OUTER SPACE FOR 2-DIMENSIONAL RAAGS AND FIXED POINT SETS OF FINITE SUBGROUPS

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In [CCV07], Charney, Crisp, and Vogtmann construct an outer space for a 2-dimensional right-angled Artin group A_Γ. It is a contractible space on which a finite index subgroup $\mathrm{Out}^0(A_\Gamma)$ of $\mathrm{Out}(A_\Gamma)$ acts properly. We construct a different outer space $\mathcal{S}(A_\Gamma)$ for A_Γ and show that non-empty fixed point sets of finite subgroups of $\mathrm{Out}^0(A_\Gamma)$ are contractible in this space. While Culler's realization theorem ([Cul84]) implies that fixed point sets of finite subgroups of $\mathrm{Out}(F_n)$ are always non-empty in the Culler-Vogtmann outer space, there is no direct counterpart to this result in the case of right-angled Artin groups and $\mathcal{S}(A_\Gamma)$. We present some methods for constructing elements in fixed point sets of finite subgroups and examine cases where such methods are applicable.

BIOGRAPHICAL SKETCH

Victor Kostyuk was born on June 11, 1983, in Kharkov, Ukraine (then the Soviet Union). He attended elementary and middle school in Israel, and graduated from the Arkansas School for Mathematics and Sciences in Hot Springs, Arkansas. He completed a bachelor's degree in computational mathematics at the Rochester Insitute of Technology, spending a year in the Budapest Semesters in Mathematics program and later a semester in the Math in Moscow program on an AMS scholarship. It was at the Independent University of Moscow that he met Prof. Yulij Ilyashenko, who spoke very highly of Cornell university and its graduate program in mathematics. This opinion Victor does not regret taking to heart and has since come to share.

To my family and to Athena.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

The following sections briefly summarize the results of this work, and then present the relevant background: right-angled Artin groups (RAAGs), classifying spaces $\underline{E}G$, basic properties of the Culler-Vogtmann outer space of a free group, and the outer space for 2-dimensional RAAGs defined by Charney, Crisp, and Vogtmann.

1.1 A brief summary

A *right-angled Artin group* (RAAG) can be defined by a simplicial graph Γ, with vertices being the generators, and $[v, w]$ being a relator if and only if vertices $v, w \in \Gamma$ are connected by an edge. A RAAG is 2-dimensional if Γ has no triangles.

In this work we attempt to construct a finite dimensional, contractible space on which a finite index subgroup of the outer automorphism group of a 2-dimensional RAAG acts properly and with contractible fixed point sets of finite subgroups. The existence of such a space, called a classifying space of proper actions, has implications for the cohomological properties of the group (Section 1.3).

Charney, Crisp, and Vogtmann have constructed an *outer space* for 2-dimensional RAAGs, i.e., a contractible space on which a finite index subgroup of the outer automorphism group acts properly ([CCV07]). It is stitched together out of outer spaces for free groups together with certain compatibility constraints arising from the action of subgroups of the RAAG on products of trees (Section 1.5). A point in this outer space can be thought of as a collection of actions of free groups on simplicial trees glued together according to the subgroup structure of the RAAG. For a generic point in this outer space, the gluing maps impose rigidity: a given tree in the collection constituting the point may not be changed without either changing most other trees (in accord with the gluing maps) or leaving the outer space. This posses a challenge in defining paths in this outer space from one point to another and hence for contracting a fixed point set of a finite subgroup.

To get around this difficulty, we restrict ourselves to a subspace of that outer space, one with points in which the gluing maps impose only "local" restrictions (Section 2.1). This subspace is itself contractible and hence itself an outer space for a 2-dimensional RAAG (Section 2.1.3). We then define paths in individual outer spaces of free groups that can be extended to paths in this outer space (Section 2.2). This extension is then used to contract a non-empty fixed point set of a finite subgroup by "working in one outer space of a free group at a time" (Section 2.3).

The contractibility thus shown assumes that the fixed point set is non-empty. The last chapter of this work is devoted to describing some methods and partial results aimed at showing that fixed point sets of finite subgroups are in fact non-empty. Our approach is to find suitable fixed points of projections of our finite group in each outer space of a free group (Section 3.1), and then to piece them together to form a fixed point in the outer space of the RAAG (Section 3.2). To accomplish the first part, we describe the sphere system model of the outer space of a free group and show how surgery can be used to find fixed points having a certain separateness property (Sections 3.1.2, 3.1.3). These methods allow us to show, for some specific types of finite subgroups, that their fixed point sets in the outer space of a RAAG are non-empty.

1.2 Right-angled Artin groups and their automorphisms

Recall that a RAAG is a group having a presentation of the form

$$A_\Gamma = \langle v_1, \dots, v_n \mid R \rangle$$

where $[v_i, v_j] \in R$ if and only if v_i is adjacent to v_j in Γ. For an empty graph Γ, the RAAG A_Γ is free while for a complete graph Γ, the group A_Γ is free abelian. Considerable investigation of RAAGs has been conducted starting in the late 70s by Baudisch ([Bau81]), Droms ([Dro87a, Dro87b]), Charney ([Cha92, Cha95]) and others. A very good survey of RAAGs is [Cha07].

The automorphism groups of free groups and of free abelian groups, the two "extreme" cases of RAAGs, share many properties. To what extent do automorphism groups of all RAAGs share properties of $\mathrm{Aut}(F_n)$ (and $\mathrm{Out}(F_n)$) and $\mathrm{GL}_n(\mathbb{Z})$ is still largely unknown. This thesis concentrates on answering this question for on one such property, the existence of a finite $\underline{E}G$, a type of classifying space defined in Section 1.3.

Laurence ([Lau95]), based on the work of Servatius ([Ser89]), has shown that the group $\mathrm{Aut}(A_\Gamma)$ is finitely generated by generators of the following types:

1. Conjugations
2. Inversions
3. Transvections

4. Partial conjugations

5. Graph symmetries

An *inversion* is generated by sending a vertex generator to its inverse. A *transvection* is an automorphism generated by taking $w \in \Gamma$ to $wv \in \Gamma$ when the link of w is contained in the star of v, that is, $\mathrm{lk}(w) \subset \mathrm{st}(v)$, where $\mathrm{st}(v)$ is the star graph $v * \mathrm{lk}(v)$. A *partial conjugation* is a conjugation of a connected component of $\Gamma \setminus \mathrm{st}(v)$ by v. A *graph symmetry* is a graph automorphism of Γ, which induces a corresponding automorphism of A_Γ.

Let $\mathrm{Aut}^0(A_\Gamma)$ be the normal subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by all the generators above except graph symmetries. Let $\mathrm{Out}^0(A_\Gamma)$ be the quotient of $\mathrm{Aut}^0(A_\Gamma)$ by the group of inner automorphisms. By examining conjugation of each type of generator above by a graph symmetry, we can see that $\mathrm{Out}^0(A_\Gamma)$ is a normal, finite index subgroup of $\mathrm{Out}(A_\Gamma)$.

Henceforth, we restrict ourselves to *2-dimensional* RAAGs A_Γ: those for which Γ contains no triangles. Consider a binary relation $\leq$ on the set of vertices of Γ defined by link containment: $v \leq w$ if and only if $\mathrm{lk}(v) \subset \mathrm{lk}(w)$. Two vertices $v, w \in V(\Gamma)$ are in the same equivalence class if $v \leq w$ and $w \leq v$, that is, if they have the same link; the equivalence class of v is denoted $[v]$. If $[v]$ contains a single vertex v, we call v a *cyclic* vertex. We choose a representative vertex in each maximal equivalence class and define the graph Γ_0 as the induced subgraph of Γ on these representatives. The following are properties of Γ_0 (Lemma 3.7 [CCV07]).

1. Γ_0 is independent of the choice of maximal vertices (up to isomorphism).

2. Γ_0 is connected.

3. No vertex of Γ_0 is a leaf of Γ.

4. Every vertex $v \in \Gamma$ is adjacent in Γ to at least one vertex $w \in \Gamma_0$.

For two graphs G_1, G_2, the *join* $G_1 * G_2$ is a graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set consists of all edges present in G_1 and G_2, and edges connecting every vertex of G_1 to every vertex of G_2. Given any vertex $v \in \Gamma$, let $J_v = [v] * \mathrm{lk}(v)$. Note that if $v \in \Gamma_0$, this join in maximal: no other complete bipartite subgraph of Γ contains it. The following two results appear as Proposition 3.2 and Lemma 3.8 in [CCV07] respectively.

Proposition 1.2.1. *For any $\varphi \in \mathrm{Aut}^0(A_\Gamma)$ and $v \in \Gamma_0$, φ maps $A_{J_v} = F([v]) \times F(\mathrm{lk}(v))$ to a conjugate of itself, and the factor $F([v])$ is preserved up to conjugacy.*

Proposition 1.2.2. *If $v, w \in \Gamma_0$ are neighbors, and $\varphi \in \mathrm{Out}^0(A_\Gamma)$ is represented by automorphisms $\varphi_v, \varphi_w \in \mathrm{Aut}^0(A_\Gamma)$ preserving A_{J_v} and A_{J_w} respectively, then there exist $g_v \in A_{J_v}$ and $g_w \in A_{J_w}$ such that $c(g_v) \circ \varphi_v = c(g_w) \circ \varphi_w$, where $c(g)$ denotes conjugation by g.*

Let

$$R = \prod_{v \in \Gamma_0} R_v : \mathrm{Out}^0(A_\Gamma) \to \prod_{v \in \Gamma_0} \mathrm{Out}(A_{J_v})$$

be a restriction homomorphism, taking an element φ of $\mathrm{Out}^0(A_\Gamma)$ to a product $\prod \varphi_v$, each $\varphi_v \in \mathrm{Out}(A_{J_v})$ being the restriction of φ to maximal join J_v, which exists by Proposition 1. Further, since φ_v preserves $A_{[v]} = F([v])$ when $v \in \Gamma_0$ (up to conjugacy), it projects to an outer automorphism of $A_{\mathrm{lk}(v)} \cong A_{J_v}/A_{[v]}$. Thus we get a projection homomorphism

$$P = \prod_{v \in \Gamma_0} P_v : \mathrm{Out}^0(A_\Gamma) \to \prod_{v \in \Gamma_0} \mathrm{Out}(A_{\mathrm{lk}(v)})$$

This projection homomorphism lets us construct actions of $\mathrm{Out}(A_\Gamma)$ using actions of $\mathrm{Out}(A_{\mathrm{lk}(v)})$ for different vertices v. In [CV86] Culler and Vogtmann defined an *outer space for a free group*, a contractible space of proper actions of an outer automorphism group of a free group on marked trees. We describe this space and its properties in Section 1.4, and later, in Section 1.5, show how Charney, Crisp and Vogtmann use it to construct contractible a space of proper actions of $\mathrm{Out}(A_\Gamma)$, an *outer space* for A_Γ.

1.3 $\underline{\mathrm{E}}G$ and contractibility of fixed point sets of finite subgroups

The main result of this thesis concerns the contractibility of fixed point sets of finite subgroups of outer automorphisms of right angled Artin groups inside a suitable outer space. One reason why this contractibility is important is because it is a key condition for a space to be a type of classifying space usually denoted $\underline{\mathrm{E}}G$.

Given a group G, there is an Eilenberg-MacLane space $K(G,1)$ with $\pi_1(K(G,1)) = G$ and trivial higher homotopy groups. If G has finite cohomological dimension, then there is a finite dimensional $K(G,1)$, and its universal cover, a *classifying space* $\mathrm{E}G$, is a finite dimensional contractible CW-complex on which G acts freely by deck transformations. When G does not have finite cohomological dimension – when G has torsion for instance – there is no finite dimensional $K(G,1)$ and hence no finite dimensional $\mathrm{E}G$.

Thus for a group G having torsion, the condition of acting freely on its classifying space must be weakened if we want a finite dimensional space. A space X is a *classifying space for proper actions*, denoted $\underline{\mathrm{E}}G$, if it is a CW-complex on which G acts with finite cell stabilizers and for each finite subgroup $H < G$, the subcomplex fixed by H is contractible. Just as $\mathrm{E}G$ classifies free actions of G, i.e. given a CW-complex Y on which G acts freely, there is a unique (up to G-homotopy) G-equivariant map $Y \to \mathrm{E}G$, a space $\underline{\mathrm{E}}G$ classifies proper actions. By this universality, any two $\underline{\mathrm{E}}G$ spaces are G-equivariantly homotopy equivalent ([LN10]).

One way to construct an $\underline{\mathrm{E}}G$ canonically is to take the poset of finite, non-empty subsets of G, on which G acts on the left. The geometric realization of this poset is an $\underline{\mathrm{E}}G$. However, this $\underline{\mathrm{E}}G$ is infinite-dimensional when G is infinite. Examples of finite dimensional $\underline{\mathrm{E}}G$ for infinite groups G are trees on which a discrete group G acts properly with finite vertex stabilizers, the Rips complex $P_d(G)$ for a δ-hyperbolic group G, where d depends on δ ([MS02]), and a $\mathrm{CAT}(0)$ complex for a group G which acts on it properly by isometries ([BH99]). Algebraically characterizing groups G which have a finite-dimensional $\underline{\mathrm{E}}G$ is an ongoing effort ([KM98, Mis01]).

A group which has a finite dimensional $\underline{\mathrm{E}}G$ has rational cohomological dimension at most the dimension of $\underline{\mathrm{E}}G$, and its n^{th} homology group is torsion for n greater than the dimension of $\underline{\mathrm{E}}G$ ([KM98]). Classifying spaces for proper actions also play an important role in the Baum-Connes conjecture, which asserts that the so-called *assembly map* from the equivariant K-homology of an $\underline{\mathrm{E}}G$ to the K-theory of the C^*-algebra of G is an isomorphism (see [BCH94] for details).

If the fixed point condition is relaxed, we arrive at the related notion of Kropholler's hierarchy ([Kro93]). Given a class of groups $\mathcal{F}$ closed under isomorphism and taking subgroups, a group is in $\mathbf{H}_1\mathcal{F}$ if it acts on some finite dimensional, contractible CW-complex X with cell stabilizers in $\mathcal{F}$. The class $\mathbf{H}\mathcal{F}$, defined as the smallest $\mathbf{H}_1$-closed class containing $\mathcal{F}$, is called *Kropholler's class of hierarchically decomposable groups* ([LN10]). When $\mathcal{F}$ is the class of finite groups, then if a group G has a finite dimensional $\underline{E}G$, it is clearly in $\mathbf{H}_1\mathcal{F}$. The question of whether all groups $G \in \mathbf{H}_1\mathcal{F}$ have finite dimensional $\underline{E}G$ is still unsettled ([KM98, LN10]).

1.4 Outer space for free groups

For a 2-dimensional RAAG A_Γ, the projection homomorphism (Section 1.2) takes a finite subgroup of $\mathrm{Out}(A_\Gamma)$ to a product of finite subgroups of $\mathrm{Out}(A_{\mathrm{lk}(v)})$, outer automorphism groups of discrete RAAGs, that is, of free groups. To define an outer space for a general 2-dimensional RAAG, Charney, Crisp, and Vogtmann use outer spaces for free groups. The following sections describe outer spaces for free groups, path in these spaces, and the contractibility of fixed point sets of finite subgroups of outer automorphisms in these spaces.

1.4.1 $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$ and $\mathcal{O}(F_n)$

Let $F_n = \langle a_1, \ldots, a_n \rangle$ denote the free group on n generators. Then $\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\operatorname{Inn}(F_n)$ is the *outer automorphism group* of F_n. Although the structure of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ has been studied since the early days of group theory, beginning with Nelson's proof that $\mathrm{Aut}(F_n)$ is finitely generated ([Nie24]), recent results about these groups have used geometric methods and techniques. One such technique has been the study of the action of $\mathrm{Out}(F_n)$ on a contractible, finite dimensional space introduced by Culler and Vogtmann ([CV86]).

Let R_n be the rose with n petals, that is, a graph with a single vertex and n oriented edges incident at both ends to the vertex, and identify $F_n \cong \pi_1(R_n)$ sending each generator to an edge of R_n. Consider a connected simplicial metric graph G, with no valence one or two vertices, and a homotopy equivalence $\tau : R_n \to G$. A point in the (projectivized) *outer space* $\mathcal{O}(F_n)$ is an equivalence class (τ, G), where $(\tau, G) \sim (\tau', G')$ if there is a homothety $f : G \to G'$ and $\tau \circ f$ is homotopic to τ'.

The universal cover of a metric marked graph (τ, G) (from now on just G whenever this does not create confusion), is a simplicial $\mathbb{R}$-tree T on which the group $F_n \cong \tau_* \pi_1(R_n)$ acts by deck transformations. Thus we may equivalently define a point in $\mathcal{O}(F_n)$ to be an equivalence class of free minimal actions of F_n on simplicial $\mathbb{R}$-trees T, where two actions $\rho : F_n \to \mathrm{Isom}(T)$, $\rho' : F_n \to \mathrm{Isom}(T')$ are equivalent is there is an equivariant homothety $f : T \to T'$, i.e. $f \circ \rho(g) = \rho(g) \circ f$ for all $g \in F_n$.

The *axis topology* on $\mathcal{O}(F_n)$ is determined by the map $L : \mathcal{O}(F_n) \to \mathbb{RP}^\infty$ sending each tightened immersed loop in G corresponding to a cyclically reduced word in F_n to its length. This map is an injection ([CM87]) and $\mathcal{O}(F_n)$ inherits the subspace topology. This topology gives $\mathcal{O}(F_n)$ the structure of a finite dimensional simplicial complex with some faces of simplices missing. See Figure 1.1 for an example illustrating the local simplicial structure.

Since some lower dimensional faces of simplices in $\mathcal{O}(F_n)$ are missing (for example all vertices in 1.1), the quotient $\mathcal{O}(F_n)/\operatorname{Out}(F_n)$ is not compact. However, using the fact that $\mathcal{O}(F_n)$ can be equivariantly deformation retracted to a finite dimensional subcomplex on which $\operatorname{Out}(F_n)$ acts cocompactly, Culler and Vogtmann showed that the virtual cohomological dimension of $\operatorname{Out}(F_n)$ is $2n-3$

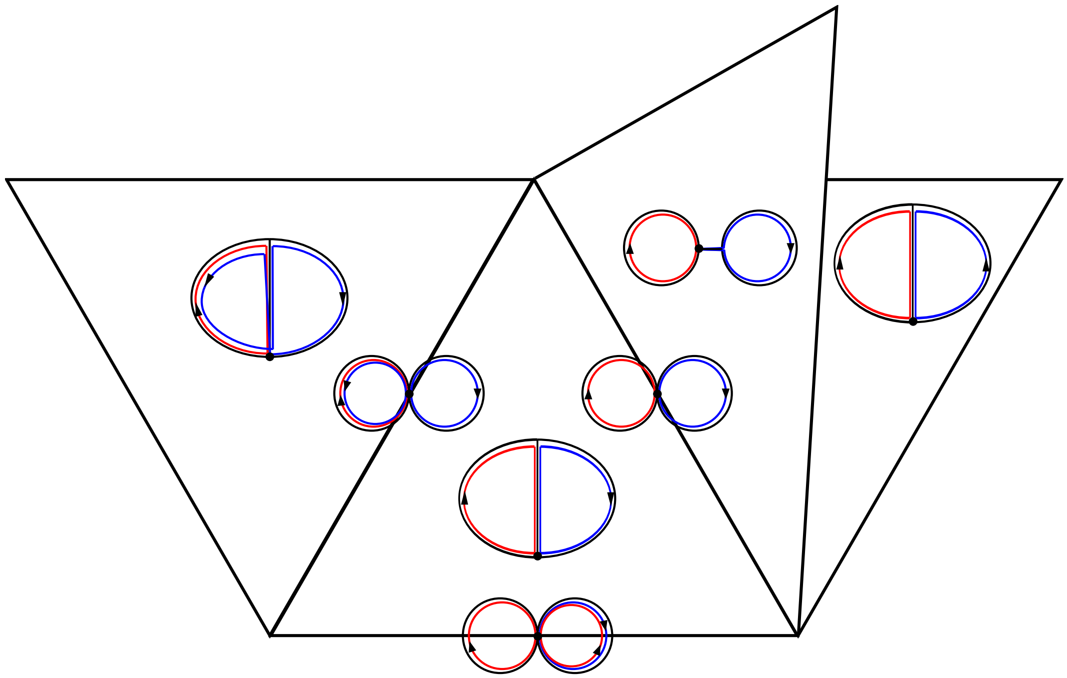

Figure 1.1: A neighborhood of a 2-cell in CV_2

([CV86]).

Bestvina and Handel ([BH92]) defined train tracks for $\mathrm{Out}(F_n)$, based on Thurston's theory of train tracks on surfaces, and used them to prove the Scott Conjecture ([DS75]): for every automorphism $\alpha \in \mathrm{Aut}(F_n)$, the fixed subgroup $\{a \in F_n \mid \alpha(a) = a\}$ has rank at most n. Since then train tracks have been used to study the dynamics of automorphisms in outer space ([Gui00]). Levitt and Lustig, for instance, showed that suitably defined irreducible automorphisms have analogous North-South dynamics in outer space as do pseudo-Anosov automorphisms acting on Teichmüller space ([LL03]). Hatcher and Vogtmann used a model of the outer space based on embedded 2-spheres inside a connected sum of $S^1 \times S^2$'s (defined and discussed in Section 3.1.2) to show that the homology groups of $\mathrm{Aut}(F_n)$ stabilize ([HV98]). That is, for $n \geq 2i + 3$, the map $H_i(\mathrm{Aut}(F_n)) \to H_i(\mathrm{Aut}(F_{n+1}))$ induced by the inclusion $\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_{n+1})$

is an isomorphism. Using this model of outer space, Hatcher and Vogtmann showed that the Dehn functions of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ for $n \geq 3$ are at most exponential ([HV96]) and recently Handel and Mosher ([HM10]) have shown that the Dehn functions are indeed exponential (for a much simpler proof of this result, see Bridson and Vogtmann [BV10]). Additionally, for $n \geq 3$, neither $\mathrm{Aut}(F_n)$ nor $\mathrm{Out}(F_n)$ can act properly and cocompactly on a $\mathrm{CAT}(0)$ space ([BV95]).

The methods and results most pertinent to this thesis concern fixed points of finite subgroups of $\mathrm{Out}(F_n)$ in the outer space $\mathcal{O}(F_n)$. These are discussed in the next section.

1.4.2 Morphisms and paths in $\mathcal{O}(F_n)$

Given two simplicial $\mathbb{R}$-trees T and T', a map $f : T \to T'$ taking vertices to vertices is called a *morphism* if it can be made simplicial by subdividing T. That is, f is a morphism if it is locally injective on $f^{-1}(T' \setminus V(T'))$, where $V(T')$ is the vertex set of T'. If additionally the trees T and T' are G trees, for some group G, then we take our morphisms to be equivariant with respect to the action of G.

We have defined the axis topology on $\mathcal{O}(F_n)$ in Section 1.4. Another way to topologize $\mathcal{O}(F_n)$, which naturally extends to the topology of the space of morphisms between F_n-trees, is by the *equivariant Gromov-Hausdorff topology*. A fundamental system of neighborhoods is defined by neighborhoods $V_T(K, P, \epsilon)$, where K is a compact subset of T, P is a finite subset of F_n, $\epsilon > 0$, and $T' \in V_T(K, P, \epsilon)$ if and only if there is a map $r\colon K \to T'$ so that $|d_T(x, gy) - d_{T'}(r(x), gr(y)| < \epsilon$ for every $x, y \in K$, $g \in P$. The Gromov-

Hausdorff and axis topologies are the same for $\mathcal{O}(F_n)$ ([GL07b]).

This topology can be extended to the space $\mathcal{M}$ of morphism between F_n-trees. Given a morphism $\varphi\colon T \to T_1$, a neighborhood $W_\varphi(K, P, \epsilon)$ includes a morphism $\psi\colon T' \to T_1'$ if and only if $T' \in V_T(K, P, \epsilon)$ and $\left|d_{T_1}(\varphi(x), g\varphi(y)) - d_{T_1'}(\psi(r(x)), g\psi(r(y)))\right| < \epsilon$.

In [Sko89] Skora defines an interpolating path between two trees T and T' for a morphism $f\colon T \to T'$. In particular, Skora constructs a partial map

$$F\colon \mathcal{O}(F_n) \times \mathcal{O}(F_n) \times \mathcal{M} \to \mathcal{O}(F_n) \times [0, \infty)$$

taking a triple $(T, T', f\colon T \to T')$ to a continuous path p where $p(0) = T$ and $p(s) = T'$ for all $s > r$, where r depends on f. This map is equivariant with respect to the action of $\mathrm{Out}(F_n)$ on $\mathcal{O}(F_n)$.

The path corresponding to $f\colon T \to T'$ is produced by defining a continuum of equivalence relations on T, where $x \sim_i y$ iff $f(x) = f(y)$ and $d_T(x, y) \leq i$. Then $p(s) = T/\sim_s$.

Definition 1.4.1. *For each $s \geq 0$, we have a quotient morphism $\varphi_s : T \to p(s)$ and a morphism $\psi_s : p(s) \to T'$ defined by $\varphi_s(x) = f(\tilde{x})$, where $\tilde{x}$ is any point in T such that $\varphi_s(\tilde{x}) = x$. We call these* Skora morphisms.

These morphisms are continuous with respect to f, and equivariant under the action of $\mathrm{Out}(F_n) \curvearrowright \mathcal{O}(F_n)$ (Theorem 4.8, [Sko89]). (See also [GL07b, GL07a].)

1.4.3 Fixed point sets of finite subgroups in $\mathcal{O}(F_n)$

Given a finite subgroup of $H < \mathrm{Out}(F_n)$, in [Cul84] Culler (also independently Khramtsov [Khr85] and Zimmermann [Zim81]) has shown that there is a graph G, with $\pi_1(G) = F_n$, such that a subgroup of the group of graph automorphisms of Γ realizes H. That is, G, as a marked graph, is fixed by H. Thus the fixed point sets of finite subgroups of $\mathrm{Out}(F_n)$ in $\mathcal{O}(F_n)$ are non-empty. White ([Whi93]), and independently Krstić and Vogtmann ([KV93]), have shown that fixed point sets of finite subgroups of $\mathrm{Out}(F_n)$ in $\mathcal{O}(F_n)$ are contractible. This proves that $\mathcal{O}(F_n)$ is an $\underline{\mathrm{E}}\,\mathrm{Out}(F_n)$.

White defines an $\mathrm{Out}(F_n)$-equivariant assignment of morphisms to pairs of trees in $\mathcal{O}(F_n)$. That is, he constructs a continuous, $\mathrm{Out}(F_n)$-equivariant function

$$\mathcal{O}(F_n) \times \mathcal{O}(F_n) \to \mathcal{M}$$

which takes trees T_0 and $\bar{T}$ to a morphism $f : T \to \bar{T}$ where T is in the same simplex of $\mathcal{O}(F_n)$ as T_0. Applying Skora's interpolation (1.4.2) produces a continuous, $\mathrm{Out}(F_n)$-equivariant map

$$\mathcal{O}(F_n) \times \mathcal{O}(F_n) \to \mathcal{O}(F_n) \times [0, \infty)$$

taking a pair of trees to a path between them (see Figure 1.2). This, together with Culler's realization theorem, implies that fixed point sets of finite subgroups of $\mathrm{Out}(F_n)$ are contractible in $\mathcal{O}(F_n)$. Namely, given two points T, T' in the fixed point set of a finite subgroup, White's theorem guarantees a path p between them. Since the assignment of paths is equivariant with respect to $\mathrm{Out}(F_n)$ action, the whole path is fixed by the finite subgroup. By the continuity of the path assignment, the fixed point set can be contracted.

$$(T_0, \bar{T}) \mapsto (f : T \to \bar{T})$$

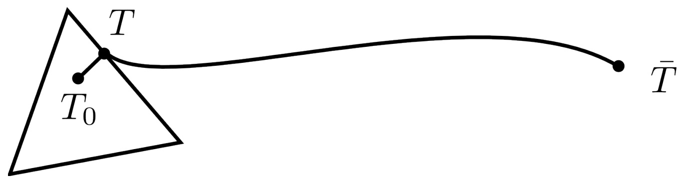

Figure 1.2: The White mapping and corresponding Skora path

Krstić and Vogtmann take a different approach to showing contractibility. For a finite subgroup $H < \mathrm{Out}(F_n)$, they construct a simplicial complex $L_H \subset \mathcal{O}(F_n)$ on which $C(H)$, the centralizer of H, acts with finite stabilizers and quotient. This complex is not the fixed point set of H, but the fixed point set equivariantly deformation retracts onto L_H. They then show that L_H is contractible and use the action $C(H) \curvearrowright L_H$ to determine the virtual cohomological dimension of $C(H)$.

In this work we will make use of White's theorem in constructing the retraction of fixed point sets of finite subgroups in the 2-dimensional RAAG case.

1.5 Outer space for 2-dimensional right-angled Artin groups

We now describe the Charney, Crisp, and Vogtmann construction of the outer space $\mathcal{O}(A_\Gamma)$ for a 2-dimensional RAAG A_Γ.

A RAAG A_Γ acts properly and cocompactly on its Cayley 2-complex, the universal cover of the presentation complex, which, for the usual presentation $\langle v_1, \ldots, v_n \mid [v_i, v_j] \rangle$, $v_i \in V(\Gamma), v_iv_j \in E(\Gamma)$, can be given a natural cube complex structure. If the graph Γ is a complete bipartite graph $V * W$, the corresponding action on the Cayley 2-complex is of a product of two free groups on the product of two simplicial trees, $F(V) \times F(W) \curvearrowright T_V \times T_W$. This example is in fact "generic": whenever $F(V) \times F(W)$ acts properly and cocompactly by isometries on a simplicial $\mathrm{CAT}(0)$ cube complex X without boundary, and both V and W have more than one element, then the action $F(V) \times F(W) \curvearrowright X$ splits as a product action $(F(V) \curvearrowright T_V) \times (F(W) \curvearrowright T_W)$ ([BH99], p.239). If V has a single element v, then all elements of $F(W)$ leave the min set of v, $\mathrm{Min}(v) = \{x \in X \mid d(vx, x) = l(v)\}$, invariant. Hence if the action on X is minimal, that is, if there is not a proper invariant subspace, $X = \mathrm{Min}(v)$ which by [BH99] Theorem 6.8 is isometric to $T \times \mathbb{R}$, where T is a simplicial $\mathbb{R}$-tree, and the action of every $w \in F(W)$ is an isometry of T followed by a translation of $\mathbb{R}$.

In the case where neither V nor W are singletons, every element of $\mathrm{Out}(A_\Gamma)$ preserves the two free factors (or perhaps interchanges them if $|V| = |W|$), and the natural outer space is then $\mathcal{O}(F(V)) \times \mathcal{O}(F(W))$. In the case where $V = \{v\}$, $\mathrm{Out}(F(\{v\}) \times F(W)) \to \mathrm{Out}(\mathbb{Z}) \times \mathrm{Out}(F(W))$ splits, since any automorphism preserves the center $\mathbb{Z}$, with the kernel being transvections $w \mapsto wv$, for every $w \in W$. Thus $\mathbb{R}^{|W|} \times \mathbb{R}^+ \times \mathcal{O}(F(W))$ can be taken as the outer space, where $\mathbb{R}^+$ is the possible translation length of v, and $\mathbb{R}^{|W|}$ are the possible translations of the $\mathbb{R}$-factor in $T \times \mathbb{R}$ for each $w \in W$. This space is contractible, and $\mathrm{Out}(\mathbb{Z} \times F(W)) \cong \mathbb{Z}^{|W|} \rtimes (\mathbb{Z}/2 \times \mathrm{Out}(F(W)))$ acts on it. For a general A_Γ, the action of subgroups of A_Γ corresponding to maximal joins in Γ on products of tress, together with compatibility conditions among the actions, encode a point in the outer space $\mathcal{O}(A_\Gamma)$.

In general, a point of $\mathcal{O}(A_\Gamma)$ is a set $\mathcal{X} = \{X_v, X_e, i_{e,v}\}_{v \in \Gamma_0}$, where $X_v = T_{\mathrm{lk}(v)} \times T_{[v]}$ is the product of $F(\mathrm{lk}(v))$ and $F([v])$-trees respectively, $X_e = T_{[v]} \times T_{[w]}$

for every edge e connecting two vertices $v, w \in \Gamma_0$, and $i_{e,v} : X_e \to X_v$ is (up to equivariant isometry) the identity on the $T_{[v]}$ factor and an $A_{[w]}$-equivariant embedding of $T_{[w]}$ into $T_{\mathrm{lk}(v)}$. The action of each $F(\mathrm{lk}(v)) \times F([v])$ on X_v is a product action unless $\mathrm{lk}(v)$ contains a leaf vertex. In this case v is cyclic, hence $T_{[v]}$ is isometric to the real line, and the action may be sheared by $\lambda : F(\mathrm{lk}(v)) \to \mathbb{R}$ taking the value 0 on non-leaf vertices:

$$(g, v^n) \cdot (x, r) = (g \cdot x, r + nt_v + \lambda(g))$$

where $x \in T_{\mathrm{lk}(v)}$, $r \in T_{[v]} \cong \mathbb{R}$, and $g \in F(\mathrm{lk}(v))$.

Definition 1.5.1. *The action of $\varphi \in \mathrm{Out}^0(A_\Gamma)$ on $\mathcal{X} = \{X_v, X_e, i_{e,v}\}_{v \in \Gamma_0}$ is given by $\varphi\mathcal{X} = \{X_v^{\varphi_v}, X_e^{\varphi_e}, t(g_v)^{-1} i_{e,v}\}$, where $X_v^{\varphi_v}$ is the action $F(\mathrm{lk}(v)) \times F([v])$ on X_v twisted by the automorphism $R_v(\varphi) = \varphi_v$, $X_e^{\varphi_e}$ is the action on X_e twisted by $\varphi_e := c(g_v) \circ \varphi_v = c(g_w) \circ \varphi_w$ if edge e connects v and w (Proposition 1.2.2), and $t(g_v)^{-1}$ is the translation by g_v^{-1} in X_v.*

A simpler way of viewing a point of $\mathcal{O}(A_\Gamma)$ is as a collection of $F(\mathrm{lk}(v))$-trees $T_{\mathrm{lk}(v)}$ and $F([v])$-trees $T_{[v]}$ for every vertex $v \in \Gamma_0$, equivariant isometries $i_{w,v} : T_{[w]} \to T_{\mathrm{lk}(v)}$ for every $v \in \Gamma_0$ and $w \in \mathrm{lk}(v) \cap \Gamma_0$, and real numbers $s(u)$ for every leaf vertex $u \in \Gamma$. The action of the free subgroups $F(\mathrm{lk}(v))$ and $F([v])$ for all $v \in \Gamma_0$ is either a simplicial action of $F(\mathrm{lk}(v)) \curvearrowright T_{\mathrm{lk}(v)}$ and $F([v]) \curvearrowright T_{[v]}$ by isometries, if $\mathrm{lk}(v)$ has no leaf vertices, or, if leaf vertices are present (in which case v is cyclic), action of the product group $F(\mathrm{lk}(v)) \times F([v])$ must be considered as above. That is, we have a shear homomorphism $\lambda : F(\mathrm{lk}(v)) \to \mathbb{R}$ taking the value 0 on non-leaf vertices and the value $s(u)$ on the leaf vertices $u \in \mathrm{lk}(v)$, and $(g, v^n) \cdot (x, r) = (g \cdot x, r + nt_v + \lambda(g))$ for $(g, v^n) \in F(\mathrm{lk}(v)) \times F([v])$. Such non-orthogonal actions must be allowed in order for leaf transvections (automorphisms of A_Γ sending a leaf vertex w connected to v to wv) to act nontrivially on a point in outer space, by “shearing” the action on $T_{\mathrm{lk}(v)} \times T_{[v]}$.

There is an injection

$$\mathcal{O}(A_\Gamma) \to \left(\prod_{v \in \Gamma_0} \mathcal{O}(F(\mathrm{lk}(v))) \right) \times \mathbb{R}^k \times \mathbb{R}^l$$

where l is the number of leaf vertices. Here $\mathbb{R}^l$ encodes the possible shear factors, and $\mathbb{R}^k$ encodes the injections $i_{v,w}$ among adjacent vertices of Γ_0 in terms of basepoints (see Proposition 4.2 in [CCV07]). The space $\mathcal{O}(A_\Gamma)$ is then given the subspace topology induced on its image.

CHAPTER 2

CONTRACTIBILITY OF NON-EMPTY FIXED POINT SETS OF FINITE SUBGROUPS

This chapter presents the main results of this thesis: the definition of a contractible subspace $\mathcal{S}(A_\Gamma) \subset \mathcal{O}(A_\Gamma)$ on which $\mathrm{Out}^0(A_\Gamma)$ acts properly and cocompactly, the analysis of paths in this space, and the proof that non-empty fixed-point sets of finite subgroups of $\mathrm{Out}^0(A_\Gamma)$ are contractible in $\mathcal{S}(A_\Gamma)$.

2.1 An outer space $\mathcal{S}(A_\Gamma)$

Given a point $\mathcal{X} \in \mathcal{O}(A_\Gamma)$, a local deformation of a single action $T_{\mathrm{lk}(v)}$, for some $v \in \Gamma_0$, need not extend to a deformation in $\mathcal{O}(A_\Gamma)$, since in general there is "rigidity" in $\mathcal{O}(A_\Gamma)$ imposed by the compatibility conditions among the actions on trees $T_{\mathrm{lk}(w)}$, $w \in \Gamma_0$. To allow such deformations of actions on individual trees to extend, we restrict the compatibility conditions. This produces a subspace $\mathcal{S}(A_\Gamma) \subset \mathcal{O}(A_\Gamma)$ equal to $\mathcal{O}(A_\Gamma)$ in dimension, but simpler in structure. We show that it is an outer space: a contractible space on which $\mathrm{Out}^0(A_\Gamma)$ acts properly.

2.1.1 A subspace $\mathcal{S}(A_\Gamma) \subset \mathcal{O}(A_\Gamma)$

A point $\mathcal{X} \in \mathcal{O}(A_\Gamma)$ is in $\mathcal{S}(A_\Gamma)$ if and only if for every $v \in \Gamma_0$ and every $w, z \in \mathrm{lk}(v) \cap \Gamma_0$ and every $g \in F(\mathrm{lk}(v))$, the intersection $\mathrm{Min}(F([w])) \cap g\,\mathrm{Min}(F([z]))$ contains at most one point, where $\mathrm{Min}(F([w]))$ is the minimal subtree of the action $F([w]) \curvearrowright T_{\mathrm{lk}(v)}$. That is, $F(\mathrm{lk}(v))$ translates of $\mathrm{Im}(i_{w,v})$, the minimal subtree of $F([w])$, and $\mathrm{Im}(i_{z,v})$, the minimal subtree of $F([z])$, intersect in at most a single point in $T_{\mathrm{lk}(v)}$, for every $v \in \Gamma_0$. For a given $v \in \Gamma_0$, we say that a tree $T_{\mathrm{lk}(v)}$ having this property is *separated*. Thus $\mathcal{X}$ is in $\mathcal{S}(A_\Gamma)$ if for every $v \in \Gamma_0$, $T_{\mathrm{lk}(v)}$ is separated.

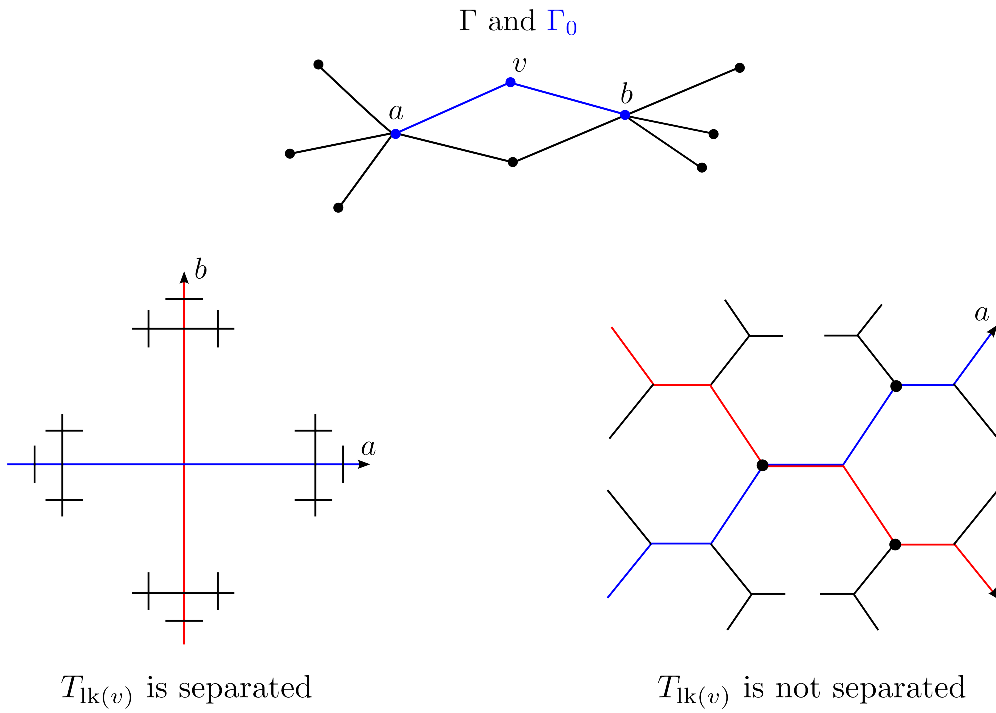

Figure 2.1: A separated tree and a non-separated tree.

Suppose $v, w \in \Gamma_0$ are adjacent vertices. Let $G_v = T_{\mathrm{lk}(v)}/A_{\mathrm{lk}(v)}$ and $G_v^w = T_{[w]}/A_{[w]}$, and let the composition map $G_v^w \hookrightarrow T_{\mathrm{lk}(v)}/A_{[w]} \to G_v$ be denoted by $\alpha_{w,v} : G_v^w \to G_v$. Then the subspace $\mathcal{S}(A_\Gamma)$ can be described as those points in $\mathcal{O}(A_\Gamma)$ in which for every $v, w, z \in \Gamma_0$ such that $w, z \in \mathrm{lk}(v)$, $\mathrm{Im}\,\alpha_{w,v} \cap \mathrm{Im}\,\alpha_{z,v}$ does not contain a non-trivial interval (i.e., is either a collection of discrete points or is empty). This and the previous definitions are equivalent since if $\mathrm{Min}(F([w]) \cap g\,\mathrm{Min}(F([z]))$ contains a non-trivial interval, so does the quotient by the $A_{\mathrm{lk}(v)}$ action, and conversely, if $\mathrm{Im}\,\alpha_{w,v} \cap \mathrm{Im}\,\alpha_{z,v}$ contains a non-trivial interval I, it implies that there is a $g \in A_{\mathrm{lk}(v)}$ such that $\mathrm{Min}(F([w])) \cap g\,\mathrm{Min}(F([z]))$ contains a non-trivial interval that maps into I by $T_{\mathrm{lk}(v)} \to G_v$.

2.1.2 The action $\mathrm{Out}^0(A_\Gamma) \curvearrowright \mathcal{S}(A_\Gamma)$ is proper

In order to show that $\mathrm{Out}^0(A_\Gamma)$ acts properly on $\mathcal{S}(A_\Gamma)$, it suffices to prove that it is a subspace invariant under the action $\mathrm{Out}^0(A_\Gamma) \curvearrowright \mathcal{O}(A_\Gamma)$ defined in Section 1.5, since that action itself is proper. To that end, we demonstrate that minimal subtrees are preserved in each $T_{\mathrm{lk}(v)}$ by the projection of any outer automorphism in $\mathrm{Out}^0(A_\Gamma)$ to the respective link (Lemma 2.1.1 below).

Lemma 2.1.1. *For every $\varphi \in \mathrm{Out}^0(A_\Gamma)$ and $w \in \Gamma_0 \cap \mathrm{lk}(v)$, $P_v(\varphi)$ translates $\mathrm{Min}(F([w]))$ in $T_{\mathrm{lk}(v)}$ by an element of $A_{\mathrm{lk}(v)}$.*

Proof. Let $\varphi \in \mathrm{Out}^0(A_\Gamma)$. By Proposition 1, there exist automorphisms φ_v and φ_w representing φ such that φ_v preserves A_{J_v} and $A_{[v]}$, and φ_w preserves A_{J_w} and $A_{[w]}$. By Proposition 1.2.2, there are $g_v \in A_{J_v}$ and $g_w \in A_{J_w}$ such that $c(g_v) \circ \varphi_v = c(g_w) \circ \varphi_w$. Thus we get $\varphi_v = c(g_v^{-1} g_w) \circ \varphi_w$.

Then $\mathrm{Min}(A_{[w]})$ in $T_{\mathrm{lk}(v)}$ is sent to the minimal set of $\varphi_v(A_{[w]}) = c(g_v^{-1}) \circ c(g_w) \circ \varphi_w(A_{[w]}) = g_v^{-1} A_{[w]} g_v$, the translation of $\mathrm{Min}(A_{[w]})$ by $g_v^{-1} \in A_{J_v}$. Since any element in A_{J_v} can be expressed as a product of generators in $A_{\mathrm{lk}(v)}$ and $A_{[v]}$, and the two sets of generators commute, $g_v^{-1} A_{[w]} g_v = h A_{[w]} h^{-1}$, where $h \in A_{\mathrm{lk}(v)}$. $\square$

Proposition 2.1.2. *The subspace $\mathcal{S}(A_\Gamma)$ is preserved under the action of $\mathrm{Out}^0(A_\Gamma)$ and the action $\mathrm{Out}^0(A_\Gamma) \curvearrowright \mathcal{S}(A_\Gamma)$ is proper.*

Proof. By definition 1.5.1, given $\varphi \in \mathrm{Out}^0(A_\Gamma)$, the action $\varphi\mathcal{X}$ takes $\{X_v, X_e, i_{e,v}\}$ to $\{X_v^{\varphi_v}, X_e^{\varphi_e}, t(g_v)^{-1} i_{e,v}\}$, where if edge e connects v to w, $\varphi_v \in \mathrm{Aut}^0(A_\Gamma)$ preserves A_{J_v} and $A_{[v]}$, while $\varphi_e \in \mathrm{Aut}^0(A_\Gamma)$ preserves both A_{J_v} and A_{J_w} (see Proposition 1.2.2). Since each φ_v preserves $A_{[v]}$ and so descends to $P_v(\varphi) : A_{\mathrm{lk}(v)} \to$

$A_{\mathrm{lk}(v)}$, in each tree $T_{\mathrm{lk}(v)}$ the action is twisted by $P_v(\varphi)$. By Lemma 2.1.1, this preserves the subtrees $\mathrm{Im}(i_{w,v})$ up to translation by an element of $A_{\mathrm{lk}(v)}$. Thus $P_v(\varphi)$ preserves the property of being separated, and hence $\mathcal{S}(A_\Gamma)$ is mapped to itself under the action of φ on $\mathcal{O}(A_\Gamma)$.

Since the action $\mathrm{Out}^0(A_\Gamma) \curvearrowright \mathcal{O}(A_\Gamma)$ is proper, and $\mathcal{S}(A_\Gamma)$ is a subspace of $\mathcal{O}(A_\Gamma)$, the action $\mathrm{Out}^0(A_\Gamma) \curvearrowright \mathcal{S}(A_\Gamma)$ is proper. □

2.1.3 $\mathcal{S}(A_\Gamma)$ is contractible

Proposition 2.1.3. *The space $\mathcal{S}(A_\Gamma)$ is contractible.*

Proof. We will show that the contraction of $\mathcal{O}(A_\Gamma)$ in Section 5 of [CCV07] restricts to a contraction of $\mathcal{S}(A_\Gamma)$. For completeness, we provide a sketch of the contraction here, applied to $\mathcal{S}(A_\Gamma)$.

Given a point $\mathcal{X} \in \mathcal{S}(A_\Gamma)$, we define an $\mathcal{X}_0 \in \mathcal{S}(A_\Gamma)$ where each tree $(T_0)_{\mathrm{lk}(v)}$ in this new point is the universal cover of a stemmed rose, a graph formed by attaching circles to each endpoint and possibly the central vertex of a star graph. Then we define a morphism $\mathcal{X}_0 \to \mathcal{X}$ (that is, an equivariant morphism for each tree $(T_0)_{\mathrm{lk}(v)} \to T_{\mathrm{lk}(v)}$) which, using Skora's interpolation (see 1.4.2), defines a path in $\mathcal{S}(A_\Gamma)$.

To define $(T_0)_{\mathrm{lk}(v)}$, we choose basepoints on each $T_{\mathrm{lk}(v)}$ from a set of projections of axes. If $x, y \in \mathrm{lk}(v)$, then $p(x, y) \in T_{\mathrm{lk}(v)}$ is the point on the oriented axis of y that is either closest to the axis of x, if the two are disjoint, or the point in the intersection of the two axes furthest along with respect to the orientation of the axis of y. If $w \in \mathrm{lk}(v) \cap \Gamma_0$ is non-cyclic, we take projections $p(t, w)$ for all

$t \in [w]$. For cyclic $w \in \mathrm{lk}(v) \cap \Gamma_0$ and for $w \in \mathrm{lk}(v)$ but not equivalent to a vertex in Γ_0, we take projections $p(t, w)$ for all $t \in \mathrm{lk}(v)$, $t \neq w$.

Thus on each axis of $u \in \mathrm{lk}(v)$ lies at least one projection. Let $b(u)$ denote the least (where the translation direction of u is positive) projection on the axis of u. Let the convex hull of the set of projections $\Pi = \{p(t, w)\}$, which is equal to the convex hull of the basepoints, be denoted by $B(\Pi, T)$; it depends continuously on T (Lemma 5.2 [CCV07]). We then take the finite tree $B(\Pi, T)$, attach an oriented circle S^1 to $b(u)$ for each $u \in \mathrm{lk}(v)$, and let the universal cover be $(T_0)_{\mathrm{lk}(v)}$ with the corresponding action of the free group $A_{\mathrm{lk}(v)}$, translation lengths of whose generators are defined to be the same as their translation lengths in $T_{\mathrm{lk}(v)}$.

We similarly define $(T_0)_{[v]}$ for every $v \in \Gamma_0$ by taking the projections $p(t, w) \in T_{[v]}$ for every $t, w \in [v]$ such that $t \neq w$ if v is not cyclic, choosing basepoints, attaching oriented S^1 to the basepoint of every axis, and taking $(T_0)_{[v]}$ to be the universal cover. If v is cyclic, $(T_0)_{[v]}$ is a line and a basepoint is chosen arbitrarily. We thus have a canonical isometric embedding $(T_0)_{[w]} \hookrightarrow (T_0)_{\mathrm{lk}(v)}$ for every $w \in \mathrm{lk}(v) \cap \Gamma_0$. The trees and embeddings are compatible with the respective actions and thus constitute a point $\mathcal{X}_0 \in \mathcal{O}(A_\Gamma)$.

By the definition of the projections, since $\mathcal{X}$ was in $\mathcal{S}(A_\Gamma)$ to begin with, the resulting trees $(T_0)_{\mathrm{lk}(v)}$ are separated. A Skora interpolation path between two separated trees does not always contain only separated trees (see Section 2.2 where this issue is dealt with). In this case, however, the Skora path for each morphism $(T_0)_{\mathrm{lk}(v)} \to T_{\mathrm{lk}(v)}$ does contain only separated trees, as the next lemma shows.

Lemma 2.1.4. *Suppose $f : T_{\mathrm{lk}(v)} \to T'_{\mathrm{lk}(v)}$ is a morphism between two separated trees and $f(T_{[w]}) \subset T'_{[w]}$ for all $w \in \mathrm{lk}(v) \cap \Gamma_0$, then all trees in the Skora interpolation path* *corresponding to f are separated as well.*

Proof. If $\hat{T}$ is in the Skora interpolation path for the morphism f, and if it is not separated, assume without loss of generality that there is an edge $e \in \hat{T}$ which is in $\hat{T}_{[u]} \cap \hat{T}_{[w]}$ for $u, w \in \mathrm{lk}(v) \cap \Gamma_0$. Then the image of e in $T'_{\mathrm{lk}(v)}$ is a non-trivial edge in $T'_{[u]} \cap T'_{[w]}$, contradicting separateness of $T'_{\mathrm{lk}(v)}$. $\square$

We can thus use the paths corresponding to morphism $f_v : (T_0)_{\mathrm{lk}(v)} \to T_{\mathrm{lk}(v)}$ and $g_w : (T_0)_{[w]} \to T_{[w]}$ to construct a compatible system of paths, since if $w \in \mathrm{lk}(v)$, g_w is the restriction of f_v to $(T_0)_{[w]} \subset (T_0)_{\mathrm{lk}(v)}$. Hence we may contract $\mathcal{S}(A_\Gamma)$ continuously to the subspace of stemmed roses (itself a subspace of $\mathcal{S}(A_\Gamma)$), in which every G_v is a stemmed rose. This subspace is contractible hence so is $\mathcal{S}(A_\Gamma)$. $\square$

Corollary 2.1.5. *The space $\mathcal{S}(A_\Gamma)$ is an outer space for two dimensional RAAGs A_Γ, that is, $\mathcal{S}(A_\Gamma)$ is contractible and $\mathrm{Out}^0(A_\Gamma)$ acts on it properly.*

While $\mathcal{S}(A_\Gamma)$ is a proper subspace of $\mathcal{O}(A_\Gamma)$, the two spaces have the same dimension. This is because marked separated graphs can correspond to top dimensional simplices in the outer space of a free group—of dimension $3n - 4$ for a free group on n generators ([CV86]). The other parameters in the topology of $\mathcal{O}(A_\Gamma)$, namely the shear constants of leaf vertices and basepoints for the injections $i_{w,v} : T_{[w]} \to T_{\mathrm{lk}(v)}$ when w is cyclic, retain their full range when restricted to $\mathcal{S}(A_\Gamma)$.

2.2 Contracting fixed point sets of finite subgroups in $\mathcal{S}(F_n)$

Before we can show that the fixed point set of any finite subgroup of $\mathrm{Out}^0(A_\Gamma)$ is contractible in $\mathcal{S}(A_\Gamma)$ for any 2-dimensional A_Γ, we must show it for Γ a discrete graph. That is, it is possible to contract the fixed point sets of finite subgroups of $\mathrm{Out}(F_n)$ in the usual Culler-Vogtmann outer space while preserving the property of being separated.

Consider a free group

$$F_n = A_1 * A_2 * \ldots * A_k * Z$$

Let $H < \mathrm{Out}(F_n)$ be a finite subgroup such that every element of H preserves each A_i (i.e., each element lifts to an automorphism sending each A_i to a conjugate of itself). Finally, let $\mathcal{S}(F_n) \subset \mathcal{O}(F_n)$ be a subset of the outer space of F_n all of whose trees are separated with respect to each A_i, that is, $\mathrm{Min}(A_i) \cap g\,\mathrm{Min}(A_j)$ is either a point or empty for every $1 \leq i, j \leq k$ and $g \in F_n$. In this section we will prove the following theorem.

Theorem 2.2.1. *The subspace* $\mathrm{Fix}(H) \cap \mathcal{S}(F_n)$ *is contractible.*

As described in Section 1.4.2, given a morphism between two trees, there is a canonical folding path between them in $\mathcal{O}(F_n)$. If the morphism is between two separated trees, however, the resulting path p between them need not stay in $\mathcal{S}(F_n)$, i.e., a tree $p(s)$ need not be separated, as is illustrated in Figure 2.2.

Figure 2.2 shows two trees F_n-trees M and N, with axes of $a \in A_1$, $b \in A_2$ and $z \in Z$, both separated. A morphism which folds as indicated in the figure has a Skora folding path which does not lie in $\mathcal{S}(F_n)$: the intermediate tree at the bottom of the figure is not separated.

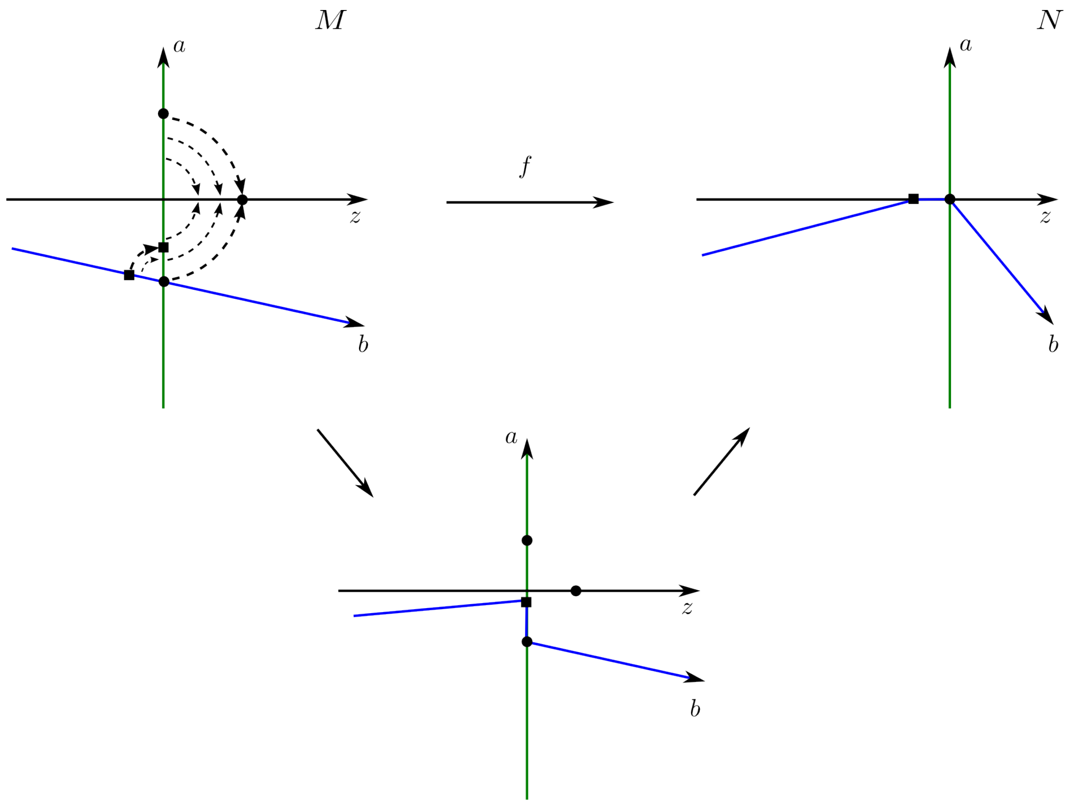

Figure 2.2: A morphism $f : M \to N$ between two separated trees

To get around this problem we define a tree $T' \in \mathcal{S}(F_n)$ and paths p_1 from T to T' and p_2 from T' to $\bar{T}$ which depend continuously on p and which both stay in $\mathcal{S}(F_n)$. We also show that if T and $\bar{T}$ are in $\mathrm{Fix}(H)$, so is T' and so are the paths p_1 and p_2. This implies the contractibility of $\mathcal{S}(F_n) \cap \mathrm{Fix}(H)$.

2.2.1 Constructing the intermediate tree T'

Informally, to construct T', we take T and equivariantly identify points of each minimal subtree $\mathrm{Min}(A_i)$ mapped to the same point of $\bar{T}$. The path from T to T' consists of folding within each minimal subtree and their translates, and the path from T' to $\bar{T}$ consists of folding the rest.

More precisely, Let $T_i = \mathrm{Min}(A_i) \subset T$. Consider the following equivalence relation: for $x, y \in T$, $x \sim y$ if and only if there is an i and a g such that $x, y \in gT_i$ and both are mapped to the same point in $\bar{T}$. Then $T' = T/\sim$. In other words, starting with T, for each $g \in F_n$ and each $1 \leq i \leq k$, identify edges in gT_i which are identified in $\bar{T}$.

There is a well-defined, free minimal action of F_n on T', since otherwise, if an $\mathrm{Axis}(u)$ (for some $u \in F_n$) becomes degenerate in T', then it must be degenerate in $\bar{T}$, contradicting $\bar{T}$ being a free, minimal F_n-tree. Also, since folding was only done within $\mathrm{Min}(A_i)$ and their translates, $T' \in \mathcal{S}(F_n)$.

Let ξ denote a morphism from T to $\bar{T}$, φ denote the quotient morphism from T to T', and ψ denote the morphism from T' to $\bar{T}$ defined by $\psi(x') = \xi(x)$, where $x \in T$ is such that $\varphi(x) = x'$.

Lemma 2.2.2. *Fix a tree $T_0 \in \mathrm{Fix}(H) \cap \mathcal{S}(F_n)$, the White morphism $\xi : T \to \bar{T}$, the tree T', morphism $\varphi : T \to T'$, and morphism $\psi : T' \to \bar{T}$ defined above depend continuously on $\bar{T}$.*

Proof. Recall that given a fixed $T_0 \in \mathcal{O}(F_n)$ and any $\bar{T}$, White ([Whi93]) constructs an equivariant morphism $\xi : T \to \bar{T}$ where T is in the same simplex as T_0. The morphism ξ, and hence also T, depends continuously on $\bar{T}$ (see Section 1.4.3).

Since T' is defined as the identification in T of all points in each $\mathrm{Min}(A_i)$ and their translates identified by ξ, the subtrees $\bar{T}_i = \mathrm{Min}(A_i) \subset \bar{T}$ are A_i-equivariantly isometric to $T'_i = \mathrm{Min}(A_i) \subset T'$. Consider an (oriented) axis of $z \in F_n \setminus A_i$ in T, and let $p(i, z)$ be the point on the axis closest to T_i. If T_i and $\mathrm{Axis}(z)$ intersect, let $p(i, z)$ be the greatest (per orientation of $\mathrm{Axis}(z)$) point of intersection. Let $d(i,z)$ be the distance between $\varphi(p(i,z))$ and T'_i in T' (see Figure 2.3). The tree T' is determined by the $T'_i = \bar{T}_i$ and the distances $d(i,z)$ for different z's. The former clearly depends continuously on $\bar{T}$. The distance $d(i,z)$ is equal to the distance between $\psi(p(i,z))$ and $\bar{T}_i$ in $\bar{T}$, since all folding in T_i is "already done" in T' and thus the segment between $\varphi(p(i,z))$ and T'_i in T' is mapped isometrically to $\bar{T}$.

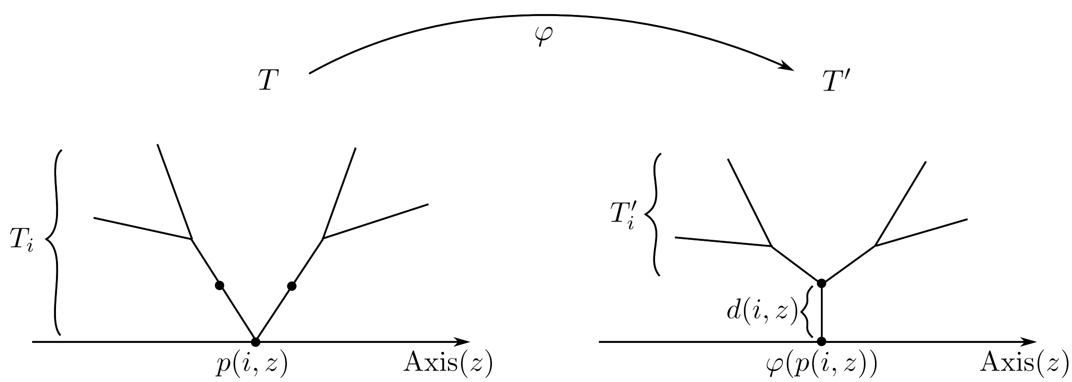

Figure 2.3: Basepoint $p(i,z)$ and distance $d(i,z)$

This shows the continuity of T'. Since $\varphi : T \to T'$ is determined by T', namely $\varphi(x) = [x]$ (where $[x]$ is the equivalence class under $\sim$ defined earlier), φ itself depends continuously on $\bar{T}$.

Finally, since $\xi = \psi \circ \varphi$, and the image of φ is all of T', the morphism ψ is determined by ξ and φ. Hence, since φ depends continuously on ξ, so does ψ. $\square$

2.2.2 Paths p_1 and p_2

Let p be the Skora path corresponding to ξ, and let p_1 be the Skora path from T to T' corresponding to φ. Let p_2 be the Skora path from T' to $\bar{T}$ corresponding to ψ, parametrized by distance of φ-preimages in T. That is, points $x, y \in T'$ are identified in $p_2(s)$ if there are φ-preimages $\tilde{x}$ and $\tilde{y}$ of x and y in T such that (1) they are mapped to the same point by ξ, and (2) they are distance s or less apart in T. These paths are illustrated in Figure 2.4.

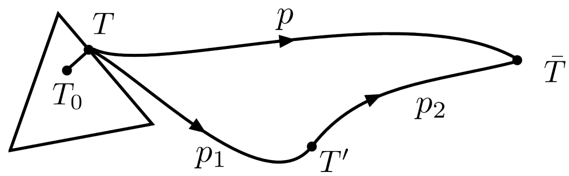

Figure 2.4: Paths p, p_1, and p_2 in $\mathcal{O}(F_n)$

Let ξ_s denote the Skora morphism (see Definition 1.4.1) from T to $p(s)$, let φ_s denote the morphism from T to $p_1(s)$, and let ψ_s denote the morphism from T to $p_2(s)$. Since ξ_s is equivariant with respect to the F_n-action, by the definition of the Skora paths p_1 and p_2, the morphisms φ_s and ψ_s are equivariant as well.

To show that the paths p_1 and p_2 are in the fixed point set of H, we will use the characterization that a tree is fixed if H is realized by a subgroup of automorphisms of the quotient graph of the tree by the free group action. Accordingly, let $G(r) = p(r)/F_n$, $G_1(r) = p_1(r)/F_n$, and $G_2(r) = p_2(r)/F_n$. The equivariant morphisms ξ_r, φ_r, and ψ_r induce morphisms $\Xi_r : T/F_n \to G(r)$, $\Phi_r : T/F_n \to G_1(r)$ and $\Psi_r : T/F_n \to G_2(r)$. Since T/F_n and $\bar{T}/F_n$ are in $\mathrm{Fix}(H)$, H acts on both graphs by graph automorphisms. By Skora's theorem, Ξ_r is equivariant with respect to the action of H. The following two facts follow directly from the continuity p_1, and p_2.

Proposition 2.2.3.

1. *If* $p_1(s) \in \mathrm{Fix}(H)$ *for* $s \leq r$, *then* Φ_r *is equivariant with respect to the action of* H.

2. *If all of* p_1 *is in* $\mathrm{Fix}(H)$ *and* $p_2(s) \in \mathrm{Fix}(H)$ *for all* $s \leq r$*, then* Ψ_r *is equivariant with respect to the action of* H*.*

Lemma 2.2.4. p_1 *is in* $\mathcal{S}(F_n) \cap \mathrm{Fix}(H)$.

Proof. Since p_1 is produced through folding within $\mathrm{Min}(A_i)$ and their translates, it is in $\mathcal{S}(F_n)$. We now show that it is in $\mathrm{Fix}(H)$, that is, the twisting of the action $F_n \curvearrowright p_1(r)$ by outer automorphisms in H quotients to an action of H on $G_1(r)$ by graph automorphisms for every r.

Let p_1 be parametrized from $p_1(0) = T$ to $p_1(t) = T'$ (i.e. no identified points are more than t apart). Consider any r such that $p_1(r) \in \mathrm{Fix}(H)$ and any $0 < \epsilon \leq$ min edge length in $p_1(r)$. We will show that $p_1(r + \epsilon) \in \mathrm{Fix}(H)$.

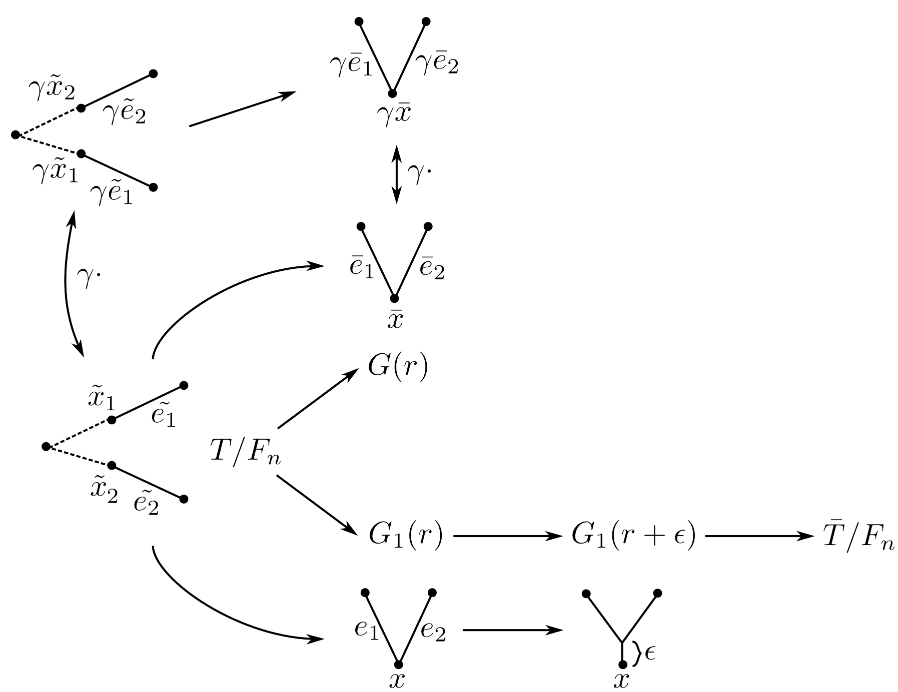

Figure 2.5: Adjacent edges in $G_1(r)$ which are ϵ folded in $G_1(r + \epsilon)$

Since $p_1(r) \in \mathrm{Fix}(H)$, we have $H \curvearrowright G_1(r)$ by graph automorphisms. To prove that $p_1(r+\epsilon) \in \mathrm{Fix}(H)$, we show that H is realized by graph automorphisms of $G_1(r+\epsilon)$.

Suppose e_1, e_2 are two edges adjacent to a common vertex x in $G_1(r)$ and an initial ϵ-segment of e_1 is folded onto an initial ϵ-segment of e_2 in $G_1(r+\epsilon)$. Consider edges $\tilde{e}_1$ and $\tilde{e}_2$ in T/F_n which are isometrically mapped by Φ_r to e_1 and e_2 respectively (see Figure 2.5). By the parametrization of the path p_1, we know that the images $\Xi(\tilde{e}_1)$ and $\Xi(\tilde{e}_2)$, denoted $\bar{e}_1$ and $\bar{e}_2$ respectively, are adjacent edges in $G(r)$. Here we may need to take a suitable subdivision of the edges of T, $p_1(r)$ and $p(r)$ for $\tilde{e}_1$ and $\tilde{e}_2$ to be edges (and not edge segments), $\Xi(\tilde{e}_1)$ and $\Xi(\tilde{e}_2)$ to be edges in $G(r)$ (and not paths) and for them to have one common vertex in $G(r)$ (and not two).

It follows from the definition of p_1 that the edges $\bar{e}_1$ and $\bar{e}_2$ ϵ-fold in $G(r+\epsilon)$ (i.e., initial segments of length ϵ of $\bar{e}_1$ and of $\bar{e}_2$ adjacent to their common vertex get folded). Since $p(r) \in \mathrm{Fix}(H)$, for any $\gamma \in H$, we have $\gamma\bar{e}_1$ and $\gamma\bar{e}_2$ ϵ-fold in $G(r+\epsilon)$. Then by Proposition 2.2.3, edges γe_1 and γe_2 also ϵ-fold in $G_1(r+\epsilon)$.

Since this is true for any $\gamma \in H$ and any ϵ-folded edges e_1, e_2 in $G_1(r)$, we conclude that $p_1(r+\epsilon) \in \mathrm{Fix}(H)$. Since ϵ was arbitrary, up to the length of the smallest edge (which is bounded away from 0), this implies that all of p_1 is in $\mathrm{Fix}(H)$. □

Lemma 2.2.5. *p_2 is in $\mathcal{S}(F_n) \cap \mathrm{Fix}(H)$.*

Proof. Note that ψ restricted to T_i' (and all translates gT_i') is an isometry, and hence so is each ψ_r. Thus, since T' separated, so is every tree $p_2(r)$.

To show that $p_2 \in \mathrm{Fix}(H)$ we use the same method as in the proof of Lemma 2.2.4. Consider $p_2(r) \in \mathrm{Fix}(H)$ and $0 < \epsilon \leq$ min edge length in $p_2(r)$. Suppose adjacent edges $e_1, e_2 \in G_2(r)$ ϵ-fold in $G_2(r+\epsilon)$.

Choose edges $\tilde{e}_1$ and $\tilde{e}_2$ in T/F_n mapping to e_1 and e_2 respectively. (As in the proof of Lemma 2.2.4, a suitable subdivision needs to be taken.) Because of our chosen parametrization of p_2, edges $\tilde{e}_1$ and $\tilde{e}_2$ are mapped to adjacent edges in $G(r)$, which we denote by $\bar{e}_1$ and $\bar{e}_2$ respectively. It follows from the definition of p_2 that these edges ϵ-fold in $G(r+\epsilon)$. Since $p(r) \in \mathrm{Fix}(H)$, $\gamma\bar{e}_1$ and $\gamma\bar{e}_2$ ϵ-fold in $G(r+\epsilon)$ as well.

By Proposition 2.2.3, this implies that γe_1 and γe_2 ϵ-fold in $G_2(r+\epsilon)$. □

Proof of Theorem 2.2.1. By Lemmas 2.2.4 and 2.2.5, both p_1 and p_2 are in $\mathrm{Fix}(H) \cap \mathcal{S}(F_n)$. Since T', φ, and ψ depend continuously on $\bar{T}$ by Lemma 2.2.2, we get that the paths p_1 and p_2 vary continuously with $\bar{T}$. Since $\bar{T} \in \mathrm{Fix}(H) \cap \mathcal{S}(F_n)$ is arbitrary, we get a deformation retraction of $\mathrm{Fix}(H) \cap \mathcal{S}(F_n)$ to T_0. Thus $\mathrm{Fix}(H) \cap \mathcal{S}(F_n)$ is contractible. □

2.3 Contracting non-empty fixed point sets of finite subgroups $H < \mathrm{Out}^0(A_\Gamma)$ in $\mathcal{S}(A_\Gamma)$

Theorem 2.3.1. *Give a finite subgroup $H < \mathrm{Out}^0(A_\Gamma)$, if the subspace $\mathrm{Fix}(H) \cap \mathcal{S}(A_\Gamma)$ is non-empty, then it is contractible.*

Proof. Let $\bar{\mathcal{X}}$ be a point in $\mathrm{Fix}(H) \cap \mathcal{S}(A_\Gamma)$. We will define a continuous deformation of $\mathrm{Fix}(H) \cap \mathcal{S}(A_\Gamma)$ to $\bar{\mathcal{X}}$.

Since $P_v(H)$ is a finite subgroup of $\mathrm{Out}(A_{\mathrm{lk}(v)})$, by Theorem 2.2.1, $\mathcal{S}(A_{\mathrm{lk}(v)}) \cap$

$\mathrm{Fix}(P_v(H))$ is contractible. That is, for any point $\mathcal{X} \in \mathrm{Fix}(H)$, the tree $T_{\mathrm{lk}(v)}$ with the corresponding action of $A_{\mathrm{lk}(v)}$ can be connected by a path within $\mathcal{S}(A_{\mathrm{lk}(v)}) \cap \mathrm{Fix}(P_v(H))$ to the tree $\bar{T}_{\mathrm{lk}(v)}$, where $\bar{T}_{\mathrm{lk}(v)}$ is the corresponding tree in $\bar{\mathcal{X}}$. We claim that this path in $\mathcal{S}(A_{\mathrm{lk}(v)}) \cap \mathrm{Fix}(P_v(H))$ extends to a path in $\mathcal{S}(A_\Gamma) \cap \mathrm{Fix}(H)$.

Note that a deformation of $T_{\mathrm{lk}(v)}$ along the path to $\bar{T}_{\mathrm{lk}(v)}$ induces a deformation of trees $T_{[w]} = \mathrm{Min}(A_{[w]}) \subset T_{\mathrm{lk}(v)}$ for every $w \in \Gamma_0 \cap \mathrm{lk}(v)$. For every $u \in \Gamma_0 \cap \mathrm{lk}(w)$, this deformation in turn extends to a deformation of $T_{\mathrm{lk}(u)}$ (see Figure 2.6).

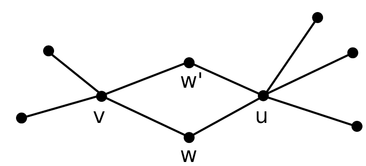

Figure 2.6: Deforming $T_{\mathrm{lk}(v)}$ induces a deformation in $T_{\mathrm{lk}(u)}$.

However, since the trees are separated, the induced deformations only affect $T_{\mathrm{lk}(u)}$ for vertices $u \in \Gamma_0$ whose distance from v is 2. Thus, since the topology on $\mathcal{O}(A_\Gamma)$ is the product topology of $\mathcal{O}(F(\mathrm{lk}(v)))$ for all $v \in \Gamma_0$ (with additional real-valued parameters) and all trees are separated, the path from $T_{\mathrm{lk}(v)}$ to $\bar{T}_{\mathrm{lk}(v)}$ extends to a path in $\mathcal{S}(A_\Gamma)$.

To see that this $\mathcal{S}(A_\Gamma)$ path stays in $\mathrm{Fix}(H)$, note that for $\varphi \in H$, $P_v(\varphi)$ fixes (up to equivariant isometry) the subtrees $T_{[w]}$ in $T_{\mathrm{lk}(v)}$ for every $w \in \mathrm{lk}(v) \cap \Gamma_0$ throughout the path from $T_{\mathrm{lk}(v)}$ to $\bar{T}_{\mathrm{lk}(v)}$. Thus $T_{[w]}$ is also fixed by $P_u(\varphi)$ as a subtree of $T_{\mathrm{lk}(u)}$ for all $u \in \mathrm{lk}(w) \cap \Gamma_0$ throughout the path in $\mathcal{S}(A_\Gamma)$. All other minimal subtrees in $T_{\mathrm{lk}(u)}$ corresponding to vertices of $(\mathrm{lk}(u) \cap \Gamma_0) \setminus \mathrm{lk}(v)$ remain unaffected since $T_{\mathrm{lk}(u)}$ is separated. Hence the $\mathcal{S}(A_\Gamma)$ path is in $\mathrm{Fix}(H) \cap \mathcal{S}(A_\Gamma)$.

This path varies continuously as a function of its starting point $\mathcal{X}$ in $\mathrm{Fix}(H)$

by the continuity of the path in $\mathrm{Fix}(P_v(H)) \cap \mathcal{S}(A_{\mathrm{lk}(v)})$ (Lemma 2.2.2). Hence we can contract $\mathrm{Fix}(H) \subset \mathcal{S}(A_\Gamma)$ to a subset in which trees corresponding to $v \in \Gamma_0$ are equivariantly isometric to $\bar{T}_{\mathrm{lk}(v)}$. We then proceed to perform the same type of deformation for trees corresponding to all other vertices $z \in \Gamma_0$, one vertex at a time. The resulting deformation takes $\mathrm{Fix}(H) \cap \mathcal{S}(A_\Gamma)$ to a single point, $\bar{\mathcal{X}}$. $\square$

CHAPTER 3

EXISTENCE OF FIXED POINTS OF FINITE SUBGROUPS

Given a two dimensional RAAG A_Γ, for a finite $H < \mathrm{Out}^0(A_\Gamma)$ the fixed point set $\mathrm{Fix}(H) \cap \mathcal{S}(A_\Gamma)$ is contractible by Theorem 2.3.1 if it is not empty. This chapter is devoted to demonstrating particular categories of H for which we can show that the fixed point sets are always non-empty. The general question of whether this is true for all finite H remains open: it is not even known if a fixed point exists in the larger space $\mathcal{O}(A_\Gamma)$.

3.1 Fixed points and sphere systems

Recall that we can project the group $\mathrm{Out}^0(A_\Gamma)$ onto each $\mathrm{Out}(A_{\mathrm{lk}(v)})$ for every $v \in \Gamma$. We may similarly project a finite $H < \mathrm{Out}^0(A_\Gamma)$ to each outer automorphism group, but just as $\mathrm{Out}^0(A_\Gamma)$ is not in general a direct product of outer automorphism groups, neither in general is H a direct product of its images. Analogously, we can realize each projection of H by a fixed graph, but it is not clear whether it is always possible to "patch" these graphs together to form a point in $S(A_\Gamma)$. The first step toward such patching is to find separated trees fixed by each projection of H to $\mathrm{Out}(A_{\mathrm{lk}(v)})$. The second step is to create a fixed point in $S(A_\Gamma)$ from these separated trees. We take up the first step in this section, and the second step in Section 3.2.

3.1.1 A simple case

If for each $v \in \Gamma_0$ the finite subgroup $P_v(H)$ of $\mathrm{Out}(A_{\mathrm{lk}(v)})$ lifts to a finite subgroup $\tilde{H} < \mathrm{Aut}(A_{\mathrm{lk}(v)})$ preserving all maximal $A_{[w]} < A_{\mathrm{lk}(v)}$ as well as the subgroup generated by non-maximal vertices, then we can easily construct a separated fixed point.

For each $v \in \Gamma_0$, let $Z_v = \langle \mathrm{lk}(v) \setminus \cup_{w \in \Gamma_0}[w] \rangle$. That is, the free group generated by vertices in the link of v which are not maximal, i.e. whose link is contained in a larger link. By Culler's Realization Theorem, there is a finite marked simplicial graph $G([w])$ realizing $\tilde{H}$ restricted to $A_{[w]}$ in which a vertex is fixed, and a graph $G(Z_v)$ realizing Z_v in which a vertex is fixed. Then we can take G_v to be the wedge product of all these graphs, wedged at their respective fixed points. The group $P_v(H)$ is realized by automorphisms of G_v, so $G_v \in \mathrm{Fix}(P_v(H))$.

Taking the universal covers T_v of each G_v and universal covers T^w of each $G([w])$, we get a point $\mathcal{X} = \{T_v, T^w, i_{w,v}, \gamma\}$ in $\mathcal{S}(A_\Gamma)$, where the maps $i_{w,v}$, are isometries between T^w and the min set of $A_{[w]}$ in T_v and γ is the zero map. By the construction of each G_v, the point $\mathcal{X}$ is fixed under the action of $H \curvearrowright \mathcal{S}(A_\Gamma)$.

We can show that $\mathcal{S}(A_{\mathrm{lk}(v)}) \cap \mathrm{Fix}(P_v(H))$ is non-empty for certain wider classes of subgroups $H < \mathrm{Out}^0(A_\Gamma)$ using the sphere system model of outer space $\mathcal{O}(F_n)$. In this model, described in Section 3.1.2, points of outer space correspond to embedded collections of spheres in a doubled handlebody with fundamental group F_n, and elements of $\mathrm{Out}(F_n)$ correspond to diffeomorphisms of the doubled handlebody. For some finite subgroups of $\mathrm{Out}(F_n)$, we can surger a sphere system corresponding to a fixed point of a finite subgroup such that the resulting sphere system is both fixed and separated (with respect to a given decomposition $F_n = A_1 * A_2 * \cdots * A_k * Z$).

3.1.2 The sphere system model of $\mathcal{O}(F_n)$

Let $F_n = A_1 * A_2 * \cdots * A_k * Z$ and $H < \mathrm{Out}(F_n)$ a finite subgroup in which every element preserves each A_i up to conjugacy. In the following paragraphs we give a sketch of a model of $\mathcal{O}(F_n)$ in which points are sphere systems inside a doubled handlebody, as described in [Hat95] and [HV98].

Consider $M = \#_n S^1 \times S^2$, the connected sum of n copies of $S^1 \times S^2$. A *sphere system* in M is a finite collection of disjoint embedded spheres S^2 in which no two spheres are isotopic and none of which bound a ball. The sphere system complex $S(M)$ is a simplicial complex whose k-simplices are isotopy classes of sphere systems with $k+1$ spheres. The barycentric coordinates in each simplex of $S(M)$ corresponding to a particular sphere system provide weights to each sphere in the sphere system. Consider the subcomplex consisting of sphere systems S such that every component of $M - S$ is simply connected. Let this subcomplex be denoted $S_0(M)$. As we describe below, the weighted sphere systems in $S_0(M)$ are in one-to-one correspondence with points in $\mathcal{O}(F_n)$.

First, we fix a homotopy equivalence $\tau : R_n \to M$, where R_n is the rose with n petals. Given a sphere system $S \in S_0(M)$, consider the graph having a vertex for each component of $M - S$ and an edge between two vertices if and only if there is a sphere in S adjacent to the components corresponding to the two vertices. The length of each edge is equal to the weight of the corresponding sphere. The homotopy equivalence τ induces a marking on the resulting graph. This defines a point in $\mathcal{O}(F_n)$. In fact $S_0(M)$ and $\mathcal{O}(F_n)$ are homeomorphic (see appendix of [Hat95]).

If we keep track of a marked basepoint p in M, diffeomorphisms of (M, p) induce automorphisms of $\pi_1(M) = F_n$. By a result of Laudenbach ([Lau74]), the mapping class group of M is generated by diffeomorphism which correspond to the Nielsen generators of $\mathrm{Aut}(F_n)$ and by 2π twists about spheres embedded in each of the $S^1 \times S^2$ summands. These twists act trivially on $S(M)$, since homotopic spheres in M are isotopic ([Lau73]). Thus, the kernel of $\mathrm{MCG}(M, p) \to \mathrm{Aut}(F_n)$ is generated by the 2π sphere twists; let this kernel be denoted by K. Then $\mathrm{MCG}(M, p)/K \cong \mathrm{Aut}(F_n)$, and if we do not keep track of a basepoint in M, then $\mathrm{MCG}(M)/K \cong \mathrm{Out}(F_n)$.

The diffeomorphism of M inducing the Nielsen automorphism $a \mapsto ab$ is a *torus Dehn twist* in a torus passing through the b handle and containing one end of the a handle (Figure 3.1). Consider M as the identification of two handlebodies H_n along their boundaries. Restricting to the Heegaard surface (the boundary along which the handlebodies are glued), this diffeomorphism restricts to two Dehn twists supported on the shaded annuli in Figure 3.1.

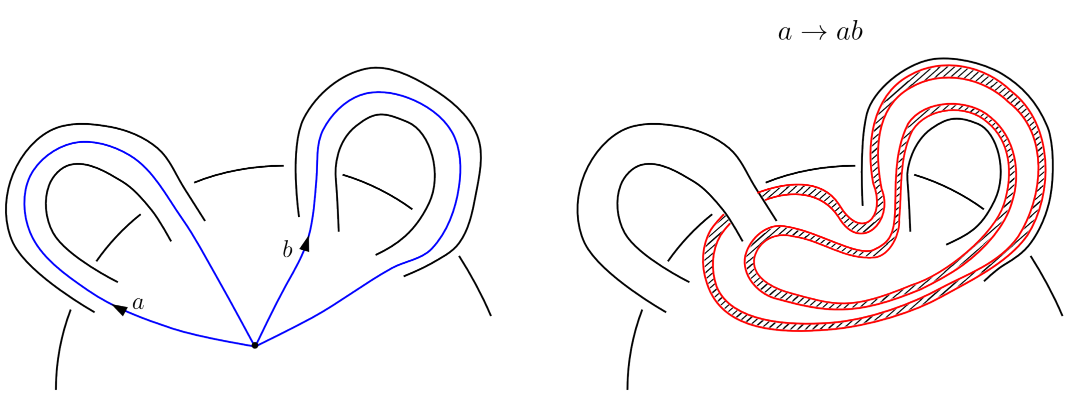

Figure 3.1: The intersection of two fixed torii with the Heegaard surface of M, and between those S^1 intersections, the support of the Dehn twist inducing $a \to ab$

By torus Dehn twist we mean a diffeomorphism fixing the boundary of an embedded torus (the outer circle in 3.2) and "twisting" a smaller torus inside it (the inner circle in 3.2) a full revolution, thus fixing it too, while twisting the region between the two torii an intermediate amount which increases continuously from 0 radians near the outer torus to 2π radians near the inner torus.

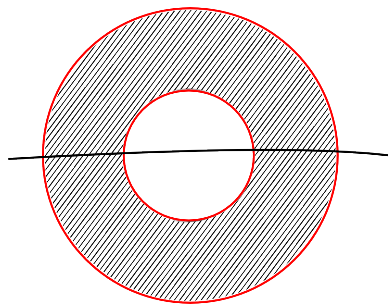

Figure 3.2: A cross section of the support of the torus Dehn twist. The horizontal segment is a cross section of the Heegaard surface in M

A convenient way of visualizing this torus Dehn twist is as "pulling" the end of $S^1 \times S^2$ corresponding to a through the $S^1 \times S^2$ corresponding to b along the b^{-1} direction, dragging nearby parts of the surface along.

The Nielsen automorphism $a \mapsto a^{-1}$ is induced by the diffeomorphism switching the "ends" of the $S^1 \times S^2$ corresponding to a, and the diffeomorphism corresponding to $a \mapsto b, b \mapsto a$ switches the two corresponding $S^1 \times S^2$'s.

Thus the automorphism which sends a to bab^{-1} and fixes the remaining generators is induced by the diffeomorphism "pulling" the whole a handle around the b handle. The support of this diffeomorphism, restricted to the Heegaard surface, is illustrated in Figure 3.3.

Given a sphere system S and a maximal sphere system Σ in M, we say that S is in *normal form* with respect to Σ if S intersects each component of $M - \Sigma$ in a

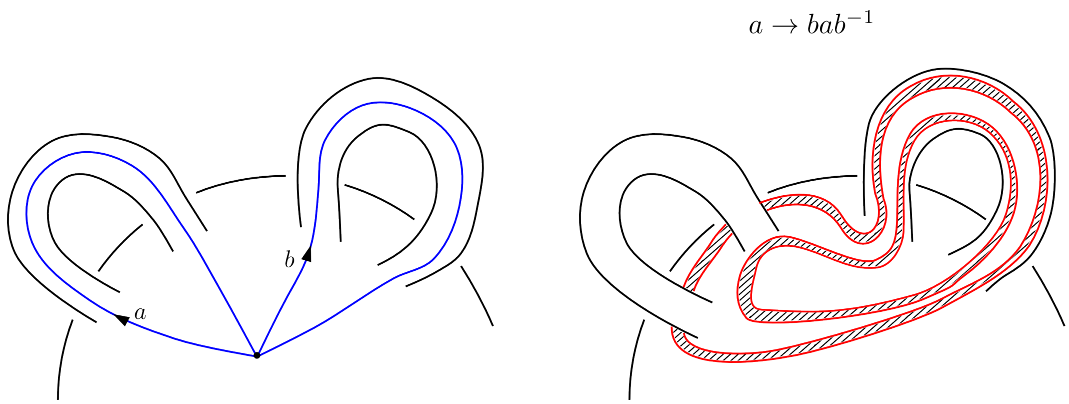

Figure 3.3: The intersection of two fixed torii with the Heegaard surface of M, and between those S^1 intersections, the support of the Dehn twist inducing $a \mapsto bab^{-1}$

collection of surfaces each having at most one boundary circle on each boundary sphere of a component of $M - \Sigma$, and no surface can be isotoped to the boundary of a component of $M - \Sigma$. That is, the number of intersections between S and Σ is minimal.

Two sphere systems S and S' in normal form with respect to Σ are *equivalent* if there's a homotopy between them fixing common spheres of Σ and S and keeping the remaining spheres in S transverse to Σ throughout the homotopy, with the circle components of $S \cap \Sigma$ varying by isotopy in Σ.

Proposition 3.1.1 (Hatcher)**.** 1. *Every sphere system S can be isotoped into normal form with respect to any maximal sphere system Σ.*

2. *Two isotopic sphere systems in normal form with respect to the same maximal sphere system are equivalent.*

The proof of the first part of this proposition proceeds by minimizing the number of transverse intersections $\Sigma \cap S$. If S is not in normal form, it can be isotoped to decrease this number of intersection. For details see [Hat95].

Hatcher uses this proposition to show that $\mathcal{O}(F_n)$ is contractible by defining a contraction of $S_0(M)$ (Theorem 2.1, [Hat95]). This contraction relies on a surgery which we'll use to construct an element of $\mathcal{S}(F_n)$ fixed by a finite subgroup of $\mathrm{Out}(F_n)$.

3.1.3 Surgery

Given a maximal sphere system Σ, and a weighted sphere system S in normal form with respect to Σ, we can surger S by a sphere $\Sigma_i \in \Sigma$. Since S is in normal form with respect to Σ, the intersection $S \cap \Sigma_i$ is a finite collection of circles. An inner-most such circle bounds either one open disk $D \subset \Sigma_i$ disjoint from S or two such disks (see Figure 3.4).

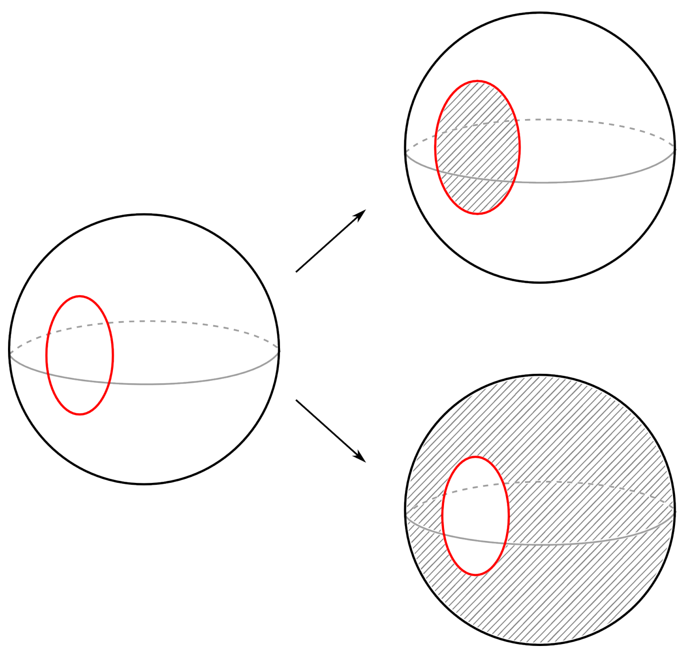

Figure 3.4: If $S \cap \Sigma_i$ is a single circle, two choices are possible for disks to surger with

If only one such disk $D \subset \Sigma_i$ disjoint from S exists, surger S across D, replacing the sphere of S in which ∂D lies with two spheres, dividing the weight of the removed sphere equally between them. If there are two choices for a disk $D \subset \Sigma_i$ disjoint from S, surger S by replacing the sphere in which ∂D lies with four spheres, dividing the weight of the removed sphere equally among them (see Figure 3.5).

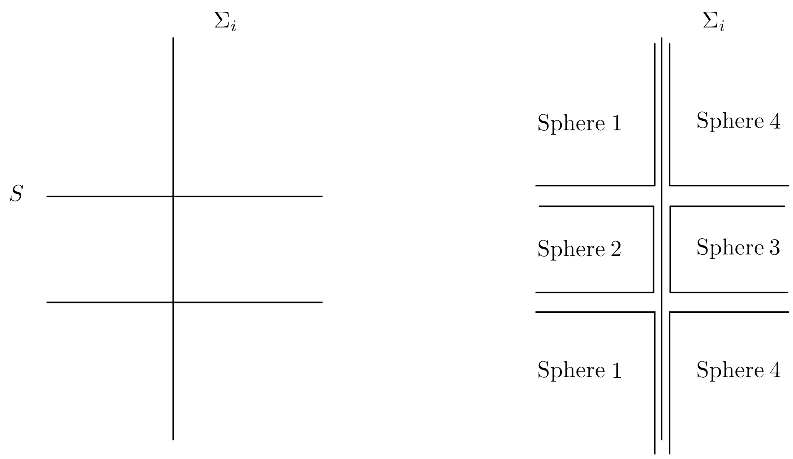

Figure 3.5: Surgering a sphere of S across Σ_i when $S \cap \Sigma_i$ is a single circle

In the resulting sphere system, the number of components of $S \cap \Sigma_i$ is decreased by one. Performing this surgery with resultant inner-most components of $S \cap \Sigma_i$, and then with all other spheres in Σ, we arrive at a sphere system S' disjoint from Σ.

This process can be made canonical by doing all inner-most surgeries at once.

3.1.4 Constructing an element in $\mathrm{Fix}(H) \subset \mathcal{S}(F_n)$

As in 2.2, let $F_n = A_1 * A_2 * \cdots * A_k * Z$, and $H < \mathrm{Out}(F_n)$ be a finite subgroup preserving $A_1, \ldots, A_k$ (i.e., each element lifts to an automorphism mapping every A_i to a conjugate of itself).

Definition 3.1.1. *To the free decomposition $A_1 * A_2 * \cdots * A_k * Z$ corresponds a single (up to isotopy)* decomposition sphere system*. It is the sphere system Σ for which $M - \Sigma$ has $k+1$ components, marked by generators A_1, A_2, ..., A_k, and Z respectively (Figure 3.6 illustrates an example).*

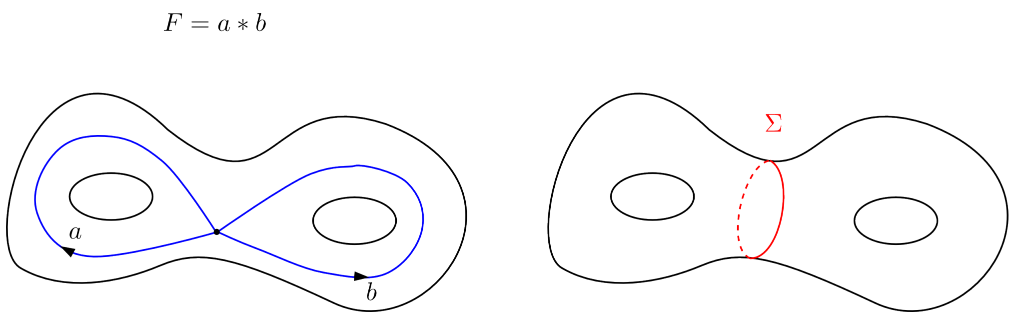

Figure 3.6: The intersection of a decomposition sphere system for $F = a * b$ with the Heegaard surface of M

Proposition 3.1.2. *Let $F_n = A_1 * A_2 * \cdots * A_k * Z$ be a free group and $H < \mathrm{Out}(F_n)$ be finite subgroup preserving $A_1, \ldots, A_k$. If the decomposition sphere system Σ is preserved by H (up to isotopy), then $\mathcal{S}(F_n) \cap \mathrm{Fix}(H)$ is non-empty.*

Proof. By Culler's Realization Theorem, there is a point in $\mathcal{O}(F_n)$ fixed by H. Let S be the sphere system corresponding to that point. Assume, without loss of generality, that it is in normal from with respect to Σ. Since both Σ and S are fixed by H, so is the system S' resulting from surgering S by Σ. The sphere system S' is also *separated*, that is, the marked graph corresponding to S' defines a point in $\mathcal{O}(F_n)$ which is separated with respect to the given decomposition of F_n. Hence S' is in $\mathcal{S}(F_n) \cap \mathrm{Fix}(H)$. □

3.1.5 Constructing a fixed point when the separating sphere system is not fixed

The hypothesis in Proposition 3.1.2 that the decomposition sphere system is fixed by H is far from necessary. For example, when all but one of the spheres in a decomposition sphere system Σ are fixed, then surgering S by the spheres that are fixed produces a separated sphere system S' in $\mathcal{S}(F_n) \cap \mathrm{Fix}(H)$. (See Figure 3.7.)

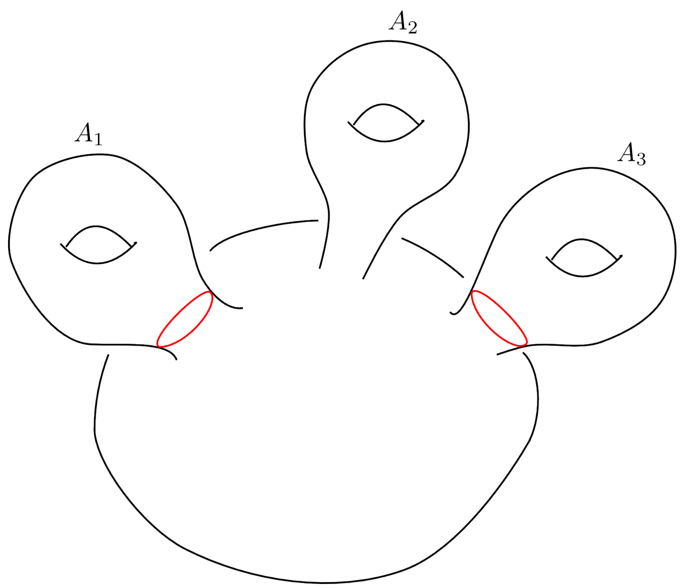

Figure 3.7: Even though only the spheres corresponding to A_1 and A_3 may be fixed (red), surgering by these spheres separates all three A_i's

Even this however, is not necessary. For example, for $F = \langle a, b, z \rangle$, (where $A_1 = \langle a \rangle$, $A_2 = \langle b \rangle$, and $Z = \langle z \rangle$), and H generated by (the conjugacy class of) the automorphism

$$a \mapsto a^{-1}$$

$$b \mapsto b^{-1}$$

$$c \mapsto acb$$

none of the spheres in the decomposition sphere system are fixed. However, the sphere system Σ illustrated in Figure 3.8 is fixed by H and it separates A_1 from A_2. Hence surgering any fixed sphere system S by Σ produces $S' \in \mathcal{S}(F) \cap \mathrm{Fix}(H)$.

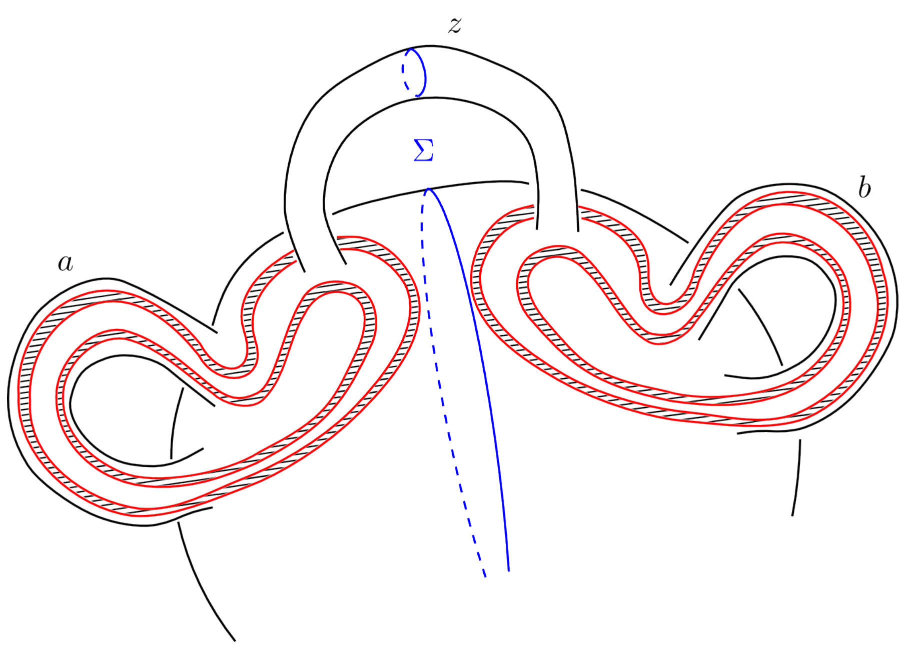

Figure 3.8: A separating sphere system Σ and support of diffeomorphism induced by H restricted to the Heegaard surface.

Unfortunately we have not found a satisfactory description of all finite $H < \mathrm{Out}(F_n)$ for which a separating sphere system exists.

Finally, we highlight one particular case in which a separating system exists here, since it will be useful in the next section. Suppose $H < \mathrm{Out}(F_n)$ is such that there is a finite $\tilde{H} < \mathrm{Aut}(F_n)$ mapping to H under $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$, and $\tilde{H}$ maps each A_i to $z_i A_i z_i^{-1}$, where $z_i \in Z$ and preserves Z. Then we say that H *strongly preserves* the free decomposition $F_n = A_1 * A_2 * \cdots * A_k * Z$. Note that all spheres in the decomposition sphere system except that corresponding to the factor Z are fixed. These fixed spheres constitute a separating sphere system.

3.2 From fixed points in $\mathcal{S}(F_n)$ to fixed points in $\mathcal{S}(A_\Gamma)$

Proposition 3.2.1. *Suppose A_Γ is a right-angled Artin group, and $H < \mathrm{Out}^0(A_\Gamma)$ a finite subgroup. If $P_v(H)$, for every $v \in \Gamma_0$, strongly preserves the decomposition $A_{\mathrm{lk}(v)} = A_{[w_1]} * A_{[w_2]} * \cdots * A_{[w_m]} * Z$, where $\{w_1, \ldots, w_m\} = \Gamma_0 \cap \mathrm{lk}(v)$, then there is a point of $\mathcal{S}(A_\Gamma)$ fixed by H.*

Proof. By the discussion in the previous section, we can find $T_{\mathrm{lk}(a)} \in \mathcal{S}(A_{\mathrm{lk}(a)}) \cap \mathrm{Fix}(P_a(H))$ for each $a \in \Gamma_0$. Suppose $w \in \Gamma_0$ is a neighbor of both v and u, both in Γ_0. Then both $\mathrm{Min}(A_{[w]}) \subset T_{\mathrm{lk}(v)}$ and $\mathrm{Min}(A_{[w]}) \subset T_{\mathrm{lk}(u)}$ are fixed by $P_v(H)|_{A_{[w]}} \cong P_u(H)|_{A_{[w]}}$ respectively. While the two minimal subtrees, one in $T_{\mathrm{lk}(v)}$ and the other in $T_{\mathrm{lk}(u)}$ need not be equivariantly isometric, they can be made equivariantly isometric, either using the White's morphism and Skora path (Section 1.4.3) or even more simply, by replacing G_u^w by G_v^w in G_u. This is possible because $P_u(H)$ and $P_v(H)$ strongly preserve $A_{\mathrm{lk}(u)}$ and $A_{\mathrm{lk}(v)}$ respectively. Hence the intersection of G_u^w with the closure of $G_u \setminus G_u^w$ is a single point, and so is the intersection of G_v^w with the closure of $G_v \setminus G_v^w$.

Thus all $\mathrm{Min}(A_{[a]})$, for all $a \in \Gamma_0$, can be made equivariantly isometric in all trees $T_{\mathrm{lk}(v)}$, $a \in \mathrm{lk}(v)$, producing a point in $\mathcal{S}(A_\Gamma)$. $\square$

Unfortunately our ability to produce separated fixed points in each individual outer space $\mathcal{S}(A_{\mathrm{lk}(v)})$, using sphere system surgery, exceeds in power our ability to put them together to produce a point in $\mathcal{S}(A_\Gamma)$. This is due to the challenge of controlling the behavior of non-maximal subgroups—those generated by elements outside Γ_0—since in general, axes of elements in non-maximal subgroups may intersect minimal trees $T_{[a]}$ for $a \in \Gamma_0$.

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