FUNCTION-INDEXED EMPIRICAL PROCESSES BASED ON AN INFINITE SOURCE POISSON TRANSMISSION STREAM

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Abstract. We study the asymptotic behavior of empirical processes generated by measurable bounded functions of an infinite source Poisson transmission process when the session length have infinite variance. In spite of the boundedness of the function, the normalized fluctuations of such an empirical process converge to a non-Gaussian stable process. This phenomenon can be viewed as caused by the long-range dependence in the transmission process. Completing previous results on the empirical mean of similar types of processes, our results on non-linear bounded functions exhibit the influence of the limit transmission rate distribution at high session lengths on the asymptotic behavior of the empirical process. As an illustration, we apply the main result to estimation of the distribution function of the steady state value of the transmission process.

1. Introduction

We consider the infinite source Poisson transmission process defined by

$$X(t) = \sum_{\ell \in \mathbb{Z}} W_\ell \, \mathbb{1}_{\{\Gamma_\ell \le t < \Gamma_\ell + Y_\ell\}}, \quad t \in \mathbb{R} \,, \tag{1}$$

where the triples $\{(\Gamma_\ell, Y_\ell, W_\ell),\ \ell \in \mathbb{Z}\}$ of session arrival times, durations and transmission rates satisfy

Assumption 1. (i) The arrival times $\{\Gamma_\ell,\ \ell \in \mathbb{Z}\}$ are the points of a homogeneous Poisson process on the real line with intensity λ, indexed in such a way that $\cdots < \Gamma_{-2} < \Gamma_{-1} < \Gamma_0 < 0 < \Gamma_1 < \Gamma_2 < \cdots$

(ii) The durations and transmission rates $\{(Y, W), (Y_\ell, W_\ell),\ \ell \in \mathbb{Z}\}$ are independent and identically distributed random pairs with values in $(0, \infty) \times [0, \infty)$ and independent of the arrival times $\{\Gamma_\ell,\ \ell \in \mathbb{Z}\}$. The random variables W_j are positive with a positive probability. The session lengths Y_j have finite expectation and infinite variance.

(iii) There exist a measure ν on $(0, \infty] \times [0, \infty]$ such that $\nu((1, \infty] \times [0, \infty]) = 1$ and, as $n \to \infty$,

$$n\mathbb{P}\left(\left(\frac{Y}{a(n)}, W\right) \in \cdot\right) \xrightarrow{v} \nu \,,$$

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where $\xrightarrow{v}$ denotes vague convergence on $(0,\infty]\times[0,\infty]$, and a is the left continuous inverse $(1/\bar{F})^{\leftarrow}$ of $1/\bar{F}$. Here F is the distribution function of Y, and $\bar{F}=1-F$ is the corresponding survival function.

Assumption 1 (iii) implies several things, listed below. See Heffernan and Resnick (2007).

- The survival function $\bar{F}$ is regularly varying with index $-\alpha$ for some $\alpha>0$. The function a is then regularly varying with index $1/\alpha$.
- The limiting measure ν is a product measure:

$$\nu=\nu_\alpha\times G\,, \tag{2}$$

where ν_α is a measure on $(0,\infty)$ satisfying $\nu_\alpha((x,\infty))=x^{-\alpha}$ for all $x>0$, and G is a probability measure on $[0,\infty]$.
- We have the following weak convergence on $[0,\infty]$, as $t\to\infty$,

$$\mathbb{P}\left(W\in\cdot\mid Y>t\right)\xrightarrow{w}G\,. \tag{3}$$

We will assume that the exponent α satisfies

$$1<\alpha<2\,. \tag{4}$$

Under Assumption 1, the process (1) is well defined and stationary, see e.g. Faÿ et al. (2007). Under additional moment assumptions, it is shown in this reference that the autocovariance function of the process X is regularly varying at infinity with index $2H-2\in(-1,0)$, where $H=(3-\alpha)/2$. Such slow rate of decay of the covariance function is often associated with long range dependence.

We are interested in studying the large time behavior of the empirical process

$$\mathcal{J}_T(\phi)=\int_0^T\phi\big(X_h(s)\big)\,\mathrm{d}s,\ T>0, \tag{5}$$

where $h>0$, $X_h(s)=\{X(s+t)\,,0\le t\le h\}$, and ϕ is a real valued measurable function defined on the space $\mathcal{D}([0,h])$ endowed with the J_1 topology, see for instance Kallenberg (2002). We notice that the $\mathcal{D}([0,h])$-valued stochastic process $\big(X_h(s),\ s\in[0,T]\big)$ is continuous in probability and, hence, has a measurable version, see Cohn (1972). In particular, $\mathcal{J}_T(\phi)$ above is a well defined random variable, as long as the function ϕ satisfies appropriate integrability assumptions, e.g. when the function ϕ is bounded.

The case $h=0$ and $\phi(x)=x$ has been considered in Mikosch et al. (2002) with $W_i\equiv1$ and by Maulik et al. (2002) in the present context of possible dependence between the session lengths and the rewards (transmission rates). These references consider the case where the intensity of the point process of arrivals is possibly increasing, which gives rise to the slow growth/fast growth dichotomy. In the slow growth case, which includes the case of constant intensity, the limit of the partial sum process is a Lévy stable process, whereas in the fast growth case, the limiting process is the fractional Brownian motion with Hurst index $H=(3-\alpha)/2$. Here we consider a fixed intensity for the sessions arrival rate, hence are restricted to the slow growth case. On the other hand we take ϕ arbitrary (but bounded) and thus obtain what appears to be the first result on the asymptotic behavior of the empirical process for this type of long range dependent shot noise process. The limit process depends on the intensity λ, the tail exponent α and the limit transmission rate distribution G defined in (3).

As an illustration, we apply the main result to the estimation of the distribution function of the steady state value of the transmission process. Moreover we allow $h > 0$. Other potential applications of our main result (e.g. to estimation of the multivariate distribution function) can be handled in a similar way, but we do not pursue them in this paper.

Our main result is stated as a a functional central limit theorem in the Skorohod M_1 topology. A convergence result in this topology was obtained in Resnick and van den Berg (2000) for a similar traffic model, but with $h = 0$ and $\phi(x) = x$. Our result can be viewed as a heavy traffic approximation of the content of a fluid queue fed with input $\phi(X(s))$. It shows, in particular, that even for ϕ boundedd (e.g. with $\phi(x) = x \wedge b$ with b denoting a maximal allowed bandwidth), the fluctuations of the asymptotic approximation of the queue content has an infinite variance. See also (Resnick and van den Berg, 2000, Section 5).

2. Notation and preliminary results

We now introduce some notation and derive certain useful properties of the empirical process (5) stated in several lemmas whose proofs are provided in Section 5.

We employ the usual queuing terminology: a time point t is said to belong to a busy period if $X(t) > 0$; it belongs to an idle period otherwise. A cycle consists of a busy period and the subsequent idle period.

Let $\{S_j, j \in \mathbb{Z}\}$ denote the successive starting times of the cycles such that $\cdots < S_{-2} < S_{-1} < 0 < S_0 < S_1 < \dots$ and let $C_j = S_j - S_{j-1}$ for all $j \in \mathbb{Z}$. Hence S_0 is the starting time of the first complete cycle starting after time 0 and $S_n = S_0 + \sum_{j=1}^n C_j$. For $T > 0$, let M_T denote the number of complete cycles initiated after time 0, and finishing before time T.

The following facts about M/G/∞ queues will be useful. The regenerative property of the cycles and (6) can be found in Hall (1988). The tail property of C_1 is proved in Section 5.

Lemma 1. *Suppose that Assumption 1 holds. Then* $\{(C_j,\, X(\cdot + S_{j-1})\mathbb{1}_{[0,C_j)}),\, j \geq 1\}$ *is an i.i.d. sequence of random pairs with values in* $(0,\infty) \times \mathcal{D}([0,\infty))$*,* C_1 *has a regularly varying tail with index* α *and*

$$\mathbb{E}[C_1] = \mathrm{e}^{\lambda \mathbb{E}[Y]}/\lambda\,, \tag{6}$$

$$\lim_{t\to\infty} t\,\mathbb{P}(C_1 > a(t)x) = \mathrm{e}^{\lambda \mathbb{E}[Y]} x^{-\alpha}\,. \tag{7}$$

Let ϕ be a measurable function defined on $\mathcal{D}([0,h])$, satisfying appropriate integrability conditions for the integral in (5) to be well defined (e.g. bounded). We decompose $\mathcal{J}_T(\phi)$ using the cycles defined above. Let us denote

$$Z_j(\phi) = \int_{S_{j-1}}^{S_j} \phi(X_h(s))\,\mathrm{d}s, \quad j = 1, 2, \dots \tag{8}$$

Then $(Z_j(\phi))_{j\geq 1}$ is a stationary sequence, but, if $h > 0$, it is not an i.i.d. sequence. Nevertheless, it is easy to see that it is strongly mixing. Define sigma-fields $\mathcal{F}_j = \sigma(X(S_j + h - s), s \geq 0)$ and $\mathcal{G}_j = \sigma(X(S_j + t), t \geq 0)$ and mixing coefficients $(\alpha_k)_{k\geq 1}$ by

$$\alpha_k = 2\sup\{|\mathrm{cov}(\mathbb{1}_A, \mathbb{1}_B)|,\ A \in \mathcal{F}_j, B \in \mathcal{G}_{j+k}\ ,\ j \geq 1\}\,.$$

Then, for all $k \geq 2$,

$$\alpha_k \leq 2\mathbb{P}(S_{j+k} - S_{j+1} \leq h) \leq \mathbb{P}(\max(C_{j+2}, \dots, C_{j+k}) \leq h) = F_C(h)^{k-1} . \tag{9}$$

where F_C denotes the distribution function of C_1. Since $F_C(h) < 1$ for any h, the mixing coefficients α_k decay exponentially fast, independently of ϕ. This property will be a key ingredient to the proof of our result since it implies that, in many aspects, the sequence $Z_j(\phi)$ has the same asymptotic properties as an i.i.d. sequence.

We will denote

$$\mathcal{E}(w, \phi) = \mathbb{E}[\phi(w + X_h(0))], \quad w \in [0, \infty) , \tag{10}$$

whenever the latter expectation is well defined, which is always the case if ϕ is bounded. It follows from the elementary renewal theorem that that $\mathbb{E}[Z_j(\phi)] = \mathbb{E}[\phi(X_h(0)]/\mathbb{E}[C_1]$. This identity is stated formally in the following lemma, which also contains another result that will be needed later.

Lemma 2. *Suppose that Assumption 1 holds. Let $h \geq 0$ and ϕ be a bounded measurable function defined on $\mathcal{D}([0, h])$. We have*

$$\mathbb{E}\left[Z_1(\phi)\right] = \mathcal{E}(0, \phi)\mathbb{E}[C_1] = \mathbb{E}[\phi(X_h(0)]\mathbb{E}[C_1] . \tag{11}$$

Moreover, for any $p \in (1, \alpha)$, there exists a constant $C > 0$ and a positive function g depending neither on ϕ nor on T such that $g(x) \to 0$ as $x \to \infty$ and

$$\mathbb{P}\left(\sup_{t \in [0,T]} |\mathcal{J}_t(\phi) - \mathbb{E}[\mathcal{J}_t(\phi)]| > x\|\phi\|_\infty\right) \leq CT^{1-p} + C\,T\,x^{-p} + g(x) . \tag{12}$$

For all $\epsilon, t > 0$ let $N_{\epsilon,t}$ be the number of sessions of length greater than $\epsilon a(t)$ arriving and ending within the first complete cycle $[S_0, S_1)$. Further, we let $Y_{\epsilon,t}$ be the length of the first session starting at or after S_0 with length greater than $\epsilon a(t)$ and let $\Gamma_{\epsilon,t}$ and $W_{\epsilon,t}$ be, correspondingly, its starting time and the transmission rate. The following lemma shows that, when $N_{\epsilon,t} \geq 1$, the process $\{\phi(X_h(s))\ , s \in [S_0, S_1)\}$ can be, in certain sense, approximated by the step function $\{\mathcal{E}(W_{\epsilon,t}, \phi)\mathbb{1}_{[\Gamma_{\epsilon,t}, \Gamma_{\epsilon,t}+Y_{\epsilon,t})}(s)\ ,\ s \in [S_0, S_1)$. (Note that by definition, if $N_{\epsilon,t} \geq 1$, then $S_0 \leq \Gamma_{\epsilon,t} \leq \Gamma_{\epsilon,t} + Y_{\epsilon,t} \leq S_1$).

Lemma 3. *Suppose that Assumption 1 holds. Let $h \geq 0$ and ϕ be a bounded measurable function defined on $\mathcal{D}([0, h])$. Let $\eta > 0$. We have, for all $\epsilon > 0$ sufficiently small,*

(13)

$$\mathbb{P}\left(\sup_{v \in [S_0, S_1]} \left|\int_{S_0}^{v} \{\phi(X_h(s)) - \mathcal{E}(W_{\epsilon,t}, \phi)\mathbb{1}_{[\Gamma_{\epsilon,t}, \Gamma_{\epsilon,t}+Y_{\epsilon,t})}(s)\}\,\mathrm{d}s\right| > \eta a(t)\,;N_{\epsilon,t} \geq 1\right) = o(t^{-1}) .$$

Let $\mathcal{W}$ be a closed subset of $[0, \infty]$ such that $\mathbb{P}(W \in \mathcal{W}) = 1$. (Note that by (3) this implies $G(\mathcal{W}) = 1$.) We introduce the following assumption.

Assumption 2. We have

$$G\left(D(\mathcal{E}(\cdot, \phi), \mathcal{W})\right) = 0 , \tag{14}$$

where $D(\mathcal{E}(\cdot, \phi), \mathcal{W})$ denotes the set of discontinuity points of the function $\mathcal{E}(\cdot, \phi)$ restricted to $\mathcal{W} \cap [0, \infty)$, and containing the point ∞ if $\infty \in \mathcal{W}$ and $\mathcal{E}(w, \phi)$ does not converge as $w \to \infty$ with $w \in \mathcal{W}$. (The notation $\mathcal{E}(\infty, \phi)$, when used in the sequel, refers to the continuous extension of $\mathcal{E}(w, \phi)$, and will be used only when such an extension exists.)

Remark 1. If the distribution of W is supported by a closed set consisting of isolated points in $[0,\infty)$ (which would be the case, for instance, if W was a nonnegative integer-valued random variable), then $D(\mathcal{E}(\cdot,\phi),\mathcal{W})$ is either empty or equal to $\{\infty\}$. In the latter case, if $G(\{\infty\})=0$, then Assumption 2 is verified.

The next lemma, which may be of independent interest, states the multivariate regular variation property of the empirical process over a cycle.

Lemma 4. *Suppose that Assumption 1 holds. Let $h\geq 0$ and $\phi_1,\dots,\phi_d$ be bounded measurable functions defined on $\mathcal{D}([0,h])$ satisfying Assumption 2 with G defined by (2). With $\mathcal{E}(w,\phi_i)=\mathbb{E}[w+\phi_i(X_h(0))]$, $i=1,\dots,d$, $w\geq 0$, we let*

$$\mathbf{Z}=\left[\int_{S_0}^{S_1}\phi_1(X_h(s))\,\mathrm{d}s,\dots,\int_{S_0}^{S_1}\phi_d(X_h(s))\,\mathrm{d}s\right]^T .$$

Then $\mathbf{Z}$ is multivariate regularly varying with index α. More precisely, the following vague convergence holds on $[-\infty,\infty]^d\setminus\{0\}$ as $t\to\infty$,

$$t\mathbb{P}\left(\frac{\mathbf{Z}}{a(t)}\in\cdot\right)\xrightarrow{v}\mathrm{e}^{\lambda\mathbb{E}[Y]}\int_{y=0}^{\infty}\mathbb{P}\left(y[\mathcal{E}(W^*,\phi_1)\ \dots\ \mathcal{E}(W^*,\phi_d)]^T\in\cdot\right)\alpha\,y^{-\alpha-1}\,\mathrm{d}y\ , \tag{15}$$

where W^* is a random variable with values in $[0,\infty]$ and distribution G.

3. Main result

As observed in Resnick and van den Berg (2000), since the limit is discontinuous, the convergence of the sequence of processes $\{\mathcal{Z}_T(\phi,t),\ t\geq 0\}$ in Theorem 5 cannot hold in $\mathcal{D}([0,\infty))$ endowed with the topology induced by Skorohod's J_1 distance. We shall prove that the convergence holds in $\mathcal{D}([0,\infty))$ endowed with the topology induced by Skorohod's M_1 distance.

Theorem 5. *Suppose that Assumption 1 holds. Let $h\geq 0$ and ϕ be a bounded measurable function on $\mathcal{D}([0,h])$ satisfying Assumption 2 with G defined by (2). Then, as $T\to\infty$, the sequence of processes $\mathcal{Z}_T(\phi,\cdot)$ defined by*

$$\mathcal{Z}_T(\phi,u)=\frac{1}{a(T)}\int_0^{Tu}\{\phi(X_h(s))-\mathbb{E}[\phi(X_h(0))]\}\,\mathrm{d}s\,,\quad u\geq 0\,, \tag{16}$$

converges weakly in $\mathcal{D}([0,h])$ endowed with the M_1 topology to a totally skewed to the right strictly α-stable Lévy motion $\big(\Lambda(\phi,u),\ u\geq 0\big)$ satisfying

$$\mathbb{E}e^{it\Lambda(\phi,u)}=\exp\left\{-u|t|^{\alpha}\lambda\,c_\alpha\mathbb{E}\Big|\mathcal{E}(W^*,\phi)-\mathcal{E}(0,\phi)\Big|^{\alpha}\{1-\mathrm{i}\,\beta\operatorname{sgn}(t)\tan(\pi\alpha/2)\}\right\} \tag{17}$$

for $u\geq 0$ and $t\in\mathbb{R}$, where $c_\alpha=-\Gamma(1-\alpha)\cos(\pi\alpha/2)$, W^* is as in Lemma 4, and

$$\beta=\frac{\mathbb{E}\Big[\Big|\mathcal{E}(W^*,\phi)-\mathcal{E}(0,\phi)\Big|^{\alpha}\operatorname{sgn}\Big(\mathcal{E}(W^*,\phi)-\mathcal{E}(0,\phi)\Big)\Big]}{\mathbb{E}\Big|\mathcal{E}(W^*,\phi)-\mathcal{E}(0,\phi)\Big|^{\alpha}}\ .$$

Remark 2. For applications of Theorem 5 it is sometimes useful to represent the limiting Lévy motion $\big(\Lambda(\phi,u),\, u\ge 0\big)$ in the form

$$\Lambda(\phi,u)=\int_0^u\int_{\mathcal{W}}\{\mathcal{E}(w,\phi)-\mathcal{E}(0,\phi)\}\, M_\alpha(\mathrm{d}s,\mathrm{d}w)\;,\; u\ge 0\,, \tag{18}$$

where M_α is a totally skewed to the right α-stable random measure on $(0,\infty)\times\mathcal{W}$ with control measure $\lambda\, c_\alpha \mathrm{Leb}\times G$; see Samorodnitsky and Taqqu (1994). The representation (18) is linear in ϕ, and this allows, for example, handling more than one function ϕ at a time.

Specifically, if Assumption 1 holds, and $\mathcal{F}$ is a class of bounded measurable functions satisfying Assumption 2, then, by linearity, Theorem 5 implies that, for any $n\ge 2$ and bounded measurable functions $\phi_1,\dots,\phi_n$ on $\mathcal{D}([0,h])$ satisfying Assumption 2, the family of $\mathbb{R}^n$-valued processes $(\mathcal{Z}_T(\phi_1,\cdot),\,\dots,\,\mathcal{Z}_T(\phi_n,\cdot))$ converges weakly to the process $(\Lambda(\phi_1,\cdot),\,\dots,\,\Lambda(\phi_n,\cdot))$ in $\mathcal{D}([0,\infty))^n$ endowed with the M_1 topology, where the limiting process is understood as to be defined by (18). Clearly, this limiting process is an $\mathbb{R}^n$-valued α-stable Lévy motion.

For another application of (18), we can write the one-dimensional weak convergence prescribed by Theorem 5 at $u=1$ in the form

$$\mathcal{Z}_T(\phi,1)\Rightarrow\Lambda_1(\phi):=\int_{\mathcal{W}}\{\mathcal{E}(w,\phi)-\mathcal{E}(0,\phi)\}\,\tilde M_\alpha(\mathrm{d}w)\,, \tag{19}$$

where this time $\tilde M_\alpha$ is a totally skewed to the right α-stable random measure on $\mathcal{W}$ with control measure $\lambda\, c_\alpha\, G$. Again, the representation of the limit in the right hand side of (19) is linear in ϕ, allowing us to handle more than one function ϕ at a time.

4. An application: the empirical process

Suppose we want to estimate the distribution function K of $X(0)$. For this purpose we consider the family of empirical processes

$$E_T(x)=T^{-1}\int_0^T \mathbb{1}_{\{X(s)\le x\}}\,\mathrm{d}s,\;\, x>0\,.$$

Let D denote the set of discontinuity points of the distribution function K restricted to $\mathcal{W}\cap[0,\infty)$. The following is an immediate corollary of Theorem 5 and (19).

Corollary 6. *Let $\mathcal{X}$ be the collection of $x>0$ such that $G(x-D)=0$. Then*

$$\Big(T\,a(T)^{-1}\big(E_T(x)-K(x)\big),\, x\in\mathcal{X}\Big)\Rightarrow\big(D(x),\, x\in\mathcal{X}\big)$$

in the sense of convergence of the finite-dimensional distributions, where

$$D(x)=\int_{\mathcal{W}}\{K(x-w)-K(x)\}\tilde M_\alpha(\mathrm{d}w),\;\, x>0\,.$$

Remark 3. Note that the set D is at most countable, and the set of atoms of G is at most countable as well. We immediately conclude that the set $\mathcal{X}$ misses at most countably many $x>0$.

Further, if the distribution of W is supported by a closed set consisting of isolated point in $[0,\infty)$, we have $D=\emptyset$ (see Remark 1), and so $\mathcal{X}=(0,\infty)$, Finally, $X(0)$ is an infinitely divisible random variable with Lévy measure μ satisfying

$$\mu\big((a,\infty)\big) = \lambda \mathbb{E}\big(Y \mathbb{1}(W > a)\big),\ a > 0 .$$

Therefore, if W does not have positive atoms, then the distribution function K has a single atom, at the origin, implying that $D = \{0\}$ and $\mathcal{X}$ misses some of the atoms of G, specifically those atoms that are not isolated points of $\mathcal{W}$.

Observe that Corollary 6 shows that "the usual" $\sqrt{T}$-rate of convergence of an empirical process does not hold in the present situation, since the actual rate of convergence is $Ta(T)^{-1}$, which is regularly varying with index $1 - \alpha^{-1} \in (0, 1/2)$. This should not be surprising since presence of long range dependence has been known to yield slower rates of convergence of the empirical process. However we note that in our case the empirical process behaves quite differently from the empirical process for other long memory processes, see e.g. Taqqu (1979); Surgailis (2004).

5. Proofs

Proof of Lemma 1. By the definition of a and regular variation of the tail of F,

$$\bar{F}(a(t)) = \mathbb{P}(Y > a(t)) \sim t^{-1} \quad \text{as } t \to \infty;$$

recall, further, that a is regularly varying at infinity with index $1/\alpha$. We will use the notation $N_{\epsilon,t}$, $Y_{\epsilon,t}$, $\Gamma_{\epsilon,t}$ and $W_{\epsilon,t}$ introduced just before Lemma 3 above. Applying Lemma 1 in Resnick and Samorodnitsky (1999) and the regular variation of $\bar{F}$, we get

$$\lim_{t\to\infty} t\mathbb{P}(N_{\epsilon,t} \geq 1) = \lim_{t\to\infty} \frac{\mathbb{P}(N_{\epsilon,t} \geq 1)}{\bar{F}(\epsilon a(t))} \frac{\bar{F}(\epsilon a(t))}{\bar{F}(a(t)} = \mathrm{e}^{\lambda \mathbb{E}[Y]} \epsilon^{-\alpha} . \tag{20}$$

Imagine, for a moment, that all sessions of the length exceeding $\epsilon a(t)$ are discarded upon arrival, and do not contribute to a busy period. Let $B_{\epsilon,t}$ denote the length of the first busy period starting at or after time S_0 and generated by the remaining sessions, those of length not exceeding $\epsilon a(t)$. Then by (Resnick and Samorodnitsky, 1999, Proposition 1), there exists a constant D independent of ϵ such that

$$\mathbb{P}\left(B_{\epsilon,t} > \epsilon D a(t)\right) = o(t^{-1}) . \tag{21}$$

We immediately conclude that

$$\lim_{t\to\infty} t\mathbb{P}(C_1 > \epsilon D a(t)\,; N_{\epsilon,t} = 0) = 0 \tag{22}$$

(keeping in mind that an idle period has an exponential distribution).

We consider now the case $N_{\epsilon,t} \geq 1$, in which case we use the decomposition

$$C_1 = \{\Gamma_{\epsilon,t} - S_0\} + Y_{\epsilon,t} + \{S_1 - (\Gamma_{\epsilon,t} + Y_{\epsilon,t})\} . \tag{23}$$

By the definition of $B_{\epsilon,t}$, we have, on $\{N_{\epsilon,t} \geq 1\}$,

$$\Gamma_{\epsilon,t} - S_0 \leq B_{\epsilon,t} .$$

Hence, by (21), for any $\eta > 0$, choosing $\epsilon > 0$ sufficiently small, we have

$$\mathbb{P}\left(\Gamma_{\epsilon,t} - S_0 > a(t)\eta; N_{\epsilon,t} \geq 1\right) = o(t^{-1}) \quad \text{as } t \to \infty . \tag{24}$$

Further, denote by $\tilde{\Gamma}_{\epsilon,t}$ the completion time of the last session with length greater than $\epsilon a(t)$ before time S_1. Notice that the infinite source Poisson process (1) is time reversible, in the sense of switching the direction of time, declaring $\Gamma_\ell + Y_\ell$ to be the arrival time of session number ℓ and Γ_ℓ to be its completion time. Therefore, by time inversion, the difference $S_1 - \tilde\Gamma_{\epsilon,t}$ has the same exponential distribution with the mean $(\lambda\bar F(\epsilon a(t)))^{-1}$ as $\Gamma_{\epsilon,t} - S_0 + I_0$, where I_0 denotes the idle period preceding S_0. Further, on the event $\{N_{\epsilon,t} = 1\}$, the random variables $\Gamma_{\epsilon,t} + Y_{\epsilon,t}$ and $\tilde\Gamma_{\epsilon,t}$ coincide. We conclude that, for all $\eta, \epsilon > 0$,

$$\mathbb{P}\left(S_1 - (\Gamma_{\epsilon,t} + Y_{\epsilon,t})) > a(t)\eta \; ; N_{\epsilon,t} = 1\right) = \mathbb{P}\left(S_1 - \tilde\Gamma_{\epsilon,t} > a(t)\eta ; N_{\epsilon,t} = 1\right) \tag{25}$$

$$\le \mathbb{P}\left(\Gamma_{\epsilon,t} - S_0 + I_0 > a(t)\eta\right) = o(t^{-1}) \quad \text{as } t \to \infty \; . \tag{26}$$

Next, by Lemma 2 in Resnick and Samorodnitsky (1999), we also have

$$\mathbb{P}(N_{\epsilon,t} \ge 2) = o(t^{-1}) \quad \text{as } t \to \infty \; . \tag{27}$$

Applying (22), (23), (24), (25) and (27), we get, for any $x > \eta > 0$, choosing ϵ small enough,

$$\begin{aligned}\liminf_{t\to\infty} t\,\mathbb{P}(Y_{\epsilon,t} > a(n)x \; ; \; N_{\epsilon,t} \ge 1) &\le \liminf_{t\to\infty} t\,\mathbb{P}(C_1 > a(t)x) \\ &\le \limsup_{t\to\infty} t\,\mathbb{P}(C_1 > a(t)x) \le \limsup_{t\to\infty} t\,\mathbb{P}(Y_{\epsilon,t} > a(n)(x-\eta) \; ; \; N_{\epsilon,t} \ge 1) \; .\end{aligned} \tag{28}$$

Note that the distribution of $Y_{\epsilon,t}$ is the conditional distribution of Y given $\{Y > \epsilon a(t)\}$. and that the event $\{N_{\epsilon,t} \ge 1\}$ is independent of $Y_{\epsilon,t}$, so that (20) yields, for any $x > 0$,

$$t\,\mathbb{P}(Y_{\epsilon,t} > a(n)x \; ; \; N_{\epsilon,t} \ge 1) \sim \mathrm{e}^{\lambda\mathbb{E}[Y]}\epsilon^{-\alpha}\mathbb{P}(Y > a(t)x \mid Y > \epsilon a(t)) \to \mathrm{e}^{\lambda\mathbb{E}[Y]}x^{-\alpha}$$

as $t \to \infty$. Applying this statement to (28) and letting $\eta \to 0$ gives (7). □

Proof of Lemma 2. Observe that the process $\{X(t), t \in \mathbb{R}\}$ is a regenerative process (it regenerates at the beginning of each busy period), hence it is ergodic. Therefore, $T^{-1}\mathcal{J}_T(\phi) \to \mathcal{E}(0,\phi)$ a.s.; see e.g. Resnick (1992). On the other hand, as seen earlier, the sequence $\big(Z_j(\phi)\big)$ is strongly mixing, hence also ergodic, and so $n^{-1}\sum_{j=1}^n Z_j(\phi)$ converges almost surely to $\mathbb{E}[Z_1(\phi)]$. Since M_T/T converges almost surely to $1/\mathbb{E}[C_1]$, we also obtain

$$\frac{1}{T}\sum_{j=1}^{M_T} Z_j(\phi) \to \mathbb{E}[Z_1(\phi)]/\mathbb{E}[C_1] \; , \quad \text{a.s.},$$

and (11) follows.

Denote $\bar\phi = \phi - \mathcal{E}(0,\phi)$. Observe that $\mathcal{J}_T(\bar\phi)$ is centered and $\|\bar\phi\|_\infty \le \|\phi\|_\infty + |\mathcal{E}(0,\phi)| \le 2\|\phi\|_\infty$. We have

$$\sup_{t\in[0,S_0]} \left|\mathcal{J}_t(\bar\phi)\right| \le S_0\|\bar\phi\|_\infty \; . \tag{29}$$

For $t \ge S_0$ we use the decomposition

$$\mathcal{J}_t(\bar\phi) = \mathcal{J}_{S_0}(\bar\phi) + \sum_{j=1}^{M_t} Z_j(\bar\phi) + \int_{S_{M_t}}^t \bar\phi(X_h(s))\mathrm{d}s \; .$$

Now, using $\|\bar\phi\|_\infty \le 2\|\phi\|_\infty$, (29) and that, for all $k = 1, \dots, M_T + 1$,

$$\sup_{u\in[S_{k-1},S_k]} \left|\int_{S_{k-1}}^u \bar\phi(X_h(s))\mathrm{d}s\right| \le \|\bar\phi\|_\infty \, C_k \; ,$$

we get, for any $T > 0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T]} \left|\mathcal{J}_t(\bar\phi)\right| > 5x\|\phi\|_\infty\right) &\le \mathbb{P}(S_0 > x) + \mathbb{P}\left(\sup_{t\in[0,T]}\left|\sum_{j=1}^{M_t} Z_j(\bar\phi)\right| > x\|\phi\|_\infty\right) \\ &\quad + \mathbb{P}\left(\max_{k=1,\dots,M_T+1} C_k > x\right) \\ &\le \mathbb{P}(S_0 > x) + 2\mathbb{P}(M_T > 2T/\mathbb{E}[C_1]) \\ &\quad + \mathbb{P}\left(\max_{1\le k\le 2T/\mathbb{E}[C_1]}\left|\sum_{j=1}^{k} Z_j(\bar\phi)\right| > x\|\phi\|_\infty\right) \\ &\quad + (2T/\mathbb{E}[C_1]+1)\mathbb{P}\left(C_1 > x\right) \; .\end{aligned}$$

Applying (11), we see that $Z_j(\bar\phi)$ is centered. Moreover $|Z_j(\bar\phi)| \le 2C_j\|\phi\|_\infty$. Let $p \in (1,\alpha)$. Applying the mixing property (9), Lemma 1 and Rio (2000, Chapitre 3, Exercice 1), there exists a constant c which depends only on the distribution of C_1 and p such that

$$\mathbb{E}\left[\max_{1\le k\le n}\left|\sum_{j=1}^{k} Z_j(\bar\phi)\right|^p\right] \le c\|\phi\|_\infty^p n \; . \tag{30}$$

Finally we bound $\mathbb{P}(M_T > 2T/\mathbb{E}[C_1])$ by noting as usual that $M_T > n$ if and only if $S_{n+1} \le T$. Thus, denoting by m the smallest integer larger than or equal to $2T/\mathbb{E}[C_1]$, we have, for some constant c only depending on the distribution of C_1 and p,

$$\mathbb{P}(M_T > 2T/\mathbb{E}[C_1]) \le \mathbb{P}(S_m \le T) \le \mathbb{P}(S_m - m\mathbb{E}[C_1] \le -T) \le cT^{1-p} \; , \tag{31}$$

where the last inequality follows from the Markov and Burkhölder inequalities. Gathering the three previous displays and using $\mathbb{P}\left(C_1 > x\right) \le \mathbb{E}[C_1^p]x^{-p}$ for any $p < \alpha$, we obtain (12) with $g(x) = P(S_0 > x)$. □

Proof of Lemma 3. We will bound the function

$$\Delta(v) = \int_{S_0}^{v} \{\phi(X_h(s)) - \mathcal{E}(W_{\epsilon,t},\phi)\mathbb{1}_{[\Gamma_{\epsilon,t},\Gamma_{\epsilon,t}+Y_{\epsilon,t})}(s)\}\,\mathrm{d}s$$

on the event $\{N_{\epsilon,t} \ge 1\}$ successively for $v \in [S_0, \Gamma_{\epsilon,t}]$, $v \in [\Gamma_{\epsilon,t}, \Gamma_{\epsilon,t}+Y_{\epsilon,t}]$ and $v \in [\Gamma_{\epsilon,t}+Y_{\epsilon,t}, S_1]$.

Step 1 For $v \in [S_0, \Gamma_{\epsilon,t}]$ we have

$$|\Delta(v)| = \left|\int_{S_0}^{v} \phi(X_h(s))\mathrm{d}s\right| \le (\Gamma_{\epsilon,t} - S_0)\|\phi\|_\infty \; .$$

Hence, using (24), for any $\eta > 0$, choosing $\epsilon > 0$ sufficiently small, we have

$$\mathbb{P}\left(\sup_{v\in[S_0,\Gamma_{\epsilon,t}]} |\Delta(v)| > a(t)\eta ; N_{\epsilon,t} \ge 1\right) = o(t^{-1}) \, . \tag{32}$$

Step 2 For $v \in [\Gamma_{\epsilon,t}, \Gamma_{\epsilon,t} + Y_{\epsilon,t}]$, we write

$$\begin{aligned} |\Delta(v)| &\le |\Delta(\Gamma_{\epsilon,t})| + |\Delta(v) - \Delta(\Gamma_{\epsilon,t})| \\ &\le \sup_{v\in[S_0,\Gamma_{\epsilon,t}]} |\Delta(v)| + \sup_{y\in[0,Y_{\epsilon,t}]} \left| \int_0^y \{\phi(X_h(\Gamma_{\epsilon,t}+s)) - \mathcal{E}(W_{\epsilon,t},\phi)\}\,\mathrm{d}s \right| . \end{aligned} \tag{33}$$

For $s > 0$, $X(\Gamma_{\epsilon,t} + s)$ can be expressed as

$$X(\Gamma_{\epsilon,t} + s) = W_{\epsilon,t} + \check{X}(s) + R(s) ,$$

where $R(s)$ is the sum of all transmission rates of the sessions that started before time $\Gamma_{\epsilon,t}$ and are still active at time s, and $\{\check{X}(s),\ s \ge 0\}$ is defined by

$$\check{X}(s) = \sum_{\ell\in\mathbb{Z}} W_\ell \mathbb{1}_{\{\Gamma_{\epsilon,t}<\Gamma_\ell\le s+\Gamma_{\epsilon,t}<\Gamma_\ell+Y_\ell\}} .$$

Since each session that arrives after time S_0 but before time $\Gamma_{\epsilon,t}$ has a length not exceeding $\epsilon a(t)$, we conclude that $R(s) = 0$ for $s > \epsilon a(t)$. Using the notation $\check{X}_h(s) = \{\check{X}(s+v)\,, 0 \le v \le h\}$, we, therefore, obtain

$$\sup_{y\in[0,Y_{\epsilon,t}]} \left| \int_0^y \{\phi(X_h(\Gamma_{\epsilon,t}+s)) - \phi(W_{\epsilon,t} + \check{X}_h(s))\}\,\mathrm{d}s \right| \le 2\|\phi\|_\infty \epsilon a(t) . \tag{34}$$

Observe that the process $\check{X}$ is independent of $(Y_{\epsilon,t}, W_{\epsilon,t}, \mathbb{1}_{N_{\epsilon,t})\ge 1})$. We preserve this independence while transforming $\check{X}$ into a stationary process, with the same law as the original process X in (1) by defining

$$\hat{X}(s) = \sum_{\ell\le 0} W'_\ell \mathbb{1}_{\{\Gamma'_\ell\le s<\Gamma'_\ell+Y'_\ell\}} + \check{X}(s) , \quad s \in \mathbb{R} ,$$

where $\{(\Gamma'_\ell, Y'_\ell, W'_\ell),\ \ell \in \mathbb{Z}\}$ is an independent copy of $\{(\Gamma_\ell, Y_\ell, W_\ell),\ \ell \in \mathbb{Z}\}$. Clearly,

$$\sup_{y\in[0,Y_{\epsilon,t}]} \left| \int_0^y \{\phi(W_{\epsilon,t} + \check{X}_h(s)) - \phi(W_{\epsilon,t} + \hat{X}_h(s))\}\,\mathrm{d}s \right| \le 2\|\phi\|_\infty \sup_{\ell\le 0}(\Gamma'_\ell + Y'_\ell) ,$$

where $\hat{X}_h(s) = \{\hat{X}(s+v)\,, 0 \le v \le h\}$. The random variable in the right hand side above is finite with probability 1 and independent of $N_{\epsilon,t}$. Therefore, it follows from (20) that for any $u > 0$,

$$\mathbb{P}\left(\sup_{\ell\le 0}(\Gamma'_\ell + Y'_\ell) > a(t)u \,;\ N_{\epsilon,t} \ge 1 \right) = o(t^{-1}) .$$

The last two displays and (34) give that, for any $\eta > 0$ and $0 < \epsilon < \eta/(2\|\phi\|_\infty)$,

$$\mathbb{P}\left(\sup_{y\in[0,Y_{\epsilon,t}]} \left| \int_0^y \{\phi(X_h(\Gamma_{\epsilon,t}+s)) - \phi(W_{\epsilon,t} + \hat{X}_h(s))\}\,\mathrm{d}s \right| > a(t)\eta; N_{\epsilon,t} \ge 1 \right) = o(t^{-1}) . \tag{35}$$

The event $\{N_{\epsilon,t} \ge 1\}$ is, clearly, independent of $(Y_{\epsilon,t}, W_{\epsilon,t})$. Furthermore, the latter pair has the conditional distribution of (Y, W) given that $\{Y > \epsilon a(t)\}$. Since $\hat{X}$ has the same law as X, we get for any $x > 0$,

$$\mathbb{P}\left(\sup_{y\in[0,Y_{\epsilon,t}]}\left|\int_0^y \left\{\phi(W_{\epsilon,t}+\hat{X}_h(s))-\mathcal{E}(W_{\epsilon,t},\phi)\right\}\mathrm{d}s\right|>x\ ;\ N_{\epsilon,t}\geq 1\right) \tag{36}$$
$$=\mathbb{P}\left(\sup_{y\in[0,Y]}\left|\int_0^y \{\phi(W+X_h(s))-\mathcal{E}(W,\phi)\}\,\mathrm{d}s\right|>x \mid Y>\epsilon a(t)\right)\times\mathbb{P}(N_{\epsilon,t}\geq 1)\ ,$$

where the pair (Y,W) in the right hand side is taken to be independent of the process X.

Recall that $\mathcal{E}(w,\phi)=\mathbb{E}[\phi(w+X_h(0))]$, that for any $w\geq 0$, $\|\phi(w+\cdot)\|_\infty\leq\|\phi\|_\infty$ and, for any $y\geq 0$, $\mathbb{E}[\mathcal{J}_y(\phi(w+\cdot))]=y\mathcal{E}(w,\phi)$. It follows from these observations and (12) in Lemma 2 that, for any $x>0$,

$$\sup_{w\geq 0}\mathbb{P}\left(\sup_{y\in[0,u]}\left|\int_0^y\{\phi(w+X_h(s))-\mathcal{E}(w,\phi)\}\,\mathrm{d}s\right|>x\|\phi\|_\infty\right)\leq C\,u^{1-p}+C\,u\,x^{-p}+g(x)\ ,$$

for $p\in(1,\alpha)$, some constant $C>0$ and $g(x)\to 0$ as $x\to\infty$. Integrating in (w,u) with respect to the distribution of (W,Y) in (36), this bound yields, for any $u>0$ and $A>0$,

$$\mathbb{P}\left(\sup_{y\in[0,Y]}\left|\int_0^y\{\phi(W+X_h(s))-\mathcal{E}(W,\phi)\}\,\mathrm{d}s\right|>uA \mid Y>A\right)$$
$$\leq C\,\mathbb{E}[Y^{1-p}\mid Y>A]+C\,\|\phi\|_\infty^p\,(uA)^{-p}\mathbb{E}[Y\mid Y>A]+g(uA/\|\phi\|_\infty)\ .$$

As $A\to\infty$, we have both $\mathbb{E}[Y^{1-p}\mid Y>A]\to 0$ and $\mathbb{E}[Y\mid Y>A]\to 0$ since Y has a regularly varying tail with $\alpha>1$. Since $p>1$, we see that the 3 terms in the previous bound converge to 0 as $A\to\infty$. This, together with (36) and (20), yields that, for any $\epsilon>0$ and $\eta>0$,

$$\mathbb{P}\left(\sup_{y\in[0,Y_{\epsilon,t}]}\left|\int_0^y\left\{\phi(W_{\epsilon,t}+\hat{X}_h(s))-\mathcal{E}(W_{\epsilon,t},\phi)\right\}\mathrm{d}s\right|>a(t)\eta\ ;\ N_{\epsilon,t}\geq 1\right)=o(t^{-1})\ .$$

Finally, gathering the last display, (35), (33) and (32), we obtain

$$\mathbb{P}\left(\sup_{v\in[\Gamma_{\epsilon,t},\Gamma_{\epsilon,t}+Y_{\epsilon,t}]}|\Delta(v)|>a(t)\eta; N_{\epsilon,t}\geq 1\right)=o(t^{-1})\,. \tag{37}$$

Step 3 If $v\in[\Gamma_{\epsilon,t}+Y_{\epsilon,t},S_1]$, we have on $\{N_{\epsilon,t}\geq 1\}$,

$$|\Delta(v)|\leq|\Delta(\Gamma_{\epsilon,t}+Y_{\epsilon,t})|+\left|\int_{\Gamma_{\epsilon,t}+Y_{\epsilon,t}}^v\phi(X_h(s))\mathrm{d}s\right|$$
$$\leq\sup_{v\in[\Gamma_{\epsilon,t},\Gamma_{\epsilon,t}+Y_{\epsilon,t}]}|\Delta(v)|+\{S_1-(\Gamma_{\epsilon,t}+Y_{\epsilon,t})\}\,\|\phi\|_\infty\ . \tag{38}$$

Using (37) (38), (25) and (27), for any $\eta>0$, we have

$$\mathbb{P}\left(\sup_{v\in[\Gamma_{\epsilon,t}+Y_{\epsilon,t},S_1]}|\Delta(v)|>a(t)\eta; N_{\epsilon,t}\geq 1\right)=o(t^{-1})\,. \tag{39}$$

□

Proof of Lemma 4. Let f a Lipschitz function with compact support in $[-\infty,\infty]^d\setminus\{0\}$, and let L be its Lipschitz constant. Let $c>0$ be small enough such that the support of f does not intersect $[-2c,2c]^d$.

Using the fact that $Z_1(\phi_i)| \le \|\phi_i\|_\infty C_1$ for each $i=1,\dots,d$, the bound (22) implies that, as $t\to\infty$,

$$\mathbb{P}\left(|Z_1(\phi_i)| > ca(t) \text{ for some } i=1,\dots,d\,;\, N_{\epsilon,t}=0\right) = o(t^{-1})$$

as long as $\epsilon>0$ is small enough relatively to c. We will show that

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty} t\mathbb{E}\left[f(\mathbf{Z}/a(t))\,;\,N_{\epsilon,t}\ge 1\right] = \lim_{\epsilon\to 0}\liminf_{t\to\infty} t\mathbb{E}\left[f(\mathbf{Z}/a(t))\,;\,N_{\epsilon,t}\ge 1\right] \tag{40}$$
$$= \mathrm{e}^{\lambda\mathbb{E}[Y]}\int_0^\infty \mathbb{E}\left[f(y[\mathcal{E}(W^*,\phi_1)\ \dots\ \mathcal{E}(W^*,\phi_1)]^T)\right]\alpha\, y^{-\alpha-1}\,\mathrm{d}y\,;$$

this will prove the required vague convergence in (15).

Write

$$t\mathbb{E}\left[f(\mathbf{Z}/a(t))\,;\,N_{\epsilon,t}\ge 1\right] = t\mathbb{E}\left[f(\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})/a(t))\,;\,N_{\epsilon,t}\ge 1\right] + t\mathbb{E}\left[\{f(\mathbf{Z}/a(t)) - f(\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})/a(t))\}\,;\,N_{\epsilon,t}\ge 1\right]\ , \tag{41}$$

where $\mathbf{\Phi}(y,w) = y\big[\mathcal{E}(w,\phi_1),\dots,\mathcal{E}(w,\phi_d)\big]^T$. Choose $0<\eta<c$ and observe that, on the the event $\cap_i\{|Z_1(\phi_i) - \mathcal{E}(W_{\epsilon,t},\phi_i)Y_{\epsilon,t}| \le \eta a(t)\}$,

$$|f(\mathbf{Z}/a(t)) - f(\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})/a(t))| \le L\eta\,\mathbb{1}\big(|\mathcal{E}(W_{\epsilon,t},\phi_i)Y_{\epsilon,t}| > \eta a(t) \text{ for some } i=1,\dots,d\big)\ .$$

Letting g be a continuous function on $[-\infty,\infty]^d$ such that $g(x)=1$ for all $x\notin[-c,c]^d$ and $g(x)=0$ in a neighborhood of the origin, we obtain

$$t\mathbb{E}\left[|f(\mathbf{Z}/a(t)) - f(\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})/a(t))|\ ;\,N_{\epsilon,t}\ge 1\right] \le L\eta\ t\mathbb{E}\left[g(\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})/a(t))\,;\,N_{\epsilon,t}\ge 1\right] +$$
$$2\|f\|_\infty\sum_{i=1}^d t\,\mathbb{P}\left(|Z_1(\phi_i) - \mathcal{E}(W_{\epsilon,t},\phi_i)Y_{\epsilon,t}| > \eta a(t)\,;\,N_{\epsilon,t}\ge 1\right)\ .$$

Recall that by Lemma 3,

$$\lim_{t\to\infty} t\mathbb{P}\left(|Z_1(\phi_i) - \mathcal{E}(W_{\epsilon,t},\phi_i)Y_{\epsilon,t}| > \eta a(t)\,;\,N_{\epsilon,t}\ge 1\right) = 0$$

for all $\epsilon>0$ small enough (relative to η). Therefore, for each $\eta>0$ and $\epsilon>0$ small enough,

$$\limsup_{t\to\infty}\left|t\mathbb{E}\left[f(\mathbf{Z}/a(t))\,;\,N_{\epsilon,t}\ge 1\right] - t\mathbb{E}\left[f(\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})/a(t))\,;\,N_{\epsilon,t}\ge 1\right]\right|$$
$$\le L\eta\limsup_{t\to\infty} t\mathbb{E}\left[g(\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})/a(t))\,;\,N_{\epsilon,t}\ge 1\right]\ .$$

We will prove below that for any $\epsilon>0$,

$$t\mathbb{P}\left(\frac{\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})}{a(t)}\in\cdot\,;\,N_{\epsilon,t}\ge 1\right)\xrightarrow{v}\mathrm{e}^{\lambda\mathbb{E}[Y]}\left(\nu_{\alpha;\epsilon}\times G\right)\circ\mathbf{\Phi}^{-1}(\cdot)\ , \tag{42}$$

where the measure $\nu_{\alpha;\epsilon}$ on $(0,\infty)$ is the restriction of the measure ν_α in (2) to (ϵ,∞), i.e. $\nu_{\alpha;\epsilon}(x,\infty) = \min\big(x^{-\alpha},\epsilon^{-\alpha}\big)$, $x>0$. Assuming this has been proved, it will follow that

$$\limsup_{t\to\infty}\left|t\,\mathbb{E}\left[\{f(\mathbf{Z}/a(t)) - f(\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})/a(t))\}\,;\,N_{\epsilon,t}\ge 1\right]\right| \le CL\eta\int g\circ\mathbf{\Phi}\,\mathrm{d}(\nu_\alpha\times G) \tag{43}$$

for some finite positive constant C independent of η and ϵ. Note that the last integral is finite. Similarly, (42) will imply that

(44)

$$\lim_{t\to\infty} t\,\mathbb{E}\left[f(\mathbf{\Phi}(Y_{\epsilon,t},W_{\epsilon,t})/a(t))\,;\,N_{\epsilon,t}\ge 1\right] = \mathrm{e}^{\lambda\mathbb{E}[Y]}\int f\circ\mathbf{\Phi}\,\mathrm{d}(\nu_{\alpha;\epsilon}\times G) = \mathrm{e}^{\lambda\mathbb{E}[Y]}\int f\circ\mathbf{\Phi}\,\mathrm{d}(\nu_\alpha\times G)$$

for all $0<\epsilon<c/\max\|\phi\|_\infty$. We combine (41), (43) and (44) by keeping η fixed and letting $\epsilon\to 0$. This shows that

$$-C\,L\eta\int g\circ\mathbf{\Phi}\,\mathrm{d}(\nu_\alpha\times G)+\mathrm{e}^{\lambda\mathbb{E}[Y]}\int f\circ\mathbf{\Phi}\,\mathrm{d}(\nu_\alpha\times G)\le\lim_{\epsilon\to0}\liminf_{t\to\infty}t\mathbb{E}\left[f(\mathbf{Z}/a(t))\,;\,N_{\epsilon,t}\ge1\right]$$

$$\le\lim_{\epsilon\to0}\limsup_{t\to\infty}t\mathbb{E}\left[f(\mathbf{Z}/a(t))\,;\,N_{\epsilon,t}\ge1\right]\le C\,L\eta\int g\circ\mathbf{\Phi}\,\mathrm{d}(\nu_\alpha\times G)+\mathrm{e}^{\lambda\mathbb{E}[Y]}\int f\circ\mathbf{\Phi}\,\mathrm{d}(\nu_\alpha\times G)\,,$$

and (40) follows by letting $\eta\to0$.

It remains to prove (42). holds. Since the event $\{N_{\epsilon,t}\ge1\}$ is independent of $(Y_{\epsilon,t},W_{\epsilon,t})$, whose distribution is the conditional distribution of (Y,W) given that $\{Y>\epsilon a(t)\}$, we have, as $t\to\infty$,

$$\begin{aligned}t\mathbb{P}\left(\mathbf{\Phi}(Y_{\epsilon,t}/a(t),W_{\epsilon,t})\in\cdot\,;\,N_{\epsilon,t}\ge1\right)&=t\,\mathbb{P}\left(N_{\epsilon,t}\ge1\right)\times\mathbb{P}\left(\mathbf{\Phi}(Y/a(t),W)\in\cdot\mid Y>\epsilon a(t)\right)\\&\sim\mathrm{e}^{\lambda\mathbb{E}[Y]}\epsilon^{-\alpha}\,\mathbb{P}\left(\mathbf{\Phi}(Y/a(t),W)\in\cdot\mid Y>\epsilon a(t)\right)\ ,\end{aligned}$$

by (20). Further, by Assumption 1 (iii),

$$\mathbb{P}\left((Y/a(t),W)\in\cdot\mid Y>\epsilon a(t)\right)\xrightarrow{v}\epsilon^\alpha\,\nu_{\alpha;\epsilon}\times G\ .$$

We extend $\mathbf{\Phi}$ to $(0,\infty)\times[0,\infty]$ by

$$\mathbf{\Phi}(y,\infty)=\lim_{w\to\infty}\mathbf{\Phi}(y,w)\ ,$$

when the limit exists, or by defining the value at infinity to be equal to 0 otherwise. Then the set of discontinuities of $\mathbf{\Phi}$ in $(0,\infty]\times\mathcal{W}$ is included in

$$(0,\infty)\times\bigcup_{i=1,\dots,d}D(\mathcal{E}(\cdot,\phi_i),\mathcal{W})\ ,$$

which has $\nu_{\alpha;\epsilon}\times G$-measure zero by (2), since each function ϕ_i satisfies Assumption 2. Now, since $\mathbf{\Phi}(y,w)/a(t)=\mathbf{\Phi}(y/a(t),w)$, (42) follows from the continuous mapping theorem. □

Proof of Theorem 5. In order to prove convergence in $\mathcal{D}([0,\infty))$ it is to prove convergence in $\mathcal{D}([0,a])$ for any $a>0$. For notational simplicity we present the argument for $a=1$.

For any bounded interval $[a,b]$ and real-valued functions x_1 and x_2 in $\mathcal{D}([a,b])$ we denote by $d_{M_1}(x_1,x_2,[a,b])$ the M_1 distance between x_1 and x_2 on $[a,b]$, and we write $d_{M_1}(x_1,x_2)$ if $[a,b]=[0,1]$. We refer the reader to Whitt (2002) for details on the M_1 and J_1 Skorohod topologies we use below.

To simplify the notation we assume that $\mathbb{E}[\phi(X_h(0))] = 0$, i.e. that $\phi = \bar{\phi}$. Define the following processes:

$$\mathcal{S}_T(u) = \frac{1}{a(T)} \sum_{j=1}^{[Tu]} Z_j(\phi) \;, \quad \xi_T(u) = \frac{1}{a(T)} (M_{Tu} - Tu/\mathbb{E}[C_1]) \;,$$

$$\tilde{\mathcal{S}}_T(u) = \mathcal{S}_T(M_{Tu}/T) = \frac{1}{a(T)} \sum_{j=1}^{M_{Tu}} Z_j(\phi) \;,$$

$$R_{0,T} = \frac{1}{a(T)} \int_0^{S_0} \phi(X_h(s))\,\mathrm{d}s \;, \quad R_T(u) = \frac{1}{a(T)} \int_{S_{M_{Tu}}}^{Tu} \phi(X_h(s))\,\mathrm{d}s$$

(recall the convention that, if $u < S_0$, then $M_t = 0$ and, hence, $\tilde{\mathcal{S}}_T(u) = 0$). Then

$$\mathcal{Z}_T(\phi, u) = R_{0,T} + \tilde{\mathcal{S}}_T(u) + R_T(u) \;.$$

We proceed through a sequence of steps. Specifically, we will prove that, as $T \to \infty$,

(i) S_T converges weakly in $\mathcal{D}([0,\infty))$ endowed with the J_1 topology to the Lévy α-stable process $(\mathbb{E}[C_1])^{1/\alpha}\Lambda(\phi,\cdot)$, where Λ is defined by (17);
(ii) ξ_T converges weakly in $\mathcal{D}([0,\infty))$ endowed with the M_1 topology to an α-stable Lévy process;
(iii) $\tilde{\mathcal{S}}_T$ converges weakly in $\mathcal{D}([0,\infty))$ endowed with the J_1 topology to the Lévy α-stable process $\Lambda(\phi,\cdot)$;
(iv) $d_{M_1}(\tilde{\mathcal{S}}_T, \mathcal{Z}_T) \to 0$ in probability.

The statement of the theorem will follow from statements (iii) and (iv), by appealing to Theorem 12.7.3 in Whitt (2002). It is interesting that the statement (iv) above holds even though R_T cannot converge to zero in either of the Skorohod topologies, since otherwise it would, as a family of continuous processes, converge to zero uniformly, and this would imply that $\mathcal{Z}_T$ converges weakly also in the J_1 topology, which is not possible since its limit is discontinuous.

We now prove (i). In the case $h = 0$, the random variables $Z_j(\phi)$ are centered and their tail behavior is given by Lemma 4. The weak convergence in the space $\mathcal{D}$ endowed with the J_1 topology of the normalized partial sum process $\mathcal{S}_T$ to the α-stable Lévy process $(\mathbb{E}[C_1])^{1/\alpha}\Lambda(\phi,\cdot)$ is well know in this case; see e.g. Resnick (2007, Corollary 7.1). We reduce the case $h > 0$ to the previous situation as follows. For $j \geq 1$, we write $Z_j(\phi) = Z_{1,j} + Z_{2,j}$ with

$$Z_{1,j} = \int_{S_{j-1}}^{(S_j - h)\vee S_{j-1}} \phi(X_h(s))\,\mathrm{d}s - \mathbb{E}\left[\int_{S_{j-1}}^{(S_j - h)\vee S_{j-1}} \phi(X_h(s))\,\mathrm{d}s\right] \;.$$

Observe that the sequence $\{Z_{1,j}\}$ is i.i.d. and centered, while the sequence $\{Z_{2,j}\}$ is centered and exponentially α-mixing by (9)). Furthermore, $|Z_{2,j}| \leq 2\|\phi\|_\infty h$. Therefore, by the maximal inequality for mixing sequences Rio (2000, Theorem 3.1), we obtain

$$\mathbb{E}\left[\max_{1\leq k\leq n} \left|\frac{1}{a(n)} \sum_{j=1}^{k} Z_{2,j}\right|^2\right] = O(n a_n^{-2}) = o(1) \;.$$

This implies that the family of processes $a(n)^{-1}\sum_{j=1}^{[n\cdot]} Z_{2,j}$ converges weakly to 0 uniformly on compact sets. Since the random variables $Z_{2,j}$ are uniformly bounded, $Z_{1,j}$ has the same tail as Z_j. Thus, as in the case $h = 0$, the family of processes $a(n)^{-1}\sum_{j=1}^{[n\cdot]} Z_{1,j}$ converges weakly in the space $\mathcal{D}$ endowed with the J_1 topology to the α-stable Lévy process $(\mathbb{E}[C_1])^{1/\alpha}\Lambda(\phi,\cdot)$. This proves (i).

By Lemma 1, M_t is the counting process associated with a renewal process whose interarrival times C_j are in the domain of attraction of a stable law with index α. More specifically, by Lemmas 1 and 4, the tails of C_1 and $Z_1(\phi)$ are equivalent. Now (ii) follows from (Whitt, 2002, Theorem 4.5.3 and Theorem 6.3.1)).

We now prove (iii) by the J_1-continuity of composition argument. Observe that $\tilde{\mathcal{S}}_T = \mathcal{S}_T \circ [M_{T\cdot}/T]$. Moreover, $M_{Tu}/T = a(T)\xi_T(u)/T + u/\mathbb{E}[C_1]$ for all $u \geq 0$. Since the supremum functional is continuous in the M_1 topology and $a(T)/T \to 0$, we can use (ii) to see that $M_{T\cdot}/T$ converges in the uniform topology on compact intervals to the linear function $\cdot/\mathbb{E}[C_1]$ in probability. By (i) and Theorem 4.4 in Billingsley (1968) we conclude that $(\mathcal{S}_T, M_{T\cdot}/T)$ converges weakly to $\big((\mathbb{E}[C_1])^{1/\alpha}\Lambda(\phi,\cdot), \cdot/\mathbb{E}[C_1]\big)$ in the product space $\mathcal{D}([0,\infty)) \times \mathcal{D}([0,\infty))$, where each of the components is endowed with the J_1 topology on compact intervals. Since the linear function is continuous and strictly increasing, we can use Theorem 13.2.2 in Whitt (2002)) to conclude that $\tilde{\mathcal{S}}_T$ converges weakly to $\big(\mathbb{E}[C_1]\big)^{1/\alpha}\Lambda(\phi, \cdot/\mathbb{E}[C_1]\big)$ in $D([0,\infty))$ endowed with the J_1 topology. By the self-similarity of centered Lévy stable motions, the latter process has the same law as $\Lambda(\phi,\cdot)$. This gives (iii).

It remains to prove (iv). Define the process $\tilde{\mathcal{Z}}_T$ by

$$\tilde{\mathcal{Z}}_T(t) = \mathcal{Z}_T(t) - \mathcal{Z}_T(S_0/T) = a(T)^{-1}\int_{S_0}^{Tt} \phi(X_h(s))\,\mathrm{d}s \,.$$

Then

$$\|\tilde{\mathcal{Z}}_T - \mathcal{Z}_T\|_\infty = \left|\frac{1}{a(T)}\int_0^{S_0} \phi(X_h(s))\right| \leq \frac{\|\phi\|_\infty\, S_0}{a(T)} = o_P(1) \,.$$

Since $\tilde{\mathcal{S}}_T(t) = 0$ for all $t \in [0, S_0/T]$, we also have

$$\sup_{t\in[0,S_0/T]} \left|\tilde{\mathcal{Z}}_T(t) - \tilde{\mathcal{S}}_T(t)\right| \leq \frac{\|\phi\|_\infty\, S_0}{a(T)} \,.$$

Next, we partition the random interval $[0, S_{M_T+1}/T] \supseteq [0,1]$ into the adjacent intervals

$$[0, S_0/T] \cup [S_0/T, S_1/T] \cup \cdots \cup [S_{i-1}/T, S_i/T] \cup \cdots \cup [S_{M_T}/T, S_{M_T+1}/T] \,.$$

Recall the following property of the M_1 metric: of $a < b < c$ and x_1, x_2 are functions in $D\big([a,c]\big)$, then

$$d_{M_1}\big(x_1, x_2, [a,c]\big) \leq \max\big[d_{M_1}\big(x_1, x_2, [a,b]\big),\, d_{M_1}\big(x_1, x_2, [b,c]\big)\big] \,.$$

We conclude that

$$\begin{aligned} d_{M_1}(\tilde{\mathcal{S}}_T, \mathcal{Z}_T) &\leq d_{M_1}(\mathcal{Z}_T, \tilde{\mathcal{Z}}_T) + d_{M_1}(\tilde{\mathcal{Z}}_T, \tilde{\mathcal{S}}_T) \\ &\leq \frac{\|\phi\|_\infty\, S_0}{a(T)} + \max_{i=1,\dots,M_T} d_{M_1}(\tilde{\mathcal{Z}}_T, \tilde{\mathcal{S}}_T, [S_{i-1}/T, S_i/T]) \\ &\quad + d_{M_1}(\tilde{\mathcal{Z}}_T, \tilde{\mathcal{S}}_T, [S_{M_T}/T, 1]) \,. \end{aligned}$$

Notice that the last term in the right hand side is bounded by $\|\phi\|_\infty C_{M_T+1}/a(T)$, and the finite mean of C_1 implies that the C_{M_T+1} converges weakly as $T\to\infty$ and, in particular, the family of the laws of (C_{M_T+1}) is tight. Observe, further, that $\tilde{\mathcal{Z}}_T$ continuously interpolates $\tilde{\mathcal{S}}_T$ at the points $t=S_i/T$, $i=0,1,2,\dots$. Hence, by (31), $\mathbb{P}(T>S_0)\to 1$ and stationarity we see that for any $\eta>0$,

$$\mathbb{P}\left(d_{M_1}(\tilde{\mathcal{S}}_T,\mathcal{Z}_T)>\eta\right)\le\frac{2T}{\mathbb{E}[C_1]}\mathbb{P}\left(d_{M_1}(\tilde{\mathcal{Z}}_T,\tilde{\mathcal{S}}_T,[S_0/T,S_1/T])>\eta/2\right)+o(1)\;.$$

Henceforth we now only consider the process $X_h(t)$ on $[S_0,S_1]$. We use the notation introduced in Section 2. First of all,

$$d_{M_1}(\tilde{\mathcal{Z}}_T,\tilde{\mathcal{S}}_T,[S_0/T,S_1/T])\le\sup_{u\in[S_0/T,S_1/T]}\left|\tilde{\mathcal{Z}}_T(u)-\tilde{\mathcal{S}}_T(u)\right|$$

$$\le a(T)^{-1}\sup_{v\in[S_0,S_1]}\int_{S_0}^{v}\phi(X_h(s))\,\mathrm{d}s\le a(T)^{-1}C_1\|\phi\|_\infty\;.$$

Combining this with (22) we see that for any $\eta>0$,

$$\mathbb{P}\left(d_{M_1}(\tilde{\mathcal{Z}}_T,\tilde{\mathcal{S}}_T,[S_0/T,S_1/T])>\eta;N_{\epsilon,T}=0\right)=o(T^{-1})\;,$$

as long as $\epsilon>0$ is chosen to be small enough.

Next we consider the event $\{N_{\epsilon,T}\ge 1\}$. Define

$$\check{\mathcal{Z}}_T(t)=a(T)^{-1}\int_{S_0}^{tT}\mathcal{E}(W_{\epsilon,T},\phi)\mathbb{1}_{[\Gamma_{\epsilon,T},\Gamma_{\epsilon,T}+Y_{\epsilon,T})}(s)\,\mathrm{d}s\;.$$

Observe that $\check{\mathcal{Z}}_T$ is monotone on $[S_0/T,S_1/T]$ and piecewise linear and $\tilde{\mathcal{S}}_T$ is constant on $[S_0/T,S_1/T)$ with a step at the point S_1/T. Using these properties it is not difficult to check that

$$d_{M_1}(\check{\mathcal{Z}}_T,\tilde{\mathcal{S}}_T,[S_0/T,S_1/T])\le\frac{C_1}{T}\vee\left|\tilde{\mathcal{S}}_T(S_1/T)-\check{\mathcal{Z}}_T(S_1/T)\right|\;.$$

On the other hand, bounding by the uniform distance gives us

$$d_{M_1}(\tilde{\mathcal{Z}}_T,\check{\mathcal{Z}}_T,[S_0/T,S_1/T])\le\sup_{t\in[S_0/T,S_1/T]}|\tilde{\mathcal{Z}}_T(t)-\check{\mathcal{Z}}_T(t)|\;.$$

Since, by Lemma 1, $\mathbb{P}(C_1>\eta T)=o(T^{-1})$, and $\tilde{\mathcal{S}}_T(S_1/T)=\tilde{\mathcal{Z}}_T(S_1/T)$, the proof of the theorem will be complete once we show that for any $\eta>0$ and $\epsilon>0$,

$$\mathbb{P}\left(\sup_{t\in[S_0/T,S_1/T]}|\tilde{\mathcal{Z}}_T(t)-\check{\mathcal{Z}}_T(t)|>\eta;N_{\epsilon,T}\ge 1\right)=o(T^{-1})\;.$$

However,

$$\sup_{t\in[S_0/T,S_1/T]}|\tilde{\mathcal{Z}}_T(t)-\check{\mathcal{Z}}_T(t)|=\frac{1}{a(T)}\sup_{v\in[S_0,S_1]}\left|\int_{S_0}^{v}\{\phi(X_h(s))-\mathcal{E}(W_{\epsilon,T},\phi)\mathbb{1}_{[\Gamma_{\epsilon,T},\Gamma_{\epsilon,T}+Y_{\epsilon,T})}(s)\}\,\mathrm{d}s\right|\;,$$

and our claim follows from Lemma 3. □

References

Patrick Billingsley. *Convergence of Probability Measures.* Wiley, New York, 1968.

Donald L. Cohn. Measurable choice of limit points and the existence of separable and measurable processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:161–165, 1972.

Gilles Fay, François Roueff, and Philippe Soulier. Estimation of the memory parameter of the infinite-source Poisson process. *Bernoulli*, 13(2):473–491, 2007.

Peter Hall. *Introduction to the Theory of Coverage Processes.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1988. ISBN 0-471-85702-5.

Janet E. Heffernan and Sidney I. Resnick. Limit laws for random vectors with an extreme component. *Ann. Appl. Probab.*, 17(2):537–571, 2007. ISSN 1050-5164.

Olaf Kallenberg. *Foundations of Modern Probability.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

Krishanu Maulik, Sidney I. Resnick, and H. Rootzén. Asymptotic independence and a network traffic model. *Journal of Applied Probability*, 39(4):671–699, 2002.

Thomas Mikosch, Sidney I. Resnick, Holger Rootzén, and Alwyn Stegeman. Is network traffic approximated by stable Levy motion or fractional Brownian motion? *Annals of Applied Probability*, 12:23–68, 2002.

Sidney Resnick and Eric van den Berg. Weak convergence of high-speed network traffic models. *J. Appl. Probab.*, 37(2):575–597, 2000. ISSN 0021-9002.

Sidney I. Resnick. *Adventures in Stochastic Processes* Birkhäuser, Boston, 1992.

Sidney I. Resnick. *Heavy-Tail Phenomena.* Springer Series in Operations Research and Financial Engineering. Springer, New York, 2007. ISBN 978-0-387-24272-9; 0-387-24272-4. Probabilistic and statistical modeling.

Sidney I. Resnick and Gennady Samorodnitsky. Activity periods of an infinite server queue and performance of certain heavy tailed fluid queues. *Queueing Systems. Theory and Applications*, 33(1-3):43–71, 1999.

Emmanuel Rio. A maximal inequality and dependent Marcinkiewicz-Zygmund strong laws. *The Annals of Probability*, 23(2):918–937, 1995.

Emmanuel Rio. *Théorie asymptotique des processus aléatoires faiblement dépendants*, volume 31 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2000.

Gennady Samorodnitsky and Murad S. Taqqu. *Stable Non-Gaussian Random Processes.* Chapman and Hall, New York, 1994.

Donatas Surgailis. Stable limits of sums of bounded functions of long-memory moving averages with finite variance. *Bernoulli*, 10(2):327–355, 2004.

Murad S. Taqqu. Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete*, 50(1):53–83, 1979. ISSN 0044-3719.

Ward Whitt. *Stochastic-Process Limits.* Springer Series in Operations Research. Springer-Verlag, New York, 2002. ISBN 0-387-95358-2. An introduction to stochastic-process limits and their application to queues.

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