Structural Morphology

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Shape and size optimization of shell structures with variable thickness

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Abstract

This paper introduces a methodology for shape and size optimization of shell structures with variable thickness. A model is defined that reduces the number of variables without losing freedom. Several optimization methods are compared. The method of the Coupled Local Minimizers (CLM) offers the certainty of the identification of the global minimum. This methodology is implemented by using MATLAB and ANSYS. It is used successfully for two instructive examples.

1 Introduction

In modern architecture, there is a tendency towards complex structures. These structures frequently have the appearance of a classical shell, but are constructed with discrete members such as trusses or curved Vierendeel beams. Although aesthetical arguments often dominate the decision to built such a shell shaped structure, structural aspects also prevail. This entails an increasing attention for the optimal design of such structures, especially shell structures with variable thickness. These shells can be considered as the continuous equivalent of space trusses with variable height. In that context, this paper describes a methodology for shape and size optimization of shell structures with variable thickness.

2 Model generation and structural analysis

A model is well-adapted for structural optimization if it is defined by a small amount of parameters and if it can represent a large range of shapes by changing the value of these parameters. These parameters will be the design variables in the optimization. Thus, the model of the shell structure and the choice of the design variables are closely related.

Imam [2] formulates modeling techniques based on Computer Aided Geometric Design (CAGD) to meet the requirements for a good model. In this research, the super curve technique is used for shape modeling, which means that the shape of the shell is defined by the shape of one or more curves. Particularly, a spline fit through the coordinates of a variable number of control points defines the edges of an automatically generated Coons patch. The dimensions are determined by a design element technique. Besides the coordinates, an additional scalar parameter is attached to some control points. The physical meaning of this parameter is the thickness in that point of the shell. The thickness of other points is determined by interpolation with Lagrange shape functions. In this way, the shape design variables (coordinates) and the size design variables (thicknesses) are clearly separated.

The model needs to be generated and analyzed for every proposed value of the design variables. In this paper, the finite element program ANSYS is used. Thanks to the use of the ANSYS Parametric Design Language (APDL), a general input file for ANSYS can be written to automate the generation of the model during the optimization proces. The obtained shell surface is meshed with SHELL93 elements and a single static load case is considered. However, programs and analysis types are easily interchangeable in the presented approach.

3 Optimization

The shape and size design variables are determined by the model of the shell structure. To complete the formulation for the optimization, an objective function and necessary constraints have to be defined. In this paper, as in Kegl *et al.* [3] and Lee *et al.* [4], the objective function is the strain energy E_s. The minimization of this function reduces the amount of bending in the shell. As in some load cases an increase of the shell thickness reduces the strain energy, it can be necessary to constrain the volume of the shell.

Similar to the finite element program, optimization methods are easily interchangeable in this methodology. As the optimization algorithm has a large influence on the efficiency of the optimization, it is useful to compare several. In the optimization module of ANSYS [1], two methods are available. In the Subproblem Approxiation method (SA), the objective function is replaced by an approximation based on the function values calculated in previous iterations. This method makes no use of derivative information. The First Order Optimization method (FOO) is a line search method in which gradient information is used to determine a decent search direction. This methods are rather limited for realistic problems, so a connection with MATLAB is established to extend the possibilities and to have better control on the proces. Two methods of the optimization toolbox of MATLAB [5] are included in the comparison. The first, `fmincon`, is a trust region method in which derivative information is used to compute a good approximation of the objective function in a small trust region. The second, `lsqnonlin`, uses specific least squares techniques. As will be clear in the examples (Section 4), only the SA-method of ANSYS is apparently less efficient, the other methods are competitive.

The mentioned methods are all local, so there is uncertainty about the nature of the optimum. Therefore, the method of the Coupled Local Minimizers (CLM) is implemented. CLM is a recently developed global optimization technique, see Suykens *et al.* [6] and Teughels *et al.* [7]. In this method, the information of several local optimizers is combined to avoid local optima. The local optimizations are started from random points over the domain, and constraints are imposed to force the search points to end up in the same point. In a successful run, this point has the lowest function value, and is the global minimum. The reliability of this method is due to the evaluation of a lot of points, spread over the domain. The advantage compared to other global methods is the use of first order information, which enforces faster convergence. To reduce calculation time, this method is used to identify the global minimum with a limited precision. When the search points have located the valley of the global minimum, the CLM-method is stopped and a local method is used until the necessary precision is reached.

In all methods, the forward finite difference method is used to calculate the gradient of the objective function.

4 Examples

4.1 Example 1: Pressure vessel

A first example discusses the shape optimization of a pressure vessel, see Kegl *et al.* [3]. A long, thin-walled vessel is loaded by an inner pressure of 1 MPa. The vessel is made of steel with Young's modulus $E = 210000$ MPa and Poisson's ratio $\nu = 0.3$. The thickness of the vessel is 10 mm. It is known that bending stresses vanish when the vessel is cylindrical, which is therefore the expected solution of the shape optimization. As the vessel is long, the analysis is limited to a small strip of 100 mm, which can be treated as a 2D-problem. Furthermore, due to the symmetry of the solution, only a quarter of this strip has to be considered.

Following the explained principles, the shell is modeled based on a super curve technique. Three control points define a spline with fixed end slope vectors. This spline defines a coons patch with a depth of 100 mm that is meshed with SHELL93 elements.

The only design variables are the coordinates $(x1, y1)$ of the middle control point. The geometry of the fixed end points imposes a radius of 1000 mm, so the expected values of the design variables are $(707.11, 707.11)$. Their starting values are $(600, 600)$ with lower and upper bounds of respectively $(550, 550)$ and $(800, 800)$. Figure 1(a) shows the starting and the optimal shape of the spline that defines the shell. The objective function is E_s and no constraints are imposed. The problem is solved with several optimization methods. A comparison between the results of the local methods of ANSYS and MATLAB for this problem is given in figure 2. It occurs that a direct method as the SA-method is less efficient, but that the other methods are competitive. The problem has no local minima, so using CLM is unnecessary.

One can see in figure 2 that the results of the optimization never reach the analytical optimum. This is because by definition, a spline can not form an exact circle. Although the difference in strain energy is small (about 4%), the influence is not negligible. Due to the small deviation of the circular shape, a limited amount of bending stresses is present in the spline model, contrary to the circular shell. This is also reflected by the shape of the deformations. As can be seen on figure 1(b,c), the spline model deforms clearly different. Hence, it should be taken into account that circular shapes are not included in the presented spline model.

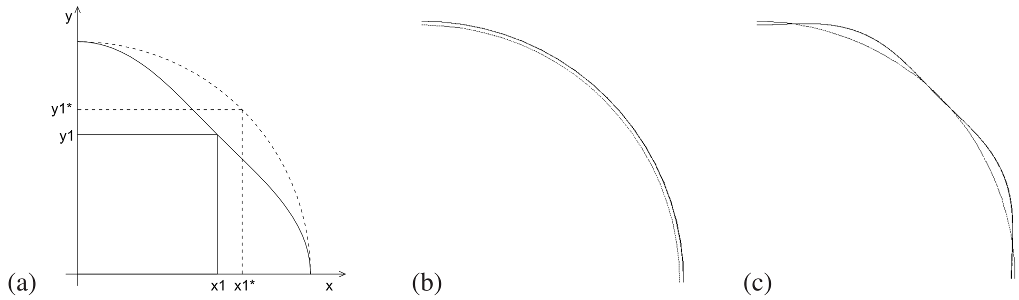

Figure 1: Shape optimization of a pressure vessel: (a) starting model and optimal shape (dotted line). The coordinates $(x1, y1)$ are the design variables, with optimal value $(x1^*, y1^*)$. Original and deformed model of (b) the analytical solution with circles and (c) the result of the optimization with the FOO-method of ANSYS with the spline model. The deformations are magnified with a factor 30. The small difference between a circle and a spline has a large influence on the shape of the deformations.

Program Method	ANSYS SA	ANSYS FOO	MATLAB `fmincon`	MATLAB `lsqnonlin`	analytical (circle)
Iterations	166	14	6	9	-
Result $x1$ (mm)	706.98	707.01	706.99	707.01	707.11
$y1$ (mm)	707.04	707.01	706.99	707.01	707.11
Strain energy (Nm)	38947	38947	38948	38947	37457

Figure 2: Shape optimization of a pressure vessel: comparison of local optimization methods and analytical result. A direct method is less efficient, but other methods are competitive. The analytical solution can not be reached by the spline model.

4.2 Example 2: tension or compression arch

This design problem considers an arch with a span of 10 m under a snow load of 7 kN/m^2 and a horizontal point load of 5 kN. To determine if either a tension or a compression arch is more efficient for this load case, the middle height Y is chosen as design variable and varied between -3 to 3 m. The model is analogous to the previous example, except the end slope vectors are now automatically calculated by ANSYS to have zero end curvature. The geometry of the arch is presented in figure 3(a).

As the problem has only one variable, it is possible to plot the strain energy as a function of the design variable Y (figure 3(b)). The function reaches a peak value when there is no curvature, i.e. $Y = 0$ m. There is a local minimum at $Y = 0.972$ m, and the global minimum is at $Y = -1.106$ m. The strain energies are respectively 3.5262 Nm and 2.8267 Nm.

This problem is optimized with the local methods of both ANSYS and MATLAB. The performance of the methods with regard to precision and calculation times are analogous to the previous example. The result of the optimization depends strongly on the starting value: the minimum in the valley of the starting value is found. In most realistic problems, it is not possible to get an image of the objective function. In that case, the observation that different local optimizations end up in different optima is an indication that global optimization is necessary. Consequently, CLM is used to optimize this problem. In figure 4, the results of a run with a population of four search points are presented. The search points are originally random spread over the domain. It can be seen that both minima are located after the first iteration. The algorithm forces the search points to make a choice which is the global minimum. Indeed, after seven iterations, all search point are located around the global minimum. A local algorithm can now precisely define the global minimum.

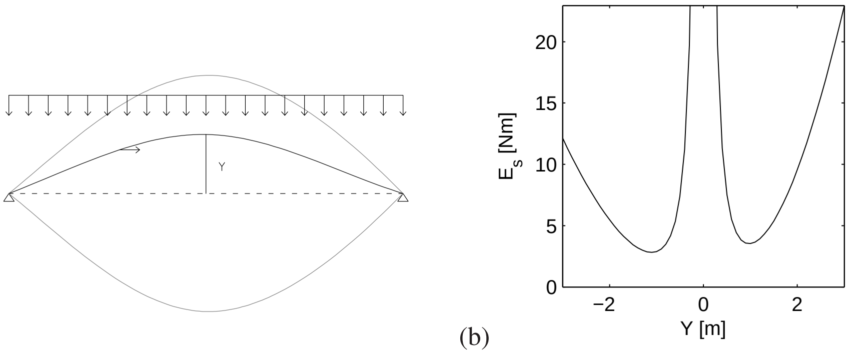

Figure 3: Tension or compression arch: (a) geometry and (b) objective function.

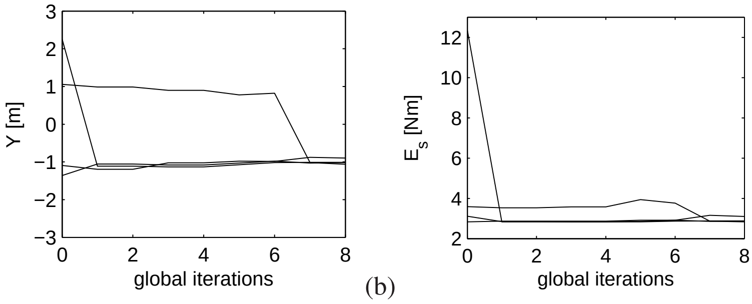

Figure 4: Tension or compression arch: (a) height Y of the arch, and (b) strain energy of the search points after every global iteration.

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Topological representation of natural and man-made structural forms

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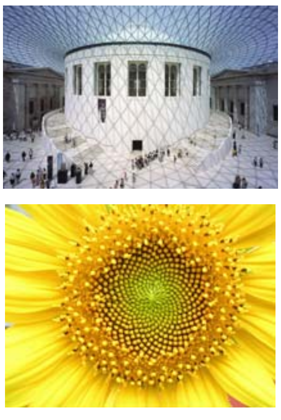
Abstract

Structural forms widely appearing in nature, have been followed by people since the very beginning of their conscious structural activity. They were regarded as a source of structural prototypes up to present times. When recent aesthetical tendencies replaced any prototypes by pure imagination and created new paradigm, these traditional rules and structural logic have to be complemented by new design tools. Many structural forms can be characterized in terms of topological models. Problems considered on this level depend not on the exact shape of the objects, but rather on the way they are put together. Topological relations concerning geometrical entities are exhaustively described in the graph theory. Projective techniques allow regarding structural geometries based on spatial polyhedral patterns as tessellations of plane i.e. graphs. Due to fundamental constraints, such as Euler formula or Eberhard formula, we can transform graphs in order to fulfill design requirements, maintaining their stability and other structural properties. From transformed graphs, a spatial (polyhedral) structure can be reconstructed by means of reciprocal projection or Gale diagrams.

1. Introduction

The roots of architecture are in the close contact of peoples with surrounding environment. Natural prototypes of structural forms, occurring in Nature, were always considered by man as an inspiration for his own work. It is easy to find examples in historical as well as in contemporary objects, Fig.1.a). (Bober and Tarczewski [1]). Traditional meaning of the "logic" of structural systems was usually connected with a clear transmission of loads and structural efficiency of elements, but also in the close symbiosis of architectural form and structural system, Fig.1.b). (Zalewski and Zablocki [14]). Some recent trends in architectural design generally called "free form design" have changed this point of view, making visual impression of the building a predominant factor. Thus, structural systems are forced to follow it.

a). 

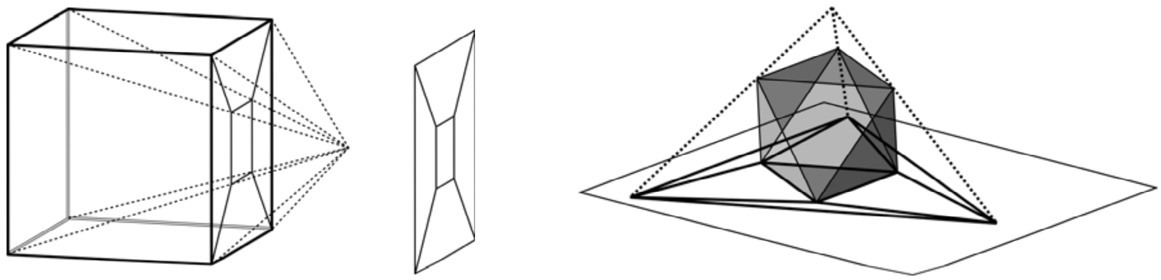
b). 

Figure 1: Examples of the close symbiosis of structural systems and architectural form

Complexity of models applied for analysis as well as increasing sophistication of numerical tools lead to widely perceptible computational obsession in design. Motto: “I design at first and then I calculate. In the case of inconsistency – I repeat calculations” became neglected. On the other hand, such a complex models are hard to be consciously controlled by single person responsible for the final result – a designer. Details are easy to manipulate but generalities – not.

2. Planar representation of structural lattices

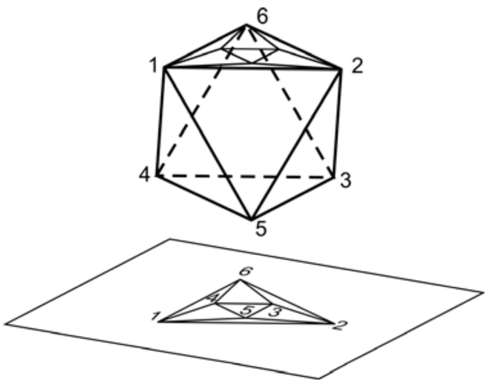
Polyhedrons and compounds of polyhedrons are basic modular elements for generation of structural nets. For this purpose a variety of cells are used: space-filling and not space-filling, regular, quasicrystal etc. (Gabriel [5]). Spatial patterns of this type have planar representation which is a projection of polyhedrons onto the plane, called Schlegel diagram. In this technique, a chosen face of polyhedron becomes a projective plane. Diagram is obtained by projecting the edges onto this face from the point, which is placed outside the polyhedron, but still very close to the center of the chosen face (Grünbaum [6], Richter-Gebert [12]). Figure 2 presents Schlegel diagrams for single cells: cube (a) and octahedron (b).



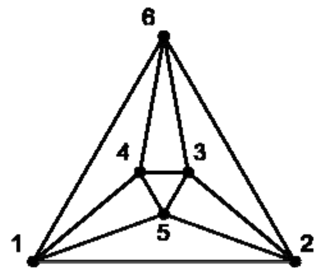
a). b).

Figure 2: Schlegel diagrams for cube and octahedron

Planar representations of structural lattices is a tiling (or tessellation) of projective plane. These tilings are highly ordered and interesting relations between them and some old decorative patterns have been found (Lu and Steinhardt [9]). Mathematical representation of linear tiling and therefore representation of polyhedron is a graph (Chartrand and Lesniak [4]). Figure 3 presents a graph of an octahedron. Numerically, graph is a countable family of closed sets of vertices *V*, edges *E* and faces *F*, which form cyclically ordered sequences (Whiteley [13]). This family of sets is called a combinatorial graph or general polyhedron. The “real”, geometrical polyhedron is regarded as a realization the combinatorial one (Richter-Gebert [12]).



a).



b).

Figure 3: Representation of polyhedra by means of graph theory

3. Topological level of representation

Graphs are objects on the lowest level of representation, regarding the number of characteristics necessarily needed for description of objects – topology. Problems considered on this level depend not on the exact shape of the objects involved, but rather on the way they are put together. They are qualitative not quantitive. From the morphological point of view, topological objects are kind of "rough sketch" of the structure. Perpendicularity, parallelism, straightness of lines and measuring of lengths – all these concepts are out of sense. Graphs preserve geometrical relations of structural components in their most general outlines. Instead of mentioned above concepts, topology deliver tools to manipulate with the internal structure of objects.

Most widely know topological restriction is Euler's formula (1), connecting number of vertices of graph *v*, number of its edges *e* and number of faces *f* with genus *g* of the surface on which graph can be embedded.

$$v - e + f = 2(1 - g) \tag{1}$$

Less known but equally powerful is Eberhard's formula (2). For given number of 3-, 4- and 5-valent faces, number of faces of higher valency is restricted by this equation. Up to now, there have been found nineteen combinations containing only faces 3-, 4-, 5- and 6-valent (Grünbaum [6]).

$$3f_3 + 2f_4 + f_5 = 12 + \sum_{k \geq 7} (k - 6) f_k \tag{2}$$

As each planar cell of the graph represents a face of its polyhedron, there is a variety of transformations of structural lattices obtained through manipulation on the structure of graphs. Figure 4 presents some examples of such transformations. In the left column we can see examples of *deleting* of the edge *E*. If one of its endpoints *c* and *d* would become 2-valent, two edges incident with this vertex are amalgamated into a single edge. Fig. 4.d). presents operation of *contraction* i.e. identification of adjacent vertices *c* and *d*. Contraction and deleting are dual operations. In the right column, Fig.4.e)f).g)., operations opposite to contractions are presented. Since after insertion of new edge, face *F* is divided onto two faces *F'* and *F''* which are coplanar, Fig.4h)., it is necessary to follow a suitable procedure to prevent it (Grünbaum and Barnett [7]). For the following steps it is important that transformations do not violate Steinitz's theorem which states that "a graph is 3D realizable if and only if it is planar and 3-connected with edges in every vertex" (Grünbaum [6], Richter-Gebert [12]). Composition of transformed graphs allows prediction of its structural properties (Laman [8]).

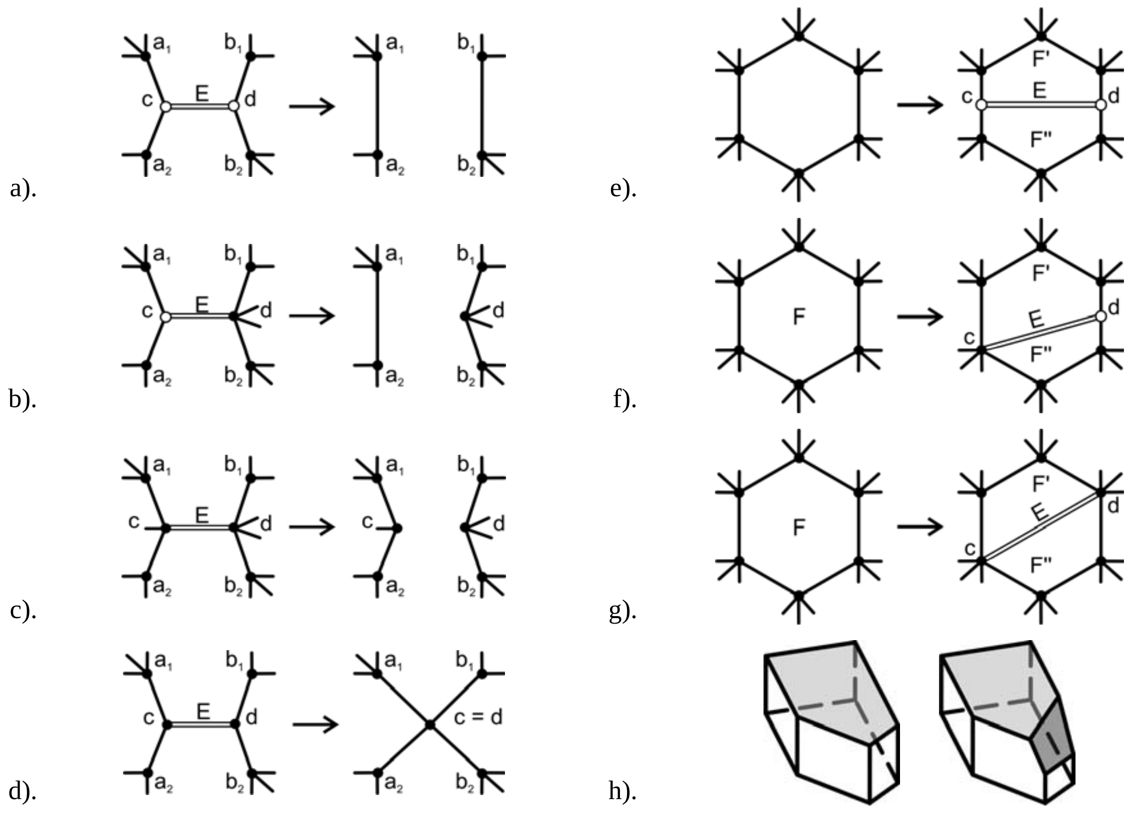


Figure 4: Examples of transformations of the internal structure of graphs

Properly transformed graphs are topological representations of some new polyhedrons. It is possible to reconstruct these polyhedrons in the space (Crapo and Whiteley [2], Croft *et al.* [3]). This can be done by means of reciprocal projection (Maxwell [10]). Another method makes use of the properties of Gale diagrams (Perles and Shephard [11]). A typical sequence to reproduce polyhedron is presented on Fig.5.

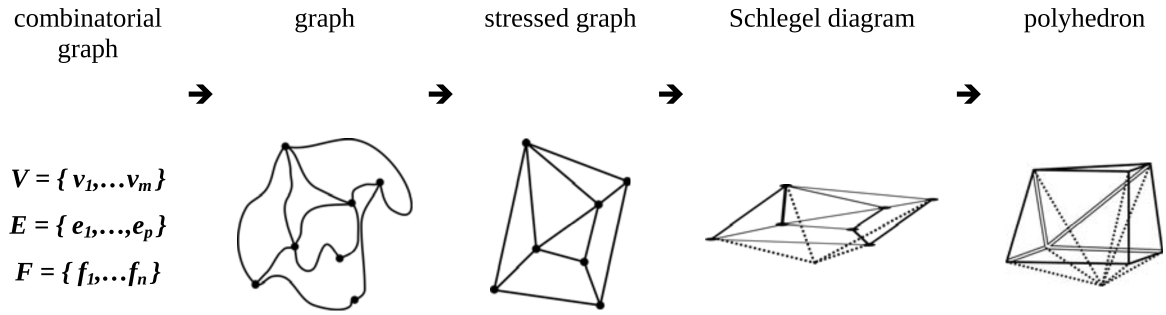


Figure 5: Sequence to reproduce a 3-polytope (polyhedron) from the combinatorial graph

3. Concluding remarks

Topological models are useful particularly in conceptual designing of spatial structures. Plane figures are much more convenient to manipulate then three-dimensional ones. If proper set of restrictions is applied, geometrical and statical properties, which are related to the internal structure of graph, can be predicted and fixed. Further research is needed to reveal if there is some kind of deeper connection between structural lattice systems and the art of tiling originated very early in the history of civilization.

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Polyhedric configurations in spatial structures

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Abstract
Polyhedra have been the subject of fascination and interest since the ancient times. They have been studied throughout the ages by mathematicians, philosophers, engineers and artists and they play an important in a number of branches of science and technology. The interest in Polyhedra in this paper stems from the fact that they provide a basis for the generation of a number of important classes of structural forms. They are efficient and appealing and are employed frequently for long span spatial structures.

The processing of polyhedric configurations in the pre-computer days was extremely difficult task. In spite of this, a number of gifted designers managed to deal with the problem and create many beautiful structures based on polyhedric configurations. The constraint of the processing difficulties, however, did not allow the designers to take full advantage of the whole spectrum of possibilities and their scope remained rather limited. Even today, the processing of polyhedric configuration is mainly carried out using computer programs that lack generality and have many limitations and shortcomings. The objective of the present paper is to introduce the concepts and constructs through which data generation for geodesic forms of all kind can be handled with ease and elegance.

Generation of geodesic forms is solved in two stages. Firstly, a function called the polyhedron function is used to generate a configuration modeled on a regualar or semi-regular polyhedron. The resulting configuration is referred to as a polyhedric form. In the next stage, a transformation referred to as the tractation retronorm is employed to obtain the projection of the polyhedric form on one or more specified surfaces. This transformation allows choice of different types of surfaces as sphere, ellipsoid and paraboloid.

The method is based on the concepts of formex algebra and its programming language Formian. In actually using the method one has to be familiar with the concepts of formex algebra and its programming language Formian. However, the present paper is written in such a way that allows a reader to follow the basic ideas without any knowledge of formex algebra and Formian.

1. Introduction
The interest in polyhedra in this paper stems from the fact that they provide a basis for the generation of a number of important classes of spatial structures. A polyhedron is a surface composed of plane polygonal surfaces, the “faces”. The sides of the polygons joining two faces are its “edges”. The corners, where three or more faces meet, are its “vertices”. The approach presented in this paper provides a methodology that allows polyhedric configurations of all kinds to be generated in a convenient manner. However, polyhedric configurations based on regular polyhedra (known as Platonic solids) or semiregular (Archimedean) polyhedra are used here to demonstrate the concepts and constructs through which polyhedric configurations may be created.

A regular polyhedron is a polyhedron whose faces are congruent regular polygons. Every vertex is to be congruent to every other vertex, that is, the faces must be arranged in same order around each vertex. There are only five regular polyhedra. The Archimedean or “semiregular polyhedra” are what is called “facially” regular polyhedra. This means that every face is a regular polygon though the faces are not all of the same kind. However, every vertex is to be congruent to every other vertex that is, the faces must be arranged in the same order around each vertex. There are fifteen semiregular polyhedra.

2. Generation of Polyhedric forms

In dealing with the formex formulation of a configuration, it is usual to begin by formulating a topological description of the configuration using formex functions. The next stage involves the employment of a transformation for associating geometric coordinates with nodes of the configuration. A transformation of this kind is referred to as a retronorm. Two categories of retronorm are employed in Formian, the standard retronorms that are incorporated in the Formian Interpreter and the supplementary retronorms. A supplementary retronormic function is introduced through a program segment which is supplied by the end user in order to create a non-standard retronorm. The program segment is linked to the body of the Formian Interpreter.

In a Formian environment the generation of a polyhedric form is achieved in two stages. Firstly, a transformation called the "polyhedron function" is used to generate a configuration modelled on a polyhedron. The resulting configuration is referred to as a "polyhedric configuration" or "polyhedric form". This term may also be used to refer to a portion of a polyhedric configuration. A polyhedron which is used as the basis for the creation of a polyhedric configuration or form is referred to as the "base polyhedron" of the polyhedric form. The polyhedron function constitutes the kernel of the problem handling strategy for the configuration processing of polyhedric forms that are in use in spatial structures.

The applications of the polyhedron function may be described with the help of an example. Consider a single layer triangular configuration which will be referred to in the sequel as the configuration. The configuration together with the normat U1-U2-U3 for the formex formulation, are shown in Figure 1. This configuration may be represented in terms of the formex variable

E=LIB(I=0,5)|RIN(1,6-I,2)|TRANID(I,I)|{[0,0;2,0],[2,0;1,1],[1,1;0,0]} (1)

Let it be required to map this configuration onto all the faces of a tetrahedron. A Formian statement describing this operation may be given as

D=POL(1,15,[0,0;12,0])|G where G=BB(1,TAN|60)|E (2)

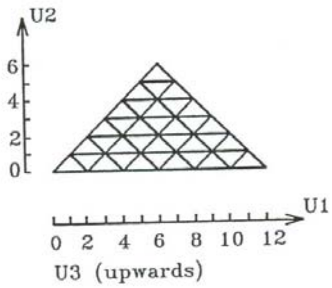

Figure 1

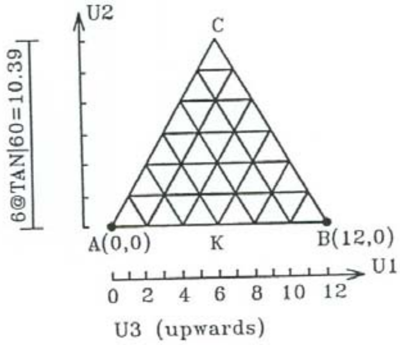

Figure 2

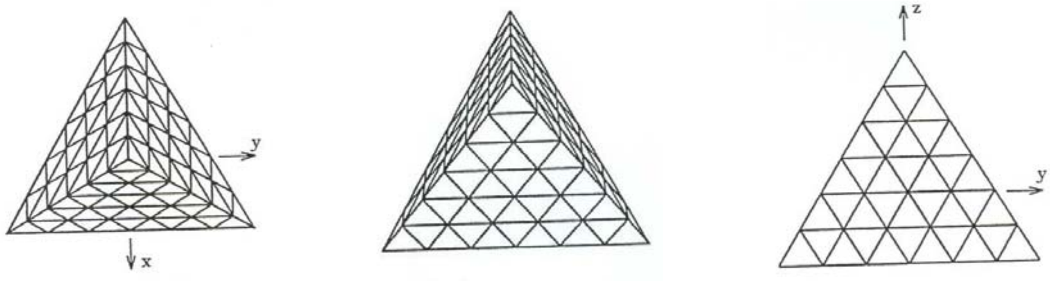

Figure 3: Plan view, perspective, elevation

A tetrahedron has four equilateral triangular faces. The triangular configuration that is described here by formex E is not equilateral. Therefore, it will be necessary to scale the configuration using the appropriate scale factors. Formex G specifies the scale factors that are used in the first and second direction to obtain the equilateral triangular configuration of Figure 2. Formex variable G represents the actual node coordinates while formex variable E represents the corresponding normat coordinates of the configuration. A graphical representation of formex variable D is as shown in Figure 3. Also, in the figure the plan view and the elevation of the polyhedric configuration have been given together with the global Cartesian coordinate system.

The construct POL(1,15,[0,0;12,0]) is a polyhedric function representing a rule for transformation of a given formex G into a formex D. The parameters 1, 15, [0,0;12,0], are parts of the rule defining the particulars of the transformation and are referred to as canonic parameters. The integer 1 in the above polyhedron function specifies a tetrahedron and is referred to as the "polyhedron code". The polyhedron code may have values from 1 to 18. Each value denotes a regular or semi-regular polyhedron. For example integer 5 represents an icosahedron and integer 14 a truncated icosahedron. The integer 15 determines the size of the polyhedron by specifying the radius of its circumsphere, that is, the sphere that contains all the vertices of the polyhedron and is referred to as the "radius specifier". The parameter [0,0;12,0] is called the "locator" and specifies the manner in which a given configuration is to be mapped onto a face of the polyhedron. To elaborate, consider the configuration shown in Figure 2. Two corners of the configuration are denoted by the letters A and B. The configuration is intended to be placed on a face of the tetrahedron in such a way that AB fits an edge of the tetrahedron. This convention is conveyed by including the U1-U2 coordinates of A and B in the locator.

3. Tractation retronorm and spatial structures

In the second stage a supplementary retronorm called the "tractation retronorm" is employed to obtain the projection of the polyhedric configuration on one or more surfaces. The "tractation retronorm" enables a polyhedric configuration to be projected on different types of surfaces such as spheres, ellipsoids, paraboloids, cylinders, hyperbolic paraboloids or planes. The term tractation is used to imply projection of a configuration on a surface or surfaces. Tractation is derived from the latin word "tractus" meaning "drawing". To explore the range of possibilities of shapes and forms four types of projections have been used. These are central, parallel, axial and radial projections and will be discussed in due course. The "tractation retronorm" allows a "polyhedric configurations" to be generated from a concise and yet readily understood formulation.

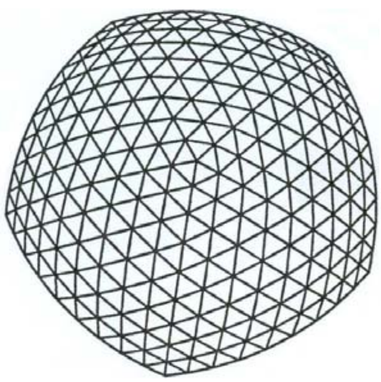

Figure 4

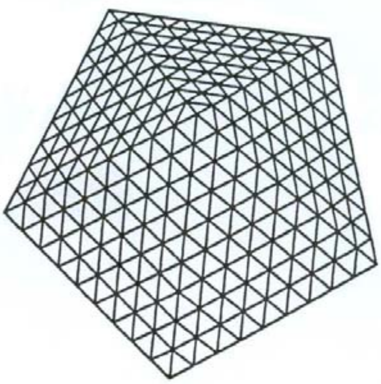

Figure 5

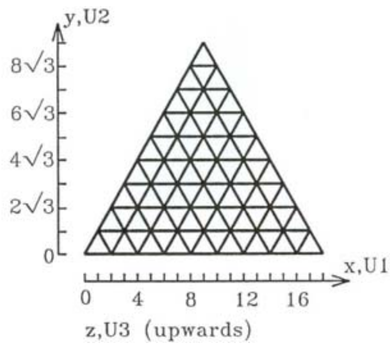

Figure 6

Consider the configuration shown in Figure 4. This is a view of a single layer geodesic form obtained by projecting the polyhedric configuration of Figure 5 on a sphere which is concentric with the icosahedron using the centre of the sphere as the centre of projection. A Formian statement describing this operation may be given as

D1=TRAC([4,1,0,0,0,12,13]|G1 (3)

Where the formex variable G1 represents the top five faces of the icosahedral configuration of Figure 5 and may be given as

G1=POL(5,10,[0,0;18,0],[1,5])|E1 (4)

Formex variable E1 represents the compret of Figure 6 and may be given as

E=GENID(9,9,2,SQRT|3,1,-1)|{[0,0;2,0],[2,0;1,SQRT|3],[1,SQRT|3;0,0]} (5)

The constituent parts of formex variable D are as follows: TRAC is an abbreviation for tractation retronorm. The integer 4 implying radial projection to be used and is called to as "projection specifier". The projection specifier may have the value 1, 2, 3 or 4 indicating central parallel, axial or radial projection respectively. The next parameter specifies the type of surface on which surface is to be made and is referred to as the "surface specifier". For formex variable D1 is given as 1 implying a sphere and this is followed by the coordinates of the centre of the sphere (0,0,0) and the number 12 that specifies the radius of the sphere. The surface specifier may have the value of 2, 3 or 4 indicating ellipsoid, elliptic paraboloid or hyperbolic paraboloid surface respectively. The integer 13 is called the selection code and specifies the course of action to be taken when the projection of a point cannot be determined uniquely.

4. Geodesic forms

Geodesic forms constitute an important family of structural systems. They are efficient and appealing and are employed frequently for spatial structures. Geodesic forms allow effective use of material and space and may be employed to create architecturally interesting and economic building structures. They are presently used in a number of specialized areas of construction such as domes for arenas, cultural centres, exhibition halls and Olympic facilities. Most of the existing geodesic domes have been obtained from the radial projection of the triangulated faces of an icosahedron on a sphere. However, in this study a geodesic dome may be obtained by projecting a polyhedric configuration on a surface. For the projection the tractation retronorm will be applied. All the surfaces and types of projections available in the tractation retronorm maybe used to generate intersecting geodesic dome configurations. The surface on which a geodesic form is produced need not necessarily be single layer or spherical. Indeed, a variety of different surfaces such as ellipsoids and paraboloids may be used for creation of single of multi layer geodesic forms, Figures 7-9. Also the type of projection need not necessarily be central and other kinds of projection, such as parallel projection, may be used instead.

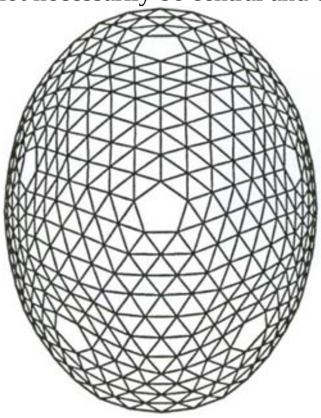

Figure 7

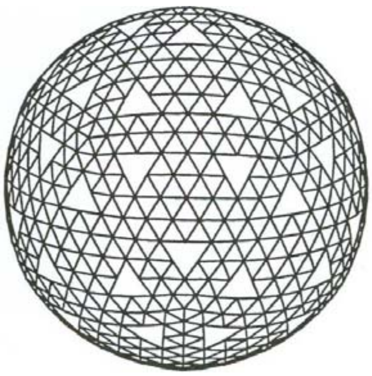

Figure 8

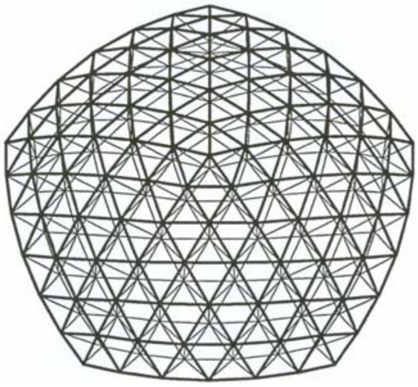

Figure 9

5. Conclusion

The widespread of geodesic forms has been obstructed by the difficulty in defining their geometry. This problem has presented a challenge for engineers and architects for decades. Many attempts have been made throughout the world to evolve techniques that deal with the data generation of geodesic forms. Polyhedric forms based on the Platonic and Archimedean polyhedra are used in this paper to demonstrate the concepts and constructs through which polyhedric configurations and geodesic forms may be created. The scope of this work, however, is much wider than the applications in relation to Platonic and Archimedean polyhedra. In fact, the approach presented in this work provides a methodology that allows data generation for polyhedric configuration and geodesic forms of all kinds to be generated in a convenient manner. The concepts of formex algebra and its associated programming language Formian have been used together with the above ideas to deal with the configuration processing of polyhedric forms and spatial structures.

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Analytical and computational form-finding

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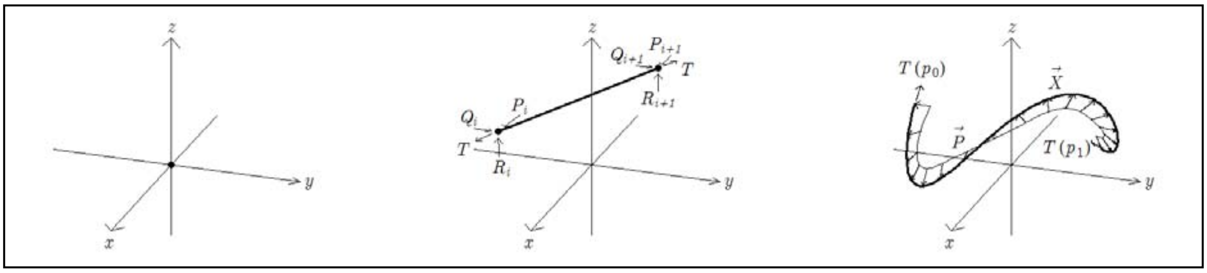
Abstract

This lecture addresses the task of finding the shape of cable and membrane structures in static equilibrium. The equations for static equilibrium are derived from the application of the Calculus of Variations (Olver [1]) to the Principal of Virtual Work from a collapsed reference configuration as motivated by Haber & Abel [2]. Restrictions on these equations lead to the 1D & 2D Laplace-Young equations (Lewis [3]) and the well-known thin-walled pressure vessel formulas (Hibbeler [4]). Analytical solution of the equilibrium equations results in several statically equilibrated shapes for uniformly stressed cable and membrane structures. To motivate the determination of additional shapes, two computational procedures, Energy Minimization (Zhang & Tabarrok [5]) and Dynamic Relaxation (Barnes [6]), are briefly summarized and supplemented with a proposed solution to the problem of tangential shape variations (Bletzinger [7]).

1. Analytical Form-Finding of Cables

1.1 Derivation of the Equation for Static Equilibrium

A loaded cable can be described pointwise by its position vector $\vec{X} = \{x, y, z\}$, load vector $\vec{P} = \{P, Q, R\}$, and the tension T, which acts tangent to the cable and is the product of the tensile stress σ and cross-sectional area A.



Figures 1a, 1b, 1c: Undeformed & deformed configurations of a loaded cable element & structure

Haber & Abel [2] investigates a "virtual work expression associated with a deformed configuration created by collapsing all nodes of the element to the global origin". Reversing this concept,

$$W = \vec{P} \cdot \vec{X} - T\,L' = P\,x + Q\,y + R\,z - \sigma\,A\sqrt{x'^2 + y'^2 + z'^2} \tag{1}$$

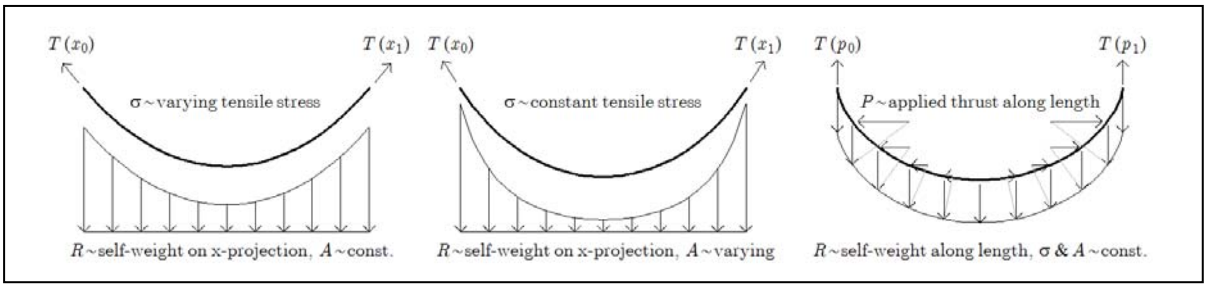
For minimum total potential energy, W must satisfy the Euler-Lagrange Equations (Olver [1]). By substitution,

$$\vec{P} + \frac{d}{dp}\left(\frac{\sigma\,A}{L'}\vec{X}'\right) = 0 \tag{2}$$

One may verify by free-body diagram that for static equilibrium of a loaded cable, Equation 2 must hold.

1.2 Three Hanging Cables

A suspended planar cable under its own weight ($y = 0, R = -\gamma A L'$) may take one of several shapes, namely the hyperbolic cosine, natural log of cosine, and cycloid in Figures 2a, 2b, and 2c, respectively.



Figures 2a, 2b, 2c: Three hanging cables, posed as analytical form-finding problems

1.3 The Laplace-Young Equation for Cables & the Cylindrical Pressure Vessel Formula

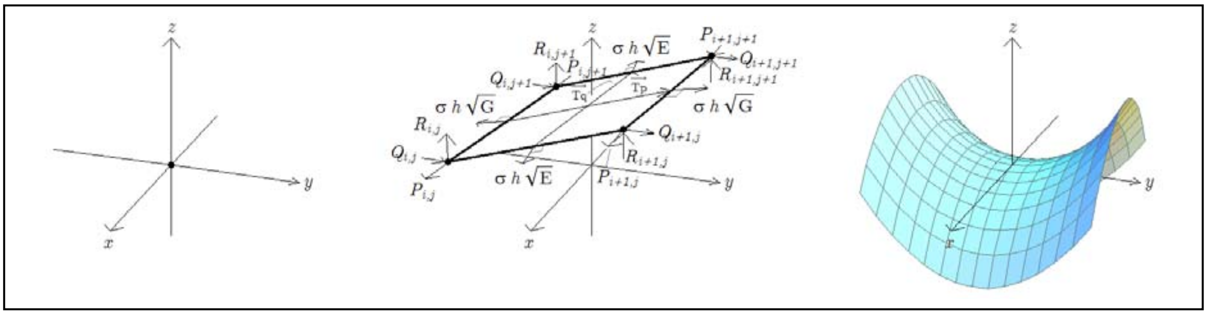
If $\sigma' = A' = 0$ & $\wp \sim P / L'$, Equation 2 reduces to Equation 3a: the Laplace-Young Equation for 1D Continua (cf. Lewis [3]). For cylindrical membranes of radius $R = 1/\kappa$ and thickness t under surface pressure $\wp$ (load per differential area), it may take the form of Equation 3b: the Cylindrical Pressure Vessel Formula (Hibbeler [4]).

$$\wp = \sigma A \kappa \qquad \& \qquad \sigma = \frac{\wp R}{t} \tag{3a \& 3b}$$

2. Analytical Form-Finding of Membranes

2.1 Derivation of the Equation for Static Equilibrium with Restrictions

What was done for cables can also be done for membranes, albeit with a few restrictions. If the thickness h is constant and the stress σ is uniform & isotropic, one may use Figure 3b in finding W for membranes.



Figures 3a, 3b, 3c: Undeformed & deformed configurations of a loaded membrane element & structure

For stress resultants of magnitudes $\sigma h \sqrt{G}$ & $\sigma h \sqrt{E}$ acting along contravariant tangent vectors $\overrightarrow{\mathrm{Vp}}$ & $\overrightarrow{\mathrm{Vq}}$,

$$W = \vec{P} \cdot \vec{X} - \sigma h \sqrt{G} \frac{\overrightarrow{Vp}}{\|\overrightarrow{Vp}\|} \cdot \sqrt{E} \frac{\overrightarrow{Tp}}{\|\overrightarrow{Tp}\|} = P x + Q y + R z - \sigma h \left\| \frac{\partial \vec{X}}{\partial p} \times \frac{\partial \vec{X}}{\partial q} \right\| \tag{4}$$

For minimum total potential energy, W must satisfy the Euler-Lagrange Equations (Olver [1]). By substitution,

$$\vec{P} + 2 \sigma h H A'' \vec{U} = 0 \tag{5}$$

for membranes with a middle surface of mean curvautre H, differential area A'', and unit surface normal $\vec{U}$.

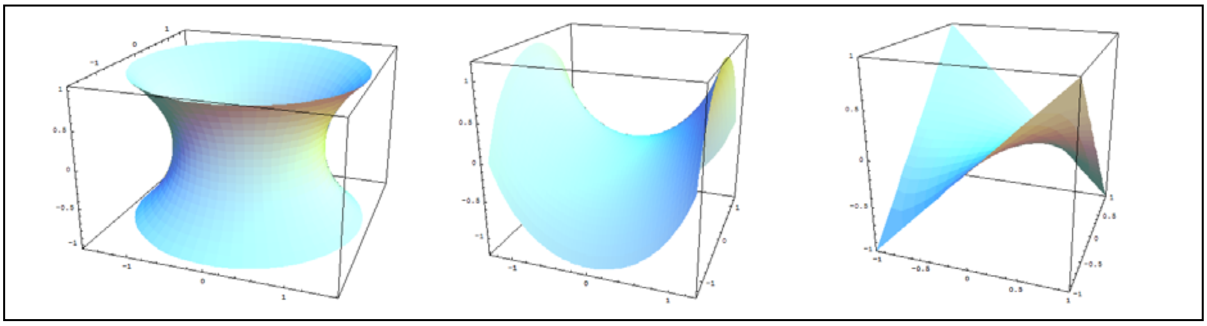
2.2 The Laplace-Young Equation for Membranes & the Spherical Pressure Vessel Formula

The scalar form of Equation 5 is Equation 6a: the Laplace-Young Equation for 2D Continua (cf. Lewis [3]). For spherical membranes of radius $R = 1/H$ and thickness $h = t$, it may take the form of Equation 6b: the Spherical Pressure Vessel Formula (Hibbeler [4]). For both equations, $\wp = \| \vec{P} \| / A''$.

$$\wp = 2\sigma h H \qquad \& \qquad \sigma = \frac{\wp R}{2t} \tag{6a \& 6b}$$

2.3 Minimal Surfaces

In addition to the restrictions made thus far, one may consider the case for which the membrane is loaded only at its boundary by tangential forces that equilibrate the stress resultants ($\wp = 0 \;\; \& \;\; \sigma h \neq 0 \;\; \Rightarrow \;\; H = 0$). For static equilibrium, the (middle) surface within that boundary must be the one with the least surface area. This is precisely the qualitative definition of a *minimal surface*, of which several are shown below in Figures 4.



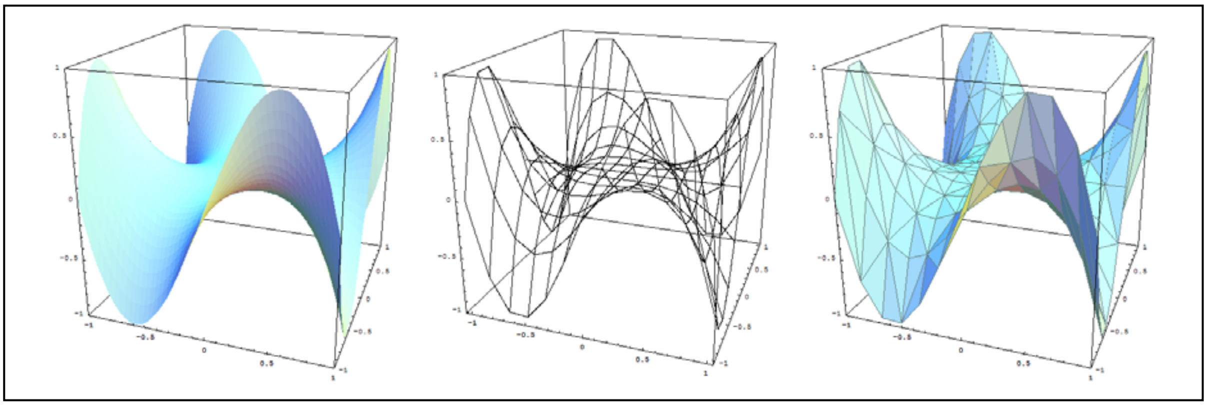
Figures 4a, 4b, 4c: Three minimal surfaces, posed as analytical form-finding problems

Analytical solutions to $H = 0$ for a surface of revolution (Figure 4a), a surface of translation (Figure 4b), and a surface bound by a regular skew quadrilateral 45 degrees out of plane (Figure 4c) involve the hyperbolic cosine, natural log of cosine, and Appell Hypergeometric functions, respectively (Gray [8], Furui and Masud [9]).

3. Computational Form-Finding of Cable & Membrane Structures

3.1 Setting

Computational form-finding is primarily characterized by geometric discretization. The shape of the structure (Figure 5a) is modeled as a mesh, which is an assemblage of elements such as line segments in the case of cable nets (Figure 5b) and planar triangles in the case of membranes (Figure 5c).



Figures 5a, 5b, 5c: Discretization of the surface form of a structure into cable & membrane elements

Discussion on cable-net form-finding (Linkwitz [10]) has been omitted in the interest of time. For both cable and membrane structures, the shape, loading, and stress distribution are typically prescribed, but for static equilibrium, the magnitude of the total resultant at each node must be zero. The fundamental concept behind computational form-finding is to adjust the shape until the magnitude of each total resultant is sufficiently small.

3.2 Energy Minimization & Dynamic Relaxation

Energy Minimization (Zhang & Tabarrok [5]) and Dynamic Relaxation (Barnes [6]) are two procedures for computational form-finding. The former entails the direct minimization of the potential energy with respect to each displacement. Shape-dependent quantities are functions of an unknown parameter, which is determined for each adjustment of the shape. The second procedure employs a technique to minimize the potential energy indirectly: targeting kinetic energy maxima over the course of displacements based on Newton's Second Law. Kinematic, constitutive, and equilibrium equations lend to numerically valued shape-dependent quantities.

3.3 Tangential Shape Variations & Deviatoric Stiffness

During an iteration, it is possible for every nodal displacement not to have a component along the normal vector to the surface, resulting in a tangential shape variation (Bletzinger [7]). If the element stiffnesses are based purely on changes in area, such cases can prevent an implementation of either procedure from reaching a unique shape. The author's solution to this problem, designed for Dynamic Relaxation, is to treat each element as a 2D isotropic material with both areal and deviatoric stiffness. If changes in both the area and the shape of each element are restricted elastically, then any tangential shape variations must also be restricted elastically.

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