that ϕ and γ are pairwise comonotonic. Then there do not exist states s and s' such that $\phi(s) > \phi(s')$ and $\gamma(s') > \gamma(s)$. Hence, there do not exist states s and s' such that $u(f(s)) - u(r(s)) > u(f(s')) - u(r(s'))$ and $u(g(s')) - u(r(s')) > u(g(s)) - u(r(s))$, so there do not exist states s and s' such that $0.5u(f(s)) + 0.5u(r(s')) > 0.5u(f(s')) + 0.5u(r(s))$ and $0.5u(g(s')) + 0.5u(r(s)) > 0.5u(g(s)) + 0.5u(r(s'))$. It follows that f and g are r-comonotonic. Now suppose that there exist pairwise comonotonic $\phi_f, \phi_g, \phi_h \in B_0(\Sigma, \mathbb{R})$ with the obvious corresponding $f, g, h \in \mathcal{F}$. Suppose $\alpha \in (0, 1)$ and $I(\phi_f) > I(\phi_g)$. Then $I(u \circ f - u \circ r) > I(u \circ g - u \circ r)$, which implies $U(f) > U(g)$ and therefore $f \succ_r$

g. Since f, g and h are pairwise r-comonotonic, r-comonotonic independence implies that $\alpha f + (1-\alpha)h \succ_r \alpha g + (1-\alpha)h$, which implies $U(\alpha f + (1-\alpha)h) > U(\alpha g + (1-\alpha)h)$ and therefore $I(u \circ [\alpha f + (1-\alpha)h] - u \circ r) > I(u \circ [\alpha g + (1-\alpha)h] - u \circ)$. This is equivalent to $I(\alpha[u \circ f - u \circ (r)] + (1-\alpha)[u \circ h - u \circ r]) > I(\alpha[u \circ g - u \circ (r)] + (1-\alpha)[u \circ h - u \circ r])$. Hence, $I(\alpha\phi_f + (1-\alpha)\phi_h) > I(\alpha\phi_g + (1-\alpha)\phi_h)$.

3. Finally, suppose that $\phi_f \geq \phi_g$. Then $u(f(s)) - u(r(s)) \geq u(g(s)) - u(r(s))$, and therefore $u(f(s)) \geq u(g(s))$, for all $s \in S$. Hence, it follows from monotonicity that $f \succeq_r g$, hence $U(f) \geq U(g)$, so $I(u \circ f - u \circ r) \geq I(u \circ g - u \circ r)$. As a result, $I(\phi_f) \geq I(\phi_g)$.

It follows from Corollary 3 and the remark following it in Schmeidler (1986) that there exists a unique capacity, v, such that for all ϕ_f and ϕ_g in $B_0(\Sigma, \mathbb{R})$, $I(\phi_f) \geq I(\phi_g)$ if and only if $\oint \phi_f dv \geq \oint \phi_g dv$. Hence,

$$f \succeq_r g \;\Leftrightarrow\; \oint_S u(f(s)) - u(r(s)) dv \geq \oint_S u(g(s)) - u(r(s)) dv \tag{A.95}$$

The proof that (2) implies (1) follows directly from the proof in Schmeidler (1986), using that ϕ_f and ϕ_g pairwise comonotonic implies that f and g are pairwise r-comonotonic. The uniqueness properties of u follows from Maccheroni et al. (2006), and the uniqueness of v follows directly from the proof of Theorem 1 in Schmeidler (1989). Likewise the proof that v is convex if and only if $\succeq_r$ satisfies UA is analogous to the proof in Schmeidler (1989). This therefore completes the proof of the first part of the theorem.

It remains to show that if every preference relation in the class $(\succeq_r)_{r \in \mathcal{F}}$ has an RCEU representation via (u_r, v_r), then the class satisfies RT if and only if for all $r, r' \in \mathcal{F}$ (1) $u_r \cong u_{r'}$ and (2) $v_r = v_{r'}$. Suppose first that the class of preference relations satisfies RT and let $r, r' \in \mathcal{F}$. Using constant additivity of the Choquet integral the proof that $u_r \cong u_{r'}$

is analogous to the proof that RT implies (1) in the proof of Proposition 3. To show that RT implies (2) as well, let $r, r' \in \mathcal{F}$ and denote by $\bar{r} := (1/2)r + (1/2)r'$. Let $\succeq_r$, $\succeq_{r'}$ and $\succeq_{\bar{r}}$ be represented by (u_r, ν_r), $(u_{r'}, \nu_{r'})$ and $(u_{\bar{r}}, \nu_{\bar{r}})$, respectively. By (1) it is without loss of generality to assume $u_r = u_{r'} = u_{\bar{r}} = u$. We have by the first part of the proposition that for all $f, g \in \mathcal{F}$

$$f \succeq_r g \;\Leftrightarrow\; \oint_S u(f(s)) - u(r(s))d\nu_r \geq \oint_S u(g(s)) - u(r(s))d\nu_r \tag{A.96}$$

By RT $f \succeq_r g$ iff $(1/2)f + (1/2)r' \succeq_{\bar{r}} (1/2)g + (1/2)r'$, and by the first part of the proposition this is equivalent to

$$\oint_S \left[\left(\frac{1}{2}u(f(s)) + \frac{1}{2}u(r'(s))\right) - \left(\frac{1}{2}u(r(s)) + \frac{1}{2}u(r'(s))\right)\right] d\nu_{\bar{r}} \quad \geq \tag{A.97}$$

$$\oint_S \left[\left(\frac{1}{2}u(g(s)) + \frac{1}{2}u(r'(s))\right) - \left(\frac{1}{2}u(r(s)) + \frac{1}{2}u(r'(s))\right)\right] \; d\nu_{\bar{r}} \tag{A.98}$$

$$\Leftrightarrow \oint_S u(f(s)) - u(r(s))d\nu_{\bar{r}} \quad \geq \quad \oint_S u(g(s)) - u(r(s))d\nu_{\bar{r}}\,. \tag{A.99}$$

Hence, $f \succeq_r g$ iff (A.99) holds. Hence, $(u, \nu_{\bar{r}})$ is an alternative representation of $\succeq_r$. It follows by the uniqueness of ν_r in the first part of the theorem that $\nu_r = \nu_{\bar{r}}$. Likewise, we can show that $\nu_{r'} = \nu_{\bar{r}}$, so $\nu_r = \nu_{r'}$.

Finally, show that if (1) and (2) hold, then the class satisfies RT. Suppose (1) and (2) hold, then the class of preference relations $(\succeq_r)_{r\in\mathcal{F}}$ can be represented by (u, ν). Let $r \in \mathcal{F}$, $f, g, h \in \mathcal{F}$ and $\alpha \in (0, 1)$. Then

$$f \quad \succeq_r \quad g \tag{A.100}$$

$$\Leftrightarrow \oint_S u(f(s)) - u(r(s))d\nu \quad \geq \quad \oint_S u(g(s)) - u(r(s))d\nu \tag{A.101}$$

$$\Leftrightarrow \oint_S \alpha u(f(s)) - \alpha u(r(s))d\nu \quad \geq \quad \oint_S \alpha u(g(s)) - \alpha u(r(s))d\nu \tag{A.102}$$

$$\Leftrightarrow \oint_S [(\alpha u(f(s)) + (1-\alpha)u(h(s))) - (\alpha u(r(s)) + (1-\alpha)u(h(s)))] \;\; d\nu \quad \geq$$

$$\oint_S [(\alpha u(g(s)) + (1-\alpha)u(h(s))) - (\alpha u(r(s)) + (1-\alpha)u(h(s)))] \;\; d\nu \tag{A.103}$$

$$\Leftrightarrow \;\; \alpha f + (1-\alpha)h \;\; \succeq_{\alpha r+(1-\alpha)h} \;\; \alpha g + (1-\alpha)h\,. \tag{A.104}$$

Since, r, r', f, g, h, α were arbitrary, (1) and (2) therefore imply that the class of preferences satisfies RT. This completes the proof. ■

A.2 Proofs of results in Chapter 3

A.2.1 Superdifferential

The proofs use a characterization of the superdifferential of an RMEU preference, derived from the relation to the variational preference model in Maccheroni et al. (2006). Let $I : B_0(\Sigma) \to \mathbb{R}$, then given any $f \in B_0(\Sigma)$ the directional derivative of V at f is the functional $I'(f,\cdot) : B_0(\Sigma) \to \mathbb{R}$ defined by

$$I'(f,h) = \lim_{t \downarrow 0} \frac{V(f+ht) - V(f)}{t} \;\forall h \in B_0(\Sigma)\,. \tag{A.105}$$

The superdifferential of a concave functional $I : B_0 \to \mathbb{R}$ at $f \in B_0(\Sigma)$ is the set $\partial I(f)$ of all linear and supnorm continuous functionals $L : B_0(\Sigma) \to \mathbb{R}$ such that

$$I'(f,h) \leq L(h) \;\forall h \in B_0(\Sigma)\,. \tag{A.106}$$

Every $L \in \partial I(f)$ is a *supergradient* of I at f.

Lemma 8 *Let $V : \mathcal{F} \to \mathbb{R}$ be the RMEU representation defined by Eq. 2.11. For all $f \in \mathcal{F}$,*

$$\partial V(f) = \left\{ u'(f)\, dm \,\middle|\, m \in \arg\min_{\pi \in \Pi} \int_S (u(f(s)) - u(r(s))\pi(ds)) \right\}. \tag{A.107}$$

Proof. The RMEU preference relation is a variational preference. As a result, the proof follows directly from Maccheroni et al. (2006). ■

When S is finite every supergradient is an element of $\mathbb{R}^S$. The normalized superdifferential of I at f consists of all supergradients L of I at f renormalized so that $L \in \Delta(S)$. For a concave function the normalized superdifferential exists and is a convex, bounded set at every f in the interior of the domain of I. Denote the normalized superdifferential by $\hat{\partial} I$. Corollary 7 follows immediately from the general characterization of the superdifferential for RMEU preferences.

Corollary 7 *Suppose that there is a finite number of states, $s = 1, ..., S$, then the normalized superdifferential of V at f is*

$$\hat{\partial} V(f) = \left\{ \left(\frac{\hat{\pi}_1 u'(f(1))}{\sum_{s=1}^S \hat{\pi}_s u'(f(s))}, ..., \frac{\hat{\pi}_S u'(f(S))}{\sum_{s=1}^S \hat{\pi}_s u'(f(s))} \right) \middle| \hat{\pi} \in \arg\min_{\pi \in \Pi} \sum_{s=1}^S \pi_s \left(u(f(s)) - u(r(s)) \right) \right\}. \quad \text{(A.108)}$$

A.2.2 Characterization of a competitive equilibrium

I first provide an elementary characterization of a competitive equilibrium of the economy $(e^i, \Pi_i, u_i)_{i \in I}$. In the following, $\partial V_i(x)$ denotes the supperdifferential of V_i at x for all $x \in \mathbb{R}^S_+$ and all $i \in \mathcal{I}$, and $\hat{\partial} V_i(x)$ denotes the normalized superdifferential x for all $x \in \mathbb{R}^S_+$ and all $i \in \mathcal{I}$.

Theorem 6 (Characterization of CE) *A price and allocation $(p^*, x^*) \in \Delta(S) \times \mathbb{R}^{IS}_+$ is a competitive equilibrium of $(e^i, \Pi_i, u_i)_{i \in I}$ if and only if (1) $(p^*, x^*) >> 0$, (2) $p^* \in \hat{\partial} V_i \left(x^{i*} \right)$ for all $i \in \mathcal{I}$, (3) $p^* x^{i*} = p^* e^{i*}$ for all $i \in \mathcal{I}$, and (4) $\sum_{i \in \mathcal{I}} x^{i*}_s = \sum_{i \in \mathcal{I}} e^i_s$ for all $s \in S$.*

Proof. Suppose that (p^*, x^*) is a competitive equilibrium. $p^* >> 0$ follows from strongly monotone preferences by standard arguments. Since $e^i >> 0$ for all $i \in \mathcal{I}$ and preferences are strongly monotone, it follows from $p^* >> 0$ and standard arguments that $p^* x^{i*} = p^* e^i$ for all $i \in \mathcal{I}$. Now suppose that $x^{i*}_s = 0$ for some $i \in \mathcal{I}$ and some $s \in S$. By

$p^* x^{i*} = p^* e^i$ there exists $x_t^{i*} > 0$. Consider the alternative allocation x^i with $x_s^i = \epsilon / p_s^*$ for some $\epsilon > 0$, $x_t^i - \epsilon / p_t^*$, and $x_k^i = x_k^{i*}$ for all $k \neq s, t$. For $\epsilon > 0$ sufficiently small $x_t^i > 0$ and $p^* x^i = p^* e^i$, hence x^i is feasible. Moreover, by the Inada condition on u_i, for all $\pi \in \Pi_i$ there exists $\epsilon > 0$ sufficiently small such that $E_{u_i}^{\pi}(x^i) > E_{u_i}^{\pi}(x^{i*})$ (where $E_{u_i}^{\pi}(x)$ denotes the expected utility of x given Bernoulli utility u_i and prior π). Hence, x^{i*} can not be optimal for consumer i at price p^*. It follows that $x^* >> 0$. Condition (4) is a market clearing condition satisfied by definition of a competitive equilibrium. Finally, since (p^*, x^*) is a competitive equilibrium, x^{i*} is optimal on $\{x | p^* x \leq p^* e^i\}$ for each consumer $i \in \mathcal{I}$. Since V_i is strictly concave, $x^{i*} >> 0$ and $p x^{i*} = p e^i$ it follows from the generalized Kuhn-Tucker first order conditions that there exists $\lambda > 0$ such that $0 \in \partial V_i(x^{i*}) - \lambda p^*$; hence $\lambda p^* \in \partial V_i(x^{i*})$. Since $p^* \in \Delta(S)_{++}$, the latter condition is equivalent to $p^* \in \hat{\partial} V_i(x^{i*})$ so condition (2) is satisfied.

Now suppose that conditions (1)-(4) are satisfied for some $(p^*, x^*) \in \Delta(S) \times \mathbb{R}_+^{IS}$. By condition (4) market clearing is satisfied. By conditions (1) and (3) all allocations are feasible for each consumer $i \in \mathcal{I}$. To show that x^{i*} is optimal in the budget of any consumer $i \in \mathcal{I}$, suppose that there exists an alternative allocation $x^i \geq 0$ with $p x^i \leq p e^i$. By condition (2) there exists $\lambda > 0$ such that $\lambda p^* \in \partial V_i(x^{i*})$ and hence $V_i(x^i) - V(x^{i*}) \leq \lambda p^*(x^i - x^{i*})$. Since $p x^{i*} = p e^i$, it follows that $\lambda p^*(x^i - x^{i*}) \leq 0$, hence $V_i(x^i) \leq V(x^{i*})$, so x^{i*} is optimal on $\{x | p^* x \leq p^* e^i\}$ for each $i \in \mathcal{I}$. Hence, (p^*, x^*) is a competitive equilibrium. ∎

A.2.3 Proofs

Proof of Theorem 3. Start with (1). By Theorem 6 (x, p) is an equilibrium of the economy if and only if $(x, p) >> 0$, $p x^i = p e^i$, $p \in \hat{\partial} V_i(x^i)$ for all $i = 1, ..., I$, and $\sum_i x^i =$

$\sum_i e^i$. By Corollary 7, if $\bigcap_i \Pi_i(e^i) \neq \emptyset$ there exists $p^* \in \bigcap_i \hat{\partial} V_i(e^i)$ and hence (p^*, e) is a competitive equilibrium. Now suppose that (x', p') is also a competitive equilibrium, but with trade. By the strict convexity of preferences $x^{i'} \succ_i e^i$ for all $i = 1, ..., I$ for whom $x^{i'} \neq e^i$. Hence, $p^* x^{i'} > p^* e^{i'}$ and therefore $p^*(\sum_i (x^{i'} - e^i)) > 0$, a contradiction. As a result, $\bigcap_i \Pi_i(e^i) \neq \emptyset$ implies that e^i is the unique competitive equilibrium allocation. To show the converse, suppose that $\bigcap_i \partial\Pi_i(e^i) = \emptyset$. Then by Corollary 7 there does not exist a $p \in \bigcap \hat{\partial} V_i(e^i)$; hence by Theorem 6 the initial endowment e^i is not an equilibrium allocation.

Next show (2). Suppose that $|\bigcap_i \Pi_i(e^i)| > 1$. Then there exist $p_1, p_2 \in \bigcap_i \Pi_i(e^i)$ with $p_1 \neq p_2$. Observe that Π_i convex implies $\Pi_i(e^i)$ convex, and hence $\bigcap_i \Pi_i(e^i)$ convex. As a result, $\alpha p_1 + (1 - \alpha) p_2 =: p_\alpha \in \bigcap_i \Pi_i(e^i)$ for all $\alpha \in [0, 1]$. By the same argument as above, (p_α, e) is a competitive equilibrium for all $\alpha \in (0, 1)$ and, hence, there is a market collapse. To show the converse, observe that if $|\bigcap_i \Pi_i(e^i)| \leq 1$, then either $|\bigcap_i \Pi_i(e^i)| = 1$ or $|\bigcap_i \Pi_i(e^i)| = 0$. From (1) we know that $|\bigcap_i \Pi_i(e^i)| = 0$ implies that there is trade in equilibrium (hence no market collapse). From (2) we conclude the existence of a unique $p^* \in \bigcap_i \Pi_i(e^i)$ and, hence, the existence of a unique no-trade equilibrium (p^*, e). As a result, there is no market collapse.

Finally show (3). Suppose that $\text{int} \bigcap_i \Pi_i(e^i) \neq \emptyset$ and let $p^* \in \text{int} \bigcap_i \Pi_i(e^i)$. Since $u_i \in C^1$, for all $\epsilon > 0$ there exists $\delta^i > 0$ such that $|e^{i'} - e^i| < \delta^i$ implies

$$\left| \left(\frac{p_1^* u_i'(e_1^{i'})}{\sum_s p^* u_i'(e_s^{i'})}, ..., \frac{p_S^* u_i'(e_S^{i'})}{\sum_s p^* u_i'(e_s^{i'})} \right) - \left(\frac{p_1^* u_i'(e_1^{i})}{\sum_s p_s^* u_i'(e_s^{i})}, ..., \frac{p_S^* u_i'(e_S^{i})}{\sum_s p_s^* u_i'(e_s^{i})} \right) \right| < \frac{\epsilon}{2}. \tag{A.109}$$

Moreover, by linearity, for all $\epsilon > 0$ there exists $\delta' > 0$ such that $|p' - p^*| < \delta'$ implies,

$$\left| \left(\frac{p_1^* u_i'(e_1^{i})}{\sum_s p^* u_i'(e_s^{i})}, ..., \frac{p_S' u_i'(e_S^{i})}{\sum_s p_s^* u_i'(e_s^{i})} \right) - \left(\frac{p_1' u_i'(e_1^{i})}{\sum_s p_s' u_i'(e_s^{i})}, ..., \frac{p_S' u_i'(e_S^{i})}{\sum_s p_s' u_i'(e_s^{i})} \right) \right| < \frac{\epsilon}{2}. \tag{A.110}$$

Let $\delta = \min_i\{\delta_i\}$, then $|e^i - e^{i'}| < \delta$ for all $i = 1, ..., I$ and $|p' - p^*| < \delta'$ implies that

$$
\begin{aligned}
& \left|\left(\frac{p_1^* u_i'(e_1^i)}{\sum_s p^* u_i'(e_s^i)}, ..., \frac{p_S^* u_i'(e_S^i)}{\sum_s p_s^* u_i'(e_s^i)}\right) - \left(\frac{p_1' u_i'(e_1^{i'})}{\sum_s p_s' u_i'(e_s^{i'})}, ..., \frac{p_S' u_i'(e_S^{i'})}{\sum_s p_s' u_i'(e_s^{i'})}\right)\right| \\
\leq & \left|\left(\frac{p_1^* u_i'(e_1^{i'})}{\sum_s p^* u_i'(e_s^{i'})}, ..., \frac{p_S^* u_i'(e_S^{i'})}{\sum_s p^* u_i'(e_s^{i'})}\right) - \left(\frac{p_1^* u_i'(e_1^i)}{\sum_s p_s^* u_i'(e_s^i)}, ..., \frac{p_S^* u_i'(e_S^i)}{\sum_s p_s^* u_i'(e_s^i)}\right)\right| \quad \text{(A.111)} \\
+ & \left|\left(\frac{p_1^* u_i'(e_1^i)}{\sum_s p^* u_i'(e_s^i)}, ..., \frac{p_S' u_i'(e_S^i)}{\sum_s p_s^* u_i'(e_s^i)}\right) - \left(\frac{p_1' u_i'(e_1^i)}{\sum_s p_s' u_i'(e_s^i)}, ..., \frac{p_S' u_i'(e_S^i)}{\sum_s p_s' u_i'(e_s^i)}\right)\right| \quad \text{(A.112)} \\
\leq & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \text{(A.113)}
\end{aligned}
$$

for all $i = 1, ..., I$. Since, $p^* \in \text{int} \bigcap_i \Pi_i(e^i)$ there exists some $\bar{\delta} > 0$ such that $|p'' - p^*| < \bar{\delta}$ implies $p'' \in \text{int} \bigcap_i \Pi_i(e^i)$. Hence, if e' is in the open ball centered at e with radius δ, $B_\delta(e)$ and $0 < |p' - p^*| < \min\{\delta', \bar{\delta}\}$, then $p' \in \text{int} \bigcap_i \Pi_i(e^{i'})$. It follows by part (2) that there is a market collapse for all $e' \in B_\delta(e)$. This concludes the proof. ■

Proof of Corollary 4. Consider first the economy $(e^i, \hat{\pi})_{i \in \mathcal{I}'}$. It follows by standard arguments that this economy has a unique competitive equilibrium with equilibrium price p^* satisfying

$$\frac{p_s^*}{p_t^*} = \frac{\hat{\pi}_s e_t^{\mathcal{I}'}}{\hat{\pi}_t e_s^{\mathcal{I}'}}. \quad \text{(A.114)}$$

It follows immediately that in any non-participation equilibrium p^* must be the equilibrium price. Moreover, it follows directly from Corollary 7 that any investors $i \in \mathcal{I}^*$ does not trade at the price p^* if and only if $p^* \in \Pi_i(e^i)$. Hence, there exists a non-participation equilibrium if and only if $p^* \in \bigcap_{i \in \mathcal{I}^*} \Pi_i(e^i)$. Hence, Condition 3.12 is necessary and sufficient for the existence of a non-participation equilibrium. ■

Proof of Theorem 4. First note that, by Corollary 7, at any full-insurance allocation x^i

the normalized superdifferential of investor i is given by

$$\begin{aligned}
\hat{\partial} V_i(x^i) &= \left\{ \left(\frac{\hat{\pi}_1 u'(x^i)}{\sum_{s=1}^{S} \hat{\pi}_s u'(x^i)}, \ldots, \frac{\hat{\pi}_S u'(x^i)}{\sum_{s=1}^{S} \hat{\pi}_s u'(x^i)} \right) \middle| \, \hat{\pi} \in \arg\min_{\pi \in \Pi_i} \sum_{s=1}^{S} \pi_s \left(u(x^i) - u(e^i(s)) \right) \right\} && \text{(A.115)} \\
&= \left\{ \hat{\pi} \middle| \, \hat{\pi} \in \arg\min_{\pi \in \Pi_i} \sum_{s=1}^{S} \pi_s \left(u(x^i) - u(e^i(s)) \right) \right\} && \text{(A.116)} \\
&= \left\{ \hat{\pi} \middle| \, \hat{\pi} \in \arg\max_{\pi \in \Pi_i} \sum_{s=1}^{S} \pi_s u(e^i(s)) \right\} && \text{(A.117)} \\
&= \hat{\Pi}_i(e^i) && \text{(A.118)}
\end{aligned}$$

Now suppose that $(p, x) \in \Delta(S) \times \mathbb{R}_{+}^{IS}$ is a full-insurance equilibrium allocation. Then by Theorem 6 $p \in \hat{\partial} V_i(x^i)$ for all $i \in \mathcal{I}$; hence, $\bigcap_{i \in \mathcal{I}} \hat{\Pi}_i(e^i) \neq \emptyset$. To show the converse, suppose that $\bigcap_{i \in \mathcal{I}} \hat{\Pi}_i(e^i) \neq \emptyset$ and, in particular, let $p^* \in \bigcap_{i \in \mathcal{I}} \hat{\Pi}_i(e^i)$. Then $p^* \in \Delta(S)_{++}$. Consider the (SEU) economy with common prior p^*, $(p^*, u_i, e^i)_{i \in \mathcal{I}}$. By standard arguments there exists a full insurance allocation $x^* >> 0$ such that (p^*, x^*) is a competitive equilibrium of the economy $(p^*, u_i, e^i)_{i \in \mathcal{I}}$. Now observe that (1) $(p^*, x^*) >> 0$, (2) $p^* \in \hat{\Pi}_i(e^i) = \hat{\partial} V_i \left(x^{i*} \right)$ for all $i \in \mathcal{I}$, (3) $p^* x^{i*} = p^* e^{i*}$ for all $i \in \mathcal{I}$, and (4) $\sum_{i \in \mathcal{I}} x_s^{i*} = \sum_{i \in \mathcal{I}} e_s^i$ for all $s \in S$. Conditions (3) and (4) follow from the fact that x^* is an equilibrium allocation in $(p^*, u_i, e^i)_{i \in \mathcal{I}}$. Hence, by Theorem 6, (p^*, x^*) is a full-insurance, competitive equilibrium allocation. ■

Proof of Corollary 5. From Theorem 4 there exists a full-insurance equilibrium if and only if $\bigcap_{i \in \mathcal{I}} \hat{\Pi}_i(e^i) \neq \emptyset$. Now suppose that for all $i \in \mathcal{I}$ $e_1^i = e_2^i$. Then $\hat{\Pi}_i(e^i) = \Pi_1$ for all i, and hence it follows from Billot et al. (2000) that there exists a full insurance equilibrium if (and only if) there is weak agreement about the likelihood of states. To prove the converse, consider the contrapositive and suppose that e is not a full-insurance insurance allocation. For sake of contradiction suppose that (p, x) is a competitive equilibrium of the economy $(e^i, u_i, \Pi_i)_{i \in \mathcal{I}}$ in which x is a full-insurance allocation. Since e is not a full-insurance allocation, it follows that for at least one $j \in \mathcal{I}$ $e_1^j < x_1^j$, and hence $e_2^j > x_2^j$. As a result, (by the market clearing condition) there must be at least one $k \in \mathcal{I}$ for whom $e_1^k > x_1^k$ and $e_2^k < x_2^k$. Let $\pi^* \in \text{int} \bigcap_{i \in \mathcal{I}} \Pi_i$ (such a distribution exists by the weak agreement assumption). It follows from $u_j(x^j_1) - u_j(e^j_1) > 0 > u(x^j_2) - u(e^j_2)$ that the argument $\pi(x^j)$ that solves $\min_{\pi\in\Pi_j} \pi_1(u_j(x^j_1) - u_j(e^j_1)) + \pi_2(u(x^j_2) - u(e^j_2))$ is unique and $\pi_1(x^j) < \pi^*_1$. Likewise, the argument $\pi(x^k)$ that solves $\min_{\pi\in\Pi_k} \pi_1(u_j(x^k_1) - u_j(e^k_1)) + \pi_2(u(x^k_2) - u(e^k_2))$ is unique and $\pi_1(x^k) > \pi^*_1$. Hence, $\bigcap_{i\in\mathcal{I}} \hat{\Pi}_i(e^i) = \emptyset$ and by Theorem 6 there is no full-insurance equilibrium allocation; a contradiction. It follows that if e is not a full-insurance allocation then there can be no equilibrium allocation with full-insurance. ■

Proof of Corollary 6. It is straightforward to show that given the assumption on priors

$$\max_{\pi\in\Pi_i(\pi^*,\epsilon)} \sum_s \pi_s u_i\left(e^i_s\right) = (1-\epsilon)\sum_S \pi^*_s u_i\left(e^i_s\right) + \epsilon \max_{s\in S}\left\{u\left(e^i_s\right)\right\} \tag{A.119}$$

for all $i \in \mathcal{I}$ (see, e.g., Epstein and Schneider, 2003). Given the particular endowment distribution e, define for each investor i $S_i(e^i) := \arg\max_{s\in S}\left\{u\left(e^i_s\right)\right\}$. Then for all $i \in \mathcal{I}$

$$\hat{\Pi}_i(e^i) = \quad \Big\{\pi \in \Delta(S)\Big| \pi_s = (1-\epsilon)\pi^*_s \;\forall\, s \in S - S_i(e^i)\;, \tag{A.120}$$

$$\pi_t = \gamma_t\epsilon + (1-\epsilon)\pi^*_t \;\forall\, t \in S_i(e^i) \text{ for some } \gamma \in \Delta(S_i(e^i))\Big\}. \tag{A.121}$$

Now suppose that for all $i \in \mathcal{I}$ and for all $s, t \in S$, $e^i_s = e^i_t$. Then $S_i(e^i) = S$ for all $i \in \mathcal{S}$ and therefore

$$\hat{\Pi}_i(e^i) = \Big\{\pi \in \Delta(S)\Big| \pi_t = \gamma_t + (1-\epsilon)\pi^*_t \;\forall\, t \in S_i(e^i) \text{ for some } \gamma \in \Delta(S)\Big\}\;. \tag{A.122}$$

In particular, $\pi^* \in \Pi_i(e^i)$ for all $i \in \mathcal{I}$. Hence, by Theorem 4 there exists a full-insurance equilibrium.

To show the converse, suppose that there exists some investor h for whom $e^h_t > e^h_k$ for two states $t, k \in S$. It then follows that for this investor $S_h(e^h) \neq S$. Moreover, by Assumption 2, for every state $s' \in S_h(e^h)$ there exists another investor $j \neq h$ for whom $s' \notin S_j(e^j)$ (otherwise $\sum_i e^i_t > \sum_i e^i_{s'}$). Now suppose for sake of contradiction that there exists $p^* \in \bigcap_i \hat{\Pi}_i(e^i)$. From the previous argument we have that $p^*_{s'} = (1-\epsilon)\pi^*_{s'}$ for all

$s' \in S_h(e^h)$. However, by (A.120) $\sum_{s' \in S_h(e^h)} p^*_{s'} = (1-\epsilon) \sum_{s' \in S_h(e^h)} \pi^*_{s'} + \epsilon$; a contradiction for all $\epsilon > 0$. Hence, $\bigcap_i \hat{\Pi}_i(e^i) = \emptyset$ and by Theorem 4 there does not exist a full-insurance equilibrium. Since, this is the contrapositive, it follows that existence of a full-insurance equilibrium implies that e is a full insurance allocation. ■

A.3 Proofs of results in Chapter 4

Proof of Theorem 5. To prove (1) implies (2), assume that the collection $\left(\succeq^r_{t,\omega}\right)_{(t,\omega)\in T\times\Omega}$ satisfies CP, RMEU, RP, FS, DC, BW and IMP, and that the collection $\left(\left(\succeq^r_{t,\omega}\right)_{(t,\omega)\in T\times\Omega}\right)_{r\in R}$ satisfies EUP and RT. For each $r \in \mathcal{F}$, identify for each $f \in \mathcal{F}$ a $u(f) - u(r) = \vartheta_r \in B_0(\Sigma)$. Define the relation $\gg_r$ on $B_0(\Sigma)$ such that $f \succeq_r g$ if and only $\vartheta_f \gg_r \vartheta_g$. Note that $\gg_r$ satisfies Axioms A1-A6 in Gilboa and Schmeidler (1989) (just observe that the RMEU representation in Theorem 1 is the unique superadditive, homogeneous of degree one, monotonic and constant-additive functional $I : B_0(\Sigma) \to \mathbb{R}$ such that $f \succeq_r g$ iff and only if $I(u \circ f - u \circ r) \geq I(u \circ g - u \circ r)$). Since $B_0(\Sigma)$ is a norm dense subspace of $B(\Sigma)$ (with respect to the sup-norm), there exists a unique continuous extension of I to $B(\Sigma)$ that is also super-additive, homogeneous of degree one, monotonic and constant-additive (Lemma 3.4 in Gilboa and Schmeidler (1989)). Hence, using BW as in Proposition 4.1 of Gilboa and Schmeidler (1989), define the extension of $\gg_r$ (also denoted $\gg_r$) from $B_0(\Sigma)$ to $B(\Sigma)$ by $\vartheta_f \gg_r \vartheta_g$ if and only if $I(u(f) - u(r)) \geq I(u(g) - u(r))$. Then $\gg_r$ satisfies Axioms A1-A6 in Gilboa and Schmeidler (1989) and $\gg_r$ on $B(\Sigma)$ is therefore the unique monotonic extension of $\gg_r$ on $B_0(\Sigma)$ (Gilboa and Schmeidler, 1989, see also Epstein and Schneider (2003)). It follows that for any $r \in \mathcal{F}$, $\succeq^r_{t,s}$ has an RMEU representation at every $(t,s) \in \mathcal{T} \times S$.

Now consider the case $r \in \mathcal{P}^\infty$. For r constant, $\succeq^r_{t,s}$ satisfies the Axioms in Epstein and Schneider (2003) at every $(t, s) \in \mathcal{T} \times S$. As a result, Theorem B.1 in Epstein and Schneider (2003) implies that $\left(\succeq^r_{t,s}\right)_{(t,s)\in\mathcal{T}\times S}$ has a recursive multiple priors representation, $V^r_t(f, s) = \min_{m\in\Pi_t(s)} \int \sum_{\tau\geq t} \beta^{\tau-t} u(f_t) dm$ with a weak*-closed, convex and $\mathcal{G}_t$-rectangular set of priors Π, with all measures in Π having full local support; a $0 < \beta < 1$ and a mixture-linear, non-constant $u : \mathcal{P} \to \mathbb{R}$, where $\max_{\mathcal{P}} u$ and $\min_{\mathcal{P}} u$ exist. Denote the implied (t, s) unambiguous preference relations $\succeq^*_{t,s}$. It is straightforward to show that $\succeq^*_{t,s}$ inherits CP, FS and DC from $\succeq^r_{t,s}$, and that $\succeq^*_{t,s}$ has a unanimity representation with respect to u, β and the $\mathcal{G}_t$-rectangular set of priors Π. Hence, for the constant reference-point, the set of priors in the unanimity representation is updated prior-by-prior from the $\mathcal{G}_t$-rectangular set of priors Π.

Observe that by EUP, $\succeq^*_{0,s}$ does not depend on r, hence is the same for all $r \in \mathcal{F}$. As a result, it follows from Theorem 1 and the earlier observations that for all $r \in \mathcal{F}$, that $\succeq^r_{0,s}$ has an RMEU representation, $\min_{m\in\Pi} \int \sum_t \beta^t [u(f_t) - u(r_t)] dm$, with a weak*-closed, convex and $\mathcal{G}_t$-rectangular set of priors Π, with all measures in Π having full local support; a $0 < \beta < 1$ and a mixture-linear, non-constant $u : \mathcal{P} \to \mathbb{R}$, where $\max_{\mathcal{P}} u$ and $\min_{\mathcal{P}} u$ exist; and where β, Π and u correspond (up to renormalization of u) to the respective counterparts obtained for a constant act. Now define for each $f, f' \in \mathcal{F}$ the act

$$f_{\mathcal{G}_t(s)} f' = \begin{cases} (f_0(s'), f_1(s'), ...) & \text{if } s' \in \mathcal{G}_t(s) \\ (f_0(s'), f_1(s'), ..., f_t(s'), f'_{t+1}(s'), f'_{t+2}(s'), ...) & \text{if } s' \notin \mathcal{G}_t(s) \end{cases} \tag{A.123}$$

Note that DC and independence of $\succeq^*$ imply that for all $r \in \mathcal{F}$, if $f \succeq^*_{t+1,s} f'$ for all $s' \in \mathcal{G}_t(s)$ then $f_{\mathcal{G}_t(s)} h \succeq^*_{t,s} f_{\mathcal{G}_t(s)} h$ for all $h \in \mathcal{F}$, and for all $(t, s) \mathcal{T} \times S$. Moreover, CP and independence imply that if $f_{\mathcal{G}_t(s)} h \succeq^*_{t,s} f_{\mathcal{G}_t(s)} h$ then $f \succeq^*_{t+1,s} f'$ for all $s' \in \mathcal{G}_t(s)$, for all $(s, t) \in \mathcal{T} \times S$. Hence, by standard arguments, the set of priors in the unanimity representation of $\succeq^*_{t,s}$ is updated by applying Bayes rule prior-by-prior for all $(s, t) \in \mathcal{T} \times S$ and for all $r \in \mathcal{F}$. It follows that the unambiguous preference relation $\succeq^*_{t,s}$

is independent of the reference point for all $(t, s) \in \mathcal{T} \times S$, since it corresponds for each time-event pair and for each reference-point $r \in \mathcal{F}$ to the unambiguous preference relation for the recursive multiple priors representation for a constant reference-point. As a result, by Theorem 1, for any reference-point $r \in \mathcal{F}$, the collection $\left(\succeq_{t,s}^{r}\right)_{(t,s)\in\mathcal{T}\times S}$ has a recursive RMEU representation $V_t^r(f, s) = \min_{m\in\Pi} \int \sum_t \beta^t [u(f_t) - u(r_t)] dm$, with a weak*-closed, convex and $\mathcal{G}_t$-rectangular set of priors Π, with all measures in Π having full local support; a $0 < \beta < 1$ and a mixture-linear, non-constant $u : \mathcal{P} \to \mathbb{R}$, where $\max_{\mathcal{P}} u$ and $\min_{\mathcal{P}} u$ exist. The uniqueness properties follow from Epstein and Schneider (2003). The proof that (2) implies (1) is standard, that (2) implies IMP is shown in Epstein and Schneider (2003). This therefore concludes the proof. ■

Proof of Proposition 6.

In general, V is not differentiable. However, since V is concave it does have a well-defined superdifferential. It will be sufficient to study the superdifferential for a basic consumption process c with $c_t(s^t) = c^*(s_t)$, and to focus on the one-period-ahead superdifferential. Let c be a basic consumption process, and denote by $H := \mathbb{R} \times C(S)$ the set of real-valued, continuous processes $h = \{h_t\}_{t=1}^{\infty}$ with $h_t \equiv 0$ for all $t \neq 1, 2$ (where $C(S)$ denotes the set of continuous real-valued functions on S). Then call V' a one-period-ahead supergradient of $V(\cdot, s)$ at c if V' is a continuous linear functional on $\mathbb{R} \times C(S)$ satisfying

$$V(c + h; s) - V(c; s) \leq V'(h_1, h_2) \tag{A.124}$$

for all $(h_1, h_2) \in H$ such that $c + h \in \mathcal{D}$. The one-period-ahead superdifferential $\partial V(c; s)$ of V at c in state s is the set of all one-period-ahead supergradients of V at c given s. Epstein and Wang (1994) provide a characterization of the supperdifferential of V at a basic consumption process c for the special case when e is a constant process. For the sequel it is sufficient to consider only the one-period-ahead superdifferential of V at the general basic endowment process e, which follows immediately from Lemma 1 in Epstein and Wang (1994):

Lemma 9 *The superdifferential of V at e given s, denoted $\partial V(e; s)$, can be viewed as a subset of $\mathbb{R}_+ \times \Delta^+(S)$ given by*

$$\partial V(e; s) = \{(u'(e^*(s)), \pi) | \pi \in \Delta^+(S),\ \exists \hat{\pi} \in \Pi(s)\ s.t.\ d\pi = u'(e^*) d\hat{\pi}\} \tag{A.125}$$

Proof. The proof follows directly from Lemma 1 of Epstein and Wang (1994) and the observation that $\int V(e; s) d\hat{\pi} = \min_{\pi \in \Pi(s)} \int V(c; s) d\pi$ for all $\hat{\pi} \in \Pi(s)$. ■

Given Lemma 9, the proof of Theorem 2(a) in Epstein and Wang (1994) demonstrates that if $\{\pi_t\}$ is a continuous selection from Π, there exists a unique price process $p \in \mathcal{D}^N$ that satisfies Equations (4.10), and that this price process is an equilibrium price. Existence of equilibrium then follows since $\Pi(s)$ is compact-valued, convex-valued and continuous, and by Lemma 1B in Epstein and Wang (1994) adapted from Theorem 3.2 in Michael (1956) therefore admits a continuous selection. As a result, an equilibrium exists for each continuous selection from Π, at least one such continuous selection exists, and any continuous selection leads to a unique price process satisfying 4.10 which is necessary for equilibrium. This therefore completes the proof of both parts of the proposition. ■

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